PFAFFIAN IDENTITIES, WITH APPLICATIONS TO FREE RESOLUTIONS, DG–ALGEBRAS, AND ALGEBRAS WITH STRAIGHTENING LAW

Andrew R. Kustin

Abstract. A pfaffian identity plays a central role in the proof that the minimal resolution of a Huneke-Ulrich deviation two Gorenstein ring is a DG$\Gamma-$algebra. A second pfaffian identity is the patch which holds together two strands of the minimal resolution of a residual intersection of a grade three Gorenstein ideal. In this paper, we establish a family of pfaffian identities from which the two previously mentioned identities can be deduced as special cases. Our proof is by induction: the base case is an identity of binomial coefficients and the inductive step is a calculation from multilinear algebra.

Two unrelated problems in commutative algebra have recently been solved by finding the appropriate pfaffian identity. Fix a commutative noetherian ring R. For the first problem, let $X_{2n\times 2n}$ and $Y_{1\times 2n}$ be matrices with entries from R. Assume that X is an alternating matrix. Huneke and Ulrich [10] showed that if the ideal $I = I_1(YX) + \operatorname{Pf}(X)$ has the maximum possible grade (namely, $2n-1$), then I is a perfect Gorenstein ideal. The minimal $R-$resolution $\mathbb{M}$ of $A = R/I$ was found in [11]. Srinivasan [15] proved $\mathbb{M}$ is a DG$\Gamma-$algebra. The pfaffian identity [15, 4.3] is the key to showing that the proposed multiplication satisfies the differential property. The algebra structure on $\mathbb{M}$ was used in [12] to show that if $(R, \mathfrak{m}, k)$ is a regular local ring of equicharacteristic zero, then the Poincaré series

$$P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(M,k) z^i$$

is a rational function for all finitely generated $A-$modules M.

For the second problem, let J be a residual intersection of a grade three Gorenstein ideal I. If the data is sufficiently generic, then the generators of J, the minimal resolution of R/J, the minimal resolution of "half" of the divisor class group of R/J, and the minimal resolution of each power I^k have all been found in [14]. Each resolution in [14] is obtained by patching together two complexes all of whose maps are linear and well understood. The patch involves maps of many different degrees and is much more difficult to understand. Pfaffian identity [14,

1991 *Mathematics Subject Classification.* 15A69, 13C40, 13D25, 13F50.

Key words and phrases. algebras with straightening law, deviation two Gorenstein ideal, DG-algebras, pfaffian, residual intersection.

The author was supported in part by the National Science Foundation DMS–9106329.

3.2] shows that patched object is a complex. (Some of the results from [14] may be obtained using other techniques: the generators of J are also calculated in [13] and the ideals I^k are also resolved in [2].)

The above mentioned pfaffian identities (see Propositions 5.1 and 4.1 in the present paper) appear to be somewhat similar, but the exact relationship between them is not immediately obvious. The existing proofs of these identities are ad hoc and unpleasant. In the present paper, we derive the two identities from a common result: identity [15, 4.3] is the left side of Theorem 3.1 when the data is arranged so that the right side consists of only one term; and [14, 3.2] is the right side of Theorem 3.1 when the left side has become trivial. The proof of Theorem 3.1 is an induction which depends on an identity of binomial coefficients (Lemma 1.3) and a fact from multilinear algebra (Lemma 2.4).

In section 6 we apply our techniques to derive a straightening formula for pfaffians. We conclude with a brief section which shows that many familiar pfaffian identities are special cases of Theorem 3.1.

1. Binomial coefficients

We often consider binomial coefficients with negative parameters; consequently, we now recall the standard definition and properties of these objects.

Definition 1.1. For integers i and m, the binomial coefficient $\binom{m}{i}$ is defined to be

$$\binom{m}{i} = \begin{cases} \dfrac{m(m-1)\cdots(m-i+1)}{i!} & \text{if } 0 < i, \\ 1 & \text{if } 0 = i, \text{ and} \\ 0 & \text{if } i < 0. \end{cases}$$

Observation 1.2. (a) *If $0 \le m < i$, then $\binom{m}{i} = 0$.*

(b) *For all integers i and m,*

$$\binom{m}{i-1} + \binom{m}{i} = \binom{m+1}{i}.$$

(c) *If i and m are integers with $0 \le m$, then $\binom{m}{i} = \binom{m}{m-i}$.*

(d) *If i is a nonnegative integer, then $\binom{-1}{i} = (-1)^i$.*

(e) *For all integers a and b,*

$$\binom{a}{b} = (-1)^b \binom{b-a-1}{b}.$$

The identities in Lemma 1.3 form the base step in the proof of our main result. Corollary 1.4 is well known. It may be proved directly using an argument similar to the proof of Lemma 1.3 or it may be deduced from Lemma 1.3 after two applications of Observation 1.2 (e).

Lemma 1.3. *Let A, B, and C be integers. If $0 \le A$, then*

(a) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}\binom{A}{k} = (-1)^A\binom{B}{A+C}, \quad \textit{and}$$

(b) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}\binom{A}{k} = \binom{B-A}{C}.$$

Proof. The proof of (a) proceeds by induction on A. If $A=0$, then the only nonzero term on the left side occurs when $k=0$, and this term is $\binom{B}{C}$, which is equal to the right side. We now suppose that the result holds for a fixed A for all values of B and C. Let

$$X = \sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}\binom{A+1}{k}.$$

Observation 1.2 (b) gives $X = X_1 + X_2$, where

$$X_1 = \sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}\binom{A}{k} \qquad\text{and}\qquad X_2 = \sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}\binom{A}{k-1}.$$

The induction hypothesis gives

$$X_1 = (-1)^A\binom{B}{A+C} \qquad\text{and}\qquad X_2 = (-1)^{A+1}\binom{B+1}{A+C+1}.$$

We conclude that

$$X = X_1+X_2 = (-1)^{A+1}\left[\binom{B+1}{A+1+C} - \binom{B}{A+C}\right] = (-1)^{A+1}\binom{B}{A+1+C},$$

as desired. Replace k with $A-k$ in order to deduce (b) from (a). □

Corollary 1.4. *Let A, B, and C be integers. If $0 \le A$, then*

$$\sum_{k\in\mathbb{Z}}\binom{B}{C-k}\binom{A}{k} = \binom{A+B}{C}. \quad \square$$

The following generalization of Observation 1.2 (c) is used in the proof of Corollary 3.3.

Lemma 1.5. *If A and B are integers, then*

$$\binom{A}{A-B} = \binom{A}{B} + (-1)^{A+B}\binom{-B-1}{-A-1}.$$

Proof. Let

$$T_1 = \binom{A}{A-B}, \quad T_2 = \binom{A}{B} \quad\text{and}\quad T_3 = (-1)^{A+B}\binom{-B-1}{-A-1}.$$

If $0 \le A$, then $T_1 = T_2$ and $T_3 = 0$. Henceforth, we assume that $A \le -1$. If $0 \le B$, then $A - B \le A \le -1$ and $T_1 = 0$. Furthermore, in this case, $0 \le B - A - 1$; consequently,

$$T_2 = \binom{A}{B} = (-1)^B\binom{B-A-1}{B} = (-1)^B\binom{B-A-1}{-A-1} = (-1)^{A+B+1}\binom{-B-1}{-A-1} = -T_3.$$

Finally, we assume that $A \le -1$ and $B \le -1$. In this case, $T_2 = 0$ and

$$T_1 = \binom{A}{A-B} = (-1)^{A+B}\binom{-B-1}{A-B} = (-1)^{A+B}\binom{-B-1}{-A-1} = T_3. \quad \square$$

2. Multilinear algebra

Data 2.1. *Let R be a commutative noetherian ring, F be a free $R-$module of finite rank, and φ be an element of $\bigwedge^2 F$.*

We make much use of the $\bigwedge^\bullet F^*$-module structure on $\bigwedge^\bullet F$, as well as the $\bigwedge^\bullet F$-module structure on $\bigwedge^\bullet F^*$. In particular, if $a_i \in \bigwedge^i F$ and $b_j \in \bigwedge^j F^*$, then

$$a_i(b_j) \in \bigwedge^{j-i} F^* \qquad \text{and} \qquad b_j(a_i) \in \bigwedge^{i-j} F.$$

The following formulas are well known and not difficult to establish; see, for example, [4, Proposition A.3].

Proposition 2.2. *Adopt Data 2.1. Let a, $b \in \bigwedge^\bullet F$ and $c \in \bigwedge^\bullet F^*$ be homogeneous elements.*

(a) *If $a \in \bigwedge^1 F$, then*

$$(a(c))\,(b) = a \wedge (c(b)) + (-1)^{1+\deg c} c(a \wedge b).$$

(b) *If $c \in \bigwedge^{\operatorname{rank} F} F^*$, then*

$$(a(c))\,(b) = (-1)^\nu\,(b(c))\,(a),$$

where $\nu = (\operatorname{rank} F - \deg a)(\operatorname{rank} F - \deg b)$. $\square$

Note. The value for ν which is given above is correct and is different than the value given in [4].

We also make heavy use of the divided power structure on $\bigwedge^\bullet F$. If a is a homogeneous element of $\bigwedge^\bullet F$ of even degree, then, for each integer k, $a^{(k)} \in \bigwedge^\ell F$, where $\ell = k \cdot \deg a$. (Of course, if $\ell < 0$ or $\operatorname{rank} F < \ell$, then $\bigwedge^\ell F = 0$.) Some of the properties of divided powers are collected below; more information may be found in [4].

Proposition 2.3. *Adopt Data 2.1.*

(a) *If k and ℓ are integers, then*

$$\varphi^{(k)} \wedge \varphi^{(\ell)} = \binom{k+\ell}{k} \varphi^{(k+\ell)}.$$

(b) *If $b \in F^*$ and k is an integer, then* $b\left(\varphi^{(k)}\right) = b(\varphi) \wedge \varphi^{(k-1)}$.

The following multilinear algebra calculation is the key to the induction step in the proof of our main result.

Lemma 2.4. *Adopt Data 2.1. Let k and ℓ be integers. If b_1 and b are homogeneous elements of $\bigwedge^\bullet F^*$ with $\deg b_1 = 1$, then*

$$b_1\left[\left(\varphi^{(k)}(b)\right)(\varphi^{(\ell)})\right] + b_1\left[\left(\varphi^{(k-1)}(b)\right)(\varphi^{(\ell+1)})\right] - b_1(\varphi^{(1)}) \wedge \left[\left(\varphi^{(k-1)}(b)\right)(\varphi^{(\ell)})\right]$$
$$= \left[\varphi^{(k)}(b_1 \wedge b)\right](\varphi^{(\ell)}).$$

Proof. We expand $S_1 = \left[\left(b_1(\varphi^{(1)})\right)\left(\varphi^{(k-1)}(b)\right)\right](\varphi^{(\ell)})$ two different ways. On the one hand, Proposition 2.2 gives $S_1 = S_2 + S_3$, where

$$S_2 = b_1(\varphi^{(1)}) \wedge \left(\varphi^{(k-1)}(b)\right)(\varphi^{(\ell)}) \text{ and } S_3 = (-1)^{1+\deg b}\left(\varphi^{(k-1)}(b)\right)\left(b_1(\varphi^{(1)}) \wedge \varphi^{(\ell)}\right).$$

Proposition 2.3, together with the module action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$, yields

$$\begin{aligned} S_3 &= (-1)^{1+\deg b}\left(\varphi^{(k-1)}(b)\right)\left(b_1(\varphi^{(\ell+1)}\right) = (-1)^{1+\deg b}\left(\varphi^{(k-1)}(b) \wedge b_1\right)(\varphi^{(\ell+1)}) \\ &= -\left[b_1 \wedge \varphi^{(k-1)}(b)\right](\varphi^{(\ell+1)}) = -\, b_1\left[\left(\varphi^{(k-1)}(b)\right)(\varphi^{(\ell+1)})\right]. \end{aligned}$$

On the other hand, Proposition 2.2 gives

$$\left(b_1(\varphi^{(k)})\right)(b) = b_1 \wedge \varphi^{(k)}(b) - \varphi^{(k)}(b_1 \wedge b);$$

and therefore,

$$\begin{aligned} S_1 &= \left(\left[b_1(\varphi^{(1)}) \wedge \varphi^{(k-1)}\right](b)\right)(\varphi^{(\ell)}) = \left[\left(b_1(\varphi^{(k)})\right)(b)\right](\varphi^{(\ell)}) \\ &= b_1\left[\left(\varphi^{(k)}(b)\right)(\varphi^{(\ell)})\right] - \left(\varphi^{(k)}(b_1 \wedge b)\right)(\varphi^{(\ell)}). \end{aligned}$$

The proof is completed by equating the two values for S_1. □

The identities in section 3 are stated and proved in a coordinate free manner; however, some of the applications in sections 4 through 7 are derived from these identites by using bases. The hypothesis in these applications is that X is an alternating matrix with entries from the commutative noetherian ring R. We recover Data 2.1 as follows. Suppose that the matrix $X = (x_{ij})$ has N rows and columns. Let $e_1, \dots, e_N$ be a basis for a free $R-$module F; let $\varepsilon_1, \dots, \varepsilon_N$ be the corresponding dual basis for F^*; and let φ be the element

$$\varphi = \sum_{1 \le i < j \le N} x_{ij}\, e_i \wedge e_j$$

of $\bigwedge^2 F$. If $I = (n_1, \dots, n_i)$ is an $i-$tuple of integers, then define $e_I \in \bigwedge^i F$ and $\varepsilon_I \in \bigwedge^i F^*$ by

$$e_I = e_{n_1} \wedge e_{n_2} \wedge \dots \wedge e_{n_i} \qquad \text{and} \qquad \varepsilon_I = \varepsilon_{n_1} \wedge \varepsilon_{n_2} \wedge \dots \wedge \varepsilon_{n_i}.$$

Calculation 2.5. *If I is an $i-$tuple of integers, then*

$$e_I(\varepsilon_I) = \varepsilon_I(e_I) = (-1)^{\frac{i(i-1)}{2}}.$$

Proof. Let $f_i = (e_1 \wedge e_2 \wedge \dots \wedge e_i)(\varepsilon_1 \wedge \varepsilon_2 \wedge \dots \wedge \varepsilon_i)$. It is clear that $f_1 = 1$ and $f_{i+1} = (-1)^i f_i$. It is also clear that $f_i = (-1)^{\frac{i(i-1)}{2}}$ solves this recurrence relation. □

Let $I=(n_1,\dots,n_i)$ be a fixed $i-$tuple of integers. If the integers $n_1,\dots,n_i$'s are distinct, then $\sigma(I)$ is the sign of the permutation which rearranges $n_1,\dots,n_i$ into ascending order. If there is a repeat among the n's, then $\sigma(I)=0$. We say that I is *an index set of size* i if $n_1<n_2<\dots<n_i$. Sometimes, we write $|I|$ for the size of the index set I. Let S be a fixed finite set of distinct integers. We write

$$\sum_{\substack{J_1\cup J_2=S\\|J_1|=j_1}}$$

to mean that the sum is taken over all j_1-element subsets J_1 of S, and the complement of J_1 in S is denoted J_2. In the above sum, J_1 and J_2 are both taken to be index sets. If $b\in\bigwedge^r F^*$, then $b(e_I)$ is the element

$$b(e_I)=\sum_{\substack{I_1\cup I_2=I\\|I_1|=r}}\sigma(I_1I_2)b(e_{I_1})e_{I_2} \tag{2.6}$$

of $\bigwedge^{i-r}F$. If $Y=(y_{ij})$ is a matrix and I_1 and J are $j-$tuples of integers, then we take $Y(I_1;J)$ to be the determinant of the submatrix of Y which consists of rows I_1 and columns J. In particular, If $f\colon F^*\to F$ is a map with $f(\varepsilon_j)=\sum\limits_i y_{ij}e_i$ and J is a $j-$tuple of integers, then

$$(\bigwedge^j f)(\varepsilon_J)=\sum_{\substack{I_1\cup I_2=\{1,\dots,N\}\\|I_1|=j}}Y(I_1;J)e_{I_1}.$$

Thus, Calculation 2.5 shows that if $|I|=|J|=j$, then

$$\varepsilon_I\left((\bigwedge^j f)(\varepsilon_J)\right)=\varepsilon_I(e_I)Y(I,J)=(-1)^{\frac{j(j-1)}{2}}Y(I;J). \tag{2.7}$$

We use the notation "X_I" to denote pfaffians; in particular,

$$X_I=\begin{cases}\text{the pfaffian of the principal submatrix of } X \text{ consisting}\\ \text{of rows and columns } n_1,\dots,n_i \text{ (in the given order)} & \text{if } 0<i,\\ 1 & \text{if } i=0, \text{ and}\\ 0 & \text{if } i<0.\end{cases}$$

It follows that if i is an even integer, then

$$\varphi^{(i/2)}=\sum_{\substack{J_1\cup J_2=\{1,\dots,N\}\\|J_1|=i}}X_{J_1}e_{J_1};$$

and therefore, Calculation 2.5 shows that

$$\varepsilon_I\left(\varphi^{(i/2)}\right)=\varepsilon_I\left(e_I\right)X_I=(-1)^{i/2}X_I. \tag{2.8}$$

3. The main results

The first main result in this paper is Theorem 3.1. Everything in sections 4 through 7, with one exception, follows from Theorem 3.1 or one of its five Corollaries. The one exception is Proposition 5.4, and this result is a consequence of Theorem 3.7, which is the other main result in the paper.

Theorem 3.1. *Adopt Data 2.1. Let A, B, C, and d be integers. If $b \in \bigwedge^d F^*$, then*

(a) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}b(\varphi^{(k)})\wedge\varphi^{(A-k)} = \sum_{k\in\mathbb{Z}}(-1)^{A+k}\binom{B+d-k}{A+C-k}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}),$$

and

(b) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}b(\varphi^{(k)})\wedge\varphi^{(A-k)} = \sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{B-A+k}{C-d+k}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

Note. If $k<0$ or $A<k$, then the corresponding terms in each of the above summations is zero.

Proof. The proof of (a) proceeds by induction on d. We first suppose that $d=0$. In this case, the left hand side is $Xb(\varphi^{(A)})$, where

$$X=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}\binom{A}{k}.$$

The only non-zero term on the right side occurs when $k=0$, and this term is $Yb(\varphi^{(A)})$, where

$$Y=(-1)^A\binom{B}{A+C}.$$

If $A\le -1$, then $\varphi^{(A)}=0$. If $0\le A$, then Lemma 1.3 shows that $X=Y$.

Henceforth, we assume that $1\le d$. We take $b=b_1\wedge b'$, with $\deg b_1=1$ and $\deg b'=d-1$. Let

$$X=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}b(\varphi^{(k)})\wedge\varphi^{(A-k)}.$$

The element b_1 of F^* acts like a graded derivation on $\bigwedge^\bullet F$; therefore,

$$X=b_1(S_1)-b_1(\varphi^{(1)})\wedge S_2,$$

where

$$S_1=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}b'(\varphi^{(k)})\wedge\varphi^{(A-k)} \quad \text{and}$$

$$S_2=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}b'(\varphi^{(k)})\wedge\varphi^{(A-1-k)}.$$

The induction hypothesis gives

$$S_1 = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^k \binom{B+d-1-k}{A+C-k} \left(\varphi^{(k)}(b')\right)(\varphi^{(A-k)}) \quad \text{and}$$

$$S_2 = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^{k+1} \binom{B+d-1-k}{A-1+C-k} \left(\varphi^{(k)}(b')\right)(\varphi^{(A-1-k)}).$$

Use Observation 1.2 (b) in order to write $S_1 = S_1' + S_1''$, where

$$S_1' = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^k \binom{B+d-k}{A+C-k} \left(\varphi^{(k)}(b')\right)(\varphi^{(A-k)}) \quad \text{and}$$

$$S_1'' = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^{k+1} \binom{B+d-1-k}{A+C-k-1} \left(\varphi^{(k)}(b')\right)(\varphi^{(A-k)}).$$

An index shift yields

$$S_1'' = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^k \binom{B+d-k}{A+C-k} \left(\varphi^{(k-1)}(b')\right)(\varphi^{(A-k+1)}) \quad \text{and}$$

$$S_2 = (-1)^A \sum_{k\in\mathbb{Z}} (-1)^k \binom{B+d-k}{A+C-k} \left(\varphi^{(k-1)}(b')\right)(\varphi^{(A-k)}).$$

We now have

$$\begin{aligned} X &= b_1(S_1' + S_1'') - b_1(\varphi^{(1)}) \wedge S_2 \\ &= (-1)^A \sum_{k\in\mathbb{Z}} (-1)^k \binom{B+d-k}{A+C-k} \begin{bmatrix} b_1\left[\left(\varphi^{(k)}(b')\right)(\varphi^{(A-k)})\right] \\ +b_1\left[\left(\varphi^{(k-1)}(b')\right)(\varphi^{(A-k+1)})\right] \\ -\left(b_1(\varphi^{(1)})\right) \wedge \left[\left(\varphi^{(k-1)}(b')\right)(\varphi^{(A-k)})\right] \end{bmatrix}. \end{aligned}$$

Apply Lemma 2.4 to complete the proof of (a). The proof of (b) is completely analogous to the proof of (a); we omit the details. □

In our first variation of Theorem 3.1, we arrange the data so that only one term on the right side of the identity survives.

Corollary 3.2. *Adopt Data 2.1. Let p, q, and d be integers. If $b \in \bigwedge^d F^*$, then*

(a) $$\sum_{k\in\mathbb{Z}} (-1)^{k+p} \binom{q-d-1+k}{k-p} b(\varphi^{(k)}) \wedge \varphi^{(p+q-k)} = \left(\varphi^{(q)}(b)\right)(\varphi^{(p)}), \quad \textit{and}$$

(b) $$\sum_{k\in\mathbb{Z}} (-1)^{k+d+q} \binom{p-1-k}{d-q-k} b(\varphi^{(k)}) \wedge \varphi^{(p+q-k)} = \left(\varphi^{(q)}(b)\right)(\varphi^{(p)}).$$

Proof. Apply Theorem 3.1 (a) with A replaced by $p+q$, B replaced by $q-d-1$, and C replaced by $-p$, in order to see that

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{q-d-1+k}{k-p}b(\varphi^{(k)})\wedge\varphi^{(p+q-k)}=\sum_{k\in\mathbb{Z}}(-1)^{q+p+k}\binom{q-1-k}{q-k}\left(\varphi^{(k)}(b)\right)(\varphi^{(p+q-k)}).$$

The proof of (a) is complete because right side of the above sum is zero unless $q=k$. Assertion (b) follows from Theorem 3.1 (b) with A replaced by $p+q$, B replaced by $p-1$ and C replaced with $d-q$. An alternate proof of (b) can be obtained from (a) by way of Lemma 1.5 and Corollary 3.4. □

In our second variation of Theorem 3.1, we remove the "k" from the bottom part of the binomial coefficients.

Corollary 3.3. *Adopt Data 2.1. Let A, B, L, and d be integers. If $b\in\bigwedge^d F^*$, then*

(a) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{L}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=\sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{B-A+k}{L+d-A}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)})\text{ and}$$

(b) $$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{L}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=\sum_{k\in\mathbb{Z}}(-1)^{A+k}\binom{d+B-k}{L+d-A}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

Proof. We first prove (a). Apply (b) of Theorem 3.1, with C replaced by $B-L$, in order to see that $L_1=R_1$, where

$$L_1=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{B-L-k}b(\varphi^{(k)})\wedge\varphi^{(A-k)}\quad\text{and}$$

$$R_1=(-1)^d\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-A+k}{B-L-d+k}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

Apply (a) of Theorem 3.1, with B replaced by $L-B-1$ and C replaced by $-B-1$, in order to see that $L_2=R_2$, where

$$L_2=\sum_{k\in\mathbb{Z}}(-1)^k\binom{L-B-1+k}{-B-1+k}b(\varphi^{(k)})\wedge\varphi^{(A-k)}\quad\text{and}$$

$$R_2=\sum_{k\in\mathbb{Z}}(-1)^{A+k}\binom{L-B-1+d-k}{-B-1+A-k}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

Apply Lemma 1.5 (with A replaced by $B-k$ and B replaced by $B-L-k$) in order to see that

$$L_1+(-1)^L L_2=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{L}b(\varphi^{(k)})\wedge\varphi^{(A-k)}.$$

Lemma 1.5 (with A replaced by $B-A+k$ and B replaced by $B-L-d+k$) shows that

$$R_1+(-1)^L R_2=\sum_{k\in\mathbb{Z}}(-1)^{k+d}\binom{B-A+k}{d+L-A}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

The proof of (a) is complete. Identity (b) follows from (a), by way of Observation 1.2 (e); replace B with $L-B-1$. (An alternate proof of Corollary 3.3 can be obtained by mimicking the proof of Theorem 3.1.) □

In our third variation of Theorem 3.1, we impose hypotheses which allow us to set the identities in Corollary 3.3 to zero, one side at a time.

Corollary 3.4. *Adopt Data 2.1. Let A, B, d, and L be integers, and let b be an element of $\bigwedge^d F^*$.*

(i) *If $A+L+1\le d$, then*

$$\sum_{k\in Z}(-1)^k\binom{B+k}{L}(\varphi^{(k)}(b))(\varphi^{(A-k)})=0,\quad \textit{and}$$
$$\sum_{k\in Z}(-1)^k\binom{B-k}{L}(\varphi^{(k)}(b))(\varphi^{(A-k)})=0.$$

(ii) *If $d\le A-L-1$, then*

$$\sum_{k\in Z}(-1)^k\binom{B-k}{L}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=0,\quad \textit{and}$$
$$\sum_{k\in Z}(-1)^k\binom{B+k}{L}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=0.$$

Proof. Identity (a) from Corollary 3.3 gives

$$\sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{A+B-k}{L-d+A}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=\sum_{k\in Z}(-1)^k\binom{B+k}{L}(\varphi^{(k)}(b))(\varphi^{(A-k)});$$

and the hypothesis ensures that $\binom{A+B-k}{L-d+A}=0$. The other three assertions are established in the same way. $\square$

In a similar manner, one can find hypotheses which set each side of the identities in Theorem 3.1 to zero. Corollary 3.5, which plays a central role in section 6, is an example of this technique.

Corollary 3.5. *Adopt Data 2.1. Let A, B, C, and d be integers. Assume that $b\in\bigwedge^d F^*$. If $\frac{-1-d}{2}\le B\le A+C-d-1$, then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{C+k}b(\varphi^{(k)})\wedge\varphi^{(A-k)}=0.$$

Proof. If S is the left side of the identity, then Theorem 3.1 shows that

$$S=\sum_{k\in\mathbb{Z}}(-1)^{A+k}\binom{B+d-k}{A+C-k}\left(\varphi^{(k)}(b)\right)(\varphi^{(A-k)}).$$

If $0\le B+d-k$, then the hypothesis guarantees that $0\le B+d-k\le A+C-k-1$, and the binomial coefficient is zero. If $B+d-k\le -1$, then the hypothesis also guarantees that $\deg\varphi^{(k)}(b)\le -1$; thus, $\varphi^{(k)}(b)=0$. It follows that $S=0$. $\square$

Our fifth variation of Theorem 3.1 involves two elements of $\bigwedge^\bullet F^*$.

Corollary 3.6. *Adopt Data 2.1. Let A, B, d, and L be integers, and let b and b' be homogeneous elements of $\bigwedge^\bullet F^*$ with $b\in \bigwedge^d F^*$. If $A+L+1\le d$, then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{L}\left(\left(b'(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)})=0.$$

Proof. The proof is by induction on the degree of b'. If $\deg b'=0$, then the result is contained in Corollary 3.4. Henceforth, we assume that $b'=b_1\wedge b''$, where $\deg b_1=1$ and $\deg b''=\deg b'-1$. Apply Proposition 2.2 in order to see that $\left(b'(\varphi^{(k)})\right)(b)$ is equal to

$$(b_1\left(b''(\varphi^{(k)})\right))(b)=b_1\wedge\left(\left(b''(\varphi^{(k)})\right)(b)\right)+(-1)^{\deg b''+1}\left(b''(\varphi^{(k)})\right)(b_1\wedge b).$$

When this identity is applied to $\varphi^{(A-k)}$, we see that $(\left(b'(\varphi^{(k)})\right)(b))(\varphi^{(A-k)})$ is equal to

$$b_1\left(\left(\left(b''(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)})\right)+(-1)^{\deg b''+1}\left(\left(b''(\varphi^{(k)})\right)(b_1\wedge b)\right)(\varphi^{(A-k)}).$$

It follows that the left side of the identity is equal to $b_1(S_1)+(-1)^{\deg b''+1}S_2$, where

$$S_1=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{L}\left(\left(b''(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)})\quad\text{and}$$

$$S_2=\sum_{k\in\mathbb{Z}}(-1)^k\binom{B+k}{L}\left(\left(b''(\varphi^{(k)})\right)(b_1\wedge b)\right)(\varphi^{(A-k)}).$$

The induction hypothesis yields $S_1=S_2=0$, because

$$1+A+L\le\deg b<\deg(b_1\wedge b)\qquad\text{and}\qquad\deg b''<\deg b'.\quad\square$$

The second main result in this paper is concerned with the data of Corollary 3.6 when the hypothesis $A+L+1\le d$ is not satisfied. Our proof of Theorem 3.7 is much like an argument from a class in Differential Equations: we show that two expressions are equal by verifying that they both are solutions of the same "Initial Value Problem".

Theorem 3.7. *Adopt Data 2.1. Let $\widetilde{\varphi}\colon F^*\to F$ be the homomorphism which is given by $\widetilde{\varphi}(b_1)=b_1(\varphi)$ for all $b_1\in F^*$. Let d and A be integers and let b and b' be homogeneous elements of $\bigwedge^\bullet F^*$. If $\deg b=d$, then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\left(\left(b'(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)})=\sum_{k\in\mathbb{Z}}(-1)^k\varphi^{(A-d-k)}\wedge\left(\varphi^{(k)}(b')\right)\left((\bigwedge^d\widetilde{\varphi})(b)\right).$$

Proof. For each fixed integer A, we define bilinear maps

$$\Phi_A\colon\bigwedge^\bullet F^*\oplus\bigwedge^\bullet F^*\to\bigwedge^\bullet F\qquad\text{and}\qquad\Psi_A\colon\bigwedge^\bullet F^*\oplus\bigwedge^\bullet F^*\to\bigwedge^\bullet F.$$

If b and b' are homogeneous elements of $\bigwedge^{\bullet} F^*$, then

$$\Phi_A(b',b) = \sum_{k\in\mathbb{Z}}(-1)^k \left(\left(b'(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)}) \qquad \text{and}$$

$$\Psi_A(b',b) = \sum_{k\in\mathbb{Z}}(-1)^k \varphi^{(A-k-\deg b)} \wedge \left(\varphi^{(k)}(b')\right)\left(\big(\bigwedge^{\deg b}\widetilde{\varphi}\big)(b)\right).$$

We prove that $\Phi_A(b',b) = \Psi_A(b',b)$ by induction on $\deg b'$. We must show

$$\Phi_A(1,b) = \varphi^{(A-\deg b)} \wedge \left(\bigwedge^{\deg b}\widetilde{\varphi}\right)(b), \tag{3.8}$$

$$\Psi_A(1,b) = \varphi^{(A-\deg b)} \wedge \left(\bigwedge^{\deg b}\widetilde{\varphi}\right)(b), \tag{3.9}$$

$$\Phi_A(b_1\wedge b',b) = b_1(\Phi_A(b',b)) + (-1)^{\deg b'+1}\Phi_A(b',b_1\wedge b), \qquad \text{and} \tag{3.10}$$

$$\Psi_A(b_1\wedge b',b) = b_1(\Psi_A(b',b)) + (-1)^{\deg b'+1}\Psi_A(b',b_1\wedge b). \tag{3.11}$$

for all $b_1 \in F^*$.

It is clear that $\Phi_A(1,1) = \varphi^{(A)}$. If $b_1 \in F^*$, then Lemma 2.4 shows that

$$\Phi_A(1,b_1\wedge b) = b_1(\Phi_A(1,b)) - b_1(\Phi_A(1,b)) + b_1(\varphi)\wedge\Phi_{A-1}(1,b) = b_1(\varphi)\wedge\Phi_{A-1}(1,b).$$

We conclude that (3.8) holds. Assertion (3.9) is obvious. The proof of Corollary 3.6 shows that $\left[\left((b_1\wedge b')(\varphi^{(k)})\right)(b)\right](\varphi^{(A-k)})$ is equal to

$$b_1\left(\left(\left(b'(\varphi^{(k)})\right)(b)\right)(\varphi^{(A-k)})\right) + (-1)^{\deg b'+1}\left(\left(b'(\varphi^{(k)})\right)(b_1\wedge b)\right)(\varphi^{(A-k)})$$

and (3.10) holds.

Now we prove (3.11). Let $d = \deg b$, and let $X = \left((\bigwedge^d \widetilde{\varphi})(b)\right)$. We expand

$$Y = \left[\left(b_1(\varphi^{(k)})\right)(b')\right](X)$$

two different ways. On the one hand, Proposition 2.2 gives

$$Y = \left[b_1\wedge\varphi^{(k)}(b')\right](X) - \left[\varphi^{(k)}(b_1\wedge b')\right](X) = b_1\left(\left[\varphi^{(k)}(b')\right](X)\right) - \left[\varphi^{(k)}(b_1\wedge b')\right](X).$$

On the other hand,

$$Y = \left[\left(b_1(\varphi)\wedge\varphi^{(k-1)}\right)(b')\right](X) = \left[(b_1(\varphi))\left(\varphi^{(k-1)}(b')\right)\right](X).$$

Apply Proposition 2.2, once again, in order to see that

$$Y = b_1(\varphi) \wedge \left(\varphi^{(k-1)}(b')\right)(X) + (-1)^{\deg b'+1}\left(\varphi^{(k-1)}(b')\right)\left(b_1(\varphi)\wedge X\right).$$

Combine the two expansions of Y in order to obtain that $\left[\varphi^{(k)}(b_1 \wedge b')\right](X)$ is equal to

$$b_1\left(\left[\varphi^{(k)}(b')\right](X)\right) - b_1(\varphi)\wedge\left(\varphi^{(k-1)}(b')\right)(X) - (-1)^{\deg b'+1}\left[\varphi^{(k-1)}(b')\right]\left(b_1(\varphi)\wedge X\right).$$

Thus, $S_1 = S_2 + S_3 + (-1)^{\deg b'+1} S_4$, where

$$S_1 = \sum_{k\in\mathbb{Z}} (-1)^k \varphi^{(A-d-k)} \wedge \left[\varphi^{(k)}(b_1\wedge b')\right](X) = \Psi_A(b_1 \wedge b', b),$$

$$S_2 = \sum_{k\in\mathbb{Z}} (-1)^k \varphi^{(A-d-k)} \wedge b_1\left(\left[\varphi^{(k)}(b')\right](X)\right),$$

$$S_3 = -\sum_{k\in\mathbb{Z}} (-1)^k \varphi^{(A-d-k)} \wedge b_1(\varphi) \wedge \left(\varphi^{(k-1)}(b')\right)(X) = \sum_{k\in\mathbb{Z}} (-1)^k b_1(\varphi^{(A-d-k)}) \wedge \left(\varphi^{(k)}(b')\right)(X),$$

and

$$\begin{aligned} S_4 &= -\sum_{k\in\mathbb{Z}} (-1)^k \varphi^{(A-d-k)} \wedge \left(\varphi^{(k-1)}(b')\right)(b_1(\varphi)\wedge X) \\ &= \sum_{k\in\mathbb{Z}} (-1)^k \varphi^{(A-d-1-k)} \wedge \left(\varphi^{(k)}(b')\right)(b_1(\varphi)\wedge X). \end{aligned}$$

Observe that $S_2 + S_3 = b_1(\Psi_A(b', b))$. Observe also that

$$b_1(\varphi) \wedge X = b_1(\varphi) \wedge \left((\bigwedge^{d} \widetilde{\varphi})(b) \right) = (\bigwedge^{d+1} \widetilde{\varphi})(b_1 \wedge b).$$

It follows that $S_4 = \Psi(b', b_1 \wedge b)$. Assertion (3.11) has been established and the proof is complete. □

4. A new proof of the Kustin-Ulrich pfaffian identity

We give a new proof of [14, 3.2]. The notation is explained in section 2.

Proposition 4.1. *Let X be an alternating matrix with entries from the commutative noetherian ring R. If A, B, C, and D are index sets of size a, b, c, and d, respectively, with $a+b+d$ even and $a+b \le d-2$, then*

$$\sum_{\substack{t\in\mathbb{Z} \\ a+c+t \text{ even}}} \sum_{\substack{E\cup F = D \\ |E|=t}} (-1)^{(a+c+t)/2} \sigma(EF) X_{ACE} X_{BCF} = 0.$$

Proof. If M is an integer and β, β', and β'' are homogeneous elements of $\bigwedge^\bullet F^*$, then Corollary 3.6 shows that

$$\sum_{k\in\mathbb{Z}} (-1)^k \left[\left(\left(\beta'(\varphi^{(k)})\right)(\beta)\right)(\varphi^{(M-k)})\right](\beta'')$$

is the zero element of the ring R, provided

$$\deg \beta'' = 2M - \deg \beta - \deg \beta' \qquad \text{and} \tag{4.2}$$

$$1 + M \le \deg \beta. \tag{4.3}$$

We may apply Corollary 3.6 with $M = c + \frac{a+b+d}{2}$, $\beta' = \varepsilon_A$, $\beta = \varepsilon_C \wedge \varepsilon_D$, and $\beta'' = \varepsilon_B \wedge \varepsilon_C$. Indeed, (4.2) is satisfied because $b + c = 2M - (a) - (c + d)$; and (4.3) is equivalent to the hypothesis $a + b + 2 \le d$. It follows that

$$\sum_{k\in\mathbb{Z}} (-1)^k \left[\left(\left(\varepsilon_A(\varphi^{(k)})\right)(\varepsilon_C \wedge \varepsilon_D)\right)(\varphi^{(M-k)})\right](\varepsilon_B \wedge \varepsilon_C) = 0. \tag{4.4}$$

We complete the proof by translating (4.4) back into the language of the original statement. For the time being, consider k to be fixed. Let $r = 2k - a$ and let Y represent the element $\varepsilon_A(\varphi^{(k)})$ of $\bigwedge^r F$. We evaluate

$$\left[(Y\,(\varepsilon_C \wedge \varepsilon_D))(\varphi^{(M-k)})\right](\varepsilon_B \wedge \varepsilon_C)\,. \tag{4.5}$$

Recall, from (2.6), that

$$Y(\varepsilon_C \wedge \varepsilon_D) = \sum_{j=0}^{r} \sum_{\substack{C_1\cup C_2=C\\ |C_1|=j}} \sum_{\substack{E\cup F=D\\ |E|=r-j}} (-1)^{|C_2|\cdot|E|}\sigma(C_1C_2)\sigma(EF)Y(\varepsilon_{C_1} \wedge \varepsilon_E) \wedge \varepsilon_{C_2} \wedge \varepsilon_F. \tag{4.6}$$

Furthermore, in (4.6), $2(M-k) - |C_2| - |F| = |B| + |C|$; consequently, when (4.6) is inserted into (4.4), we have

$$\left[(\varepsilon_{C_2} \wedge \varepsilon_F)(\varphi^{(M-k)})\right](\varepsilon_B \wedge \varepsilon_C) = (\varepsilon_B \wedge \varepsilon_C \wedge \varepsilon_{C_2} \wedge \varepsilon_F)(\varphi^{(M-k)}).$$

If C_2 is nonempty, then $\varepsilon_C \wedge \varepsilon_{C_2} = 0$. It follows that (4.5) is equal to

$$\sum_{\substack{E\cup F=D\\ |E|=r-c}} \sigma(EF)Y(\varepsilon_C \wedge \varepsilon_E)\cdot(\varepsilon_B \wedge \varepsilon_C \wedge \varepsilon_F)(\varphi^{(M-k)}). \tag{4.7}$$

Furthermore, in (4.7), we have $\deg Y = \deg(\varepsilon_C \wedge \varepsilon_E)$; therefore,

$$Y(\varepsilon_C \wedge \varepsilon_E) = (\varepsilon_C \wedge \varepsilon_E)Y = (\varepsilon_C \wedge \varepsilon_E)\left(\varepsilon_A(\varphi^{(k)})\right) = (-1)^a(\varepsilon_A \wedge \varepsilon_C \wedge \varepsilon_E)(\varphi^{(k)}).$$

We now see that (4.4) is

$$\sum_{k\in\mathbb{Z}} (-1)^k \sum_{\substack{E\cup F=D\\ |E|=2k-a-c}} \sigma(EF)(\varepsilon_A \wedge \varepsilon_C \wedge \varepsilon_E)(\varphi^{(k)})\cdot(\varepsilon_B \wedge \varepsilon_C \wedge \varepsilon_F)(\varphi^{(M-k)}) = 0.$$

Apply (2.8) to see that

$$\sum_{k\in\mathbb{Z}} (-1)^k \sum_{\substack{E\cup F=D\\ |E|=2k-a-c}} \sigma(EF)X_{ACE}X_{BCF} = 0.$$

Replace k by $\frac{t+a+c}{2}$ in order to complete the proof. $\square$

5. Srinivasan's identities

In this section we show that the identities of [15, 16] are special cases of our results. We begin by deducing [15, 4.3] from Theorem 3.1. Proposition 4.4 in [15] is an immediate consequence of Corollary 3.3; results 4.5, 4.6, 4.7, and 4.8 in [15] are immediate consequences of Lemma 1.3. Proposition 5.3 provides a new proof of [15, 4.2]. In Proposition 5.4, we give a new proof of Srinivasan's decomposition [15, page 447] of the pfaffian X_{IJ} into products of the form $X_{IJ_1}\cdot X_{J_2}$ and $X_{I_1}\cdot X(I_2;J)$. Proposition 5.5 is a reformulation of identity (b) from [15, page 447].

Proposition 5.1. *Adopt Data 2.1. Let F have rank $2n$, and let i, j, and t be integers with $t+j$ even. If ξ is an element of $\bigwedge^{2n} F^*$ and a is an element of $\bigwedge^t F$, then*

$$\left(\left(\varphi^{(n-\frac{t+j}{2})}\wedge a\right)(\xi)\right)(\varphi^{([[\frac{i+j}{2}]])})$$

is equal to

$$\sum_{\substack{w=0\\ j-w \text{ even}}}^{j}\binom{n-[[\frac{i+t}{2}]]}{\frac{j-w}{2}}\left[\left(\varphi^{(n-\frac{t-w}{2})}(\xi)\right)(a)\right]\wedge\varphi^{([[\frac{i-w}{2}]])}.$$

Proof. Write $i=i'+\varepsilon$, where $i'+t$ is even and ε is either 0 or 1. Notice that

$$\left[\left[\frac{i+j}{2}\right]\right]=\frac{i'+j}{2},\qquad \left[\left[\frac{i+t}{2}\right]\right]=\frac{i'+t}{2},\qquad\text{and}\qquad \left[\left[\frac{i-w}{2}\right]\right]=\frac{i'-w}{2}. \tag{5.2}$$

In identity (b) from Corollary 3.2, replace p with $\frac{i'+j}{2}$, q with $n-\frac{t+j}{2}$, d with $2n-t$, and k with $n+\frac{w-t}{2}$, in order to see that

$$\left(\varphi^{(n-\frac{t+j}{2})}(b)\right)(\varphi^{(\frac{i'+j}{2})})$$

is equal to

$$\sum_{\substack{w\in\mathbb{Z}\\ j-w \text{ even}}}(-1)^{\frac{w-j}{2}}\binom{\frac{j-w}{2}-n+\frac{i'+t}{2}-1}{\frac{j-w}{2}}b(\varphi^{(n+\frac{w-t}{2})})\wedge\varphi^{(\frac{i'-w}{2})}.$$

Observation 1.2 (e) yields

$$\binom{\frac{j-w}{2}-n+\frac{i'+t}{2}-1}{\frac{j-w}{2}}=(-1)^{\frac{j-w}{2}}\binom{n-\frac{i'+t}{2}}{\frac{j-w}{2}};$$

and therefore,

$$\left(\varphi^{(n-\frac{t+j}{2})}(b)\right)(\varphi^{(\frac{i'+j}{2})})=\sum_{\substack{w\in\mathbb{Z}\\ j-w \text{ even}}}\binom{n-\frac{i'+t}{2}}{\frac{j-w}{2}}b(\varphi^{(n+\frac{w-t}{2})})\wedge\varphi^{(\frac{i'-w}{2})}.$$

If $j<w$, then the binomial coefficient

$$\binom{n-\frac{i'+t}{2}}{\frac{j-w}{2}}$$

is equal to zero. If $w<0$, then

$$b(\varphi^{(n+\frac{w-t}{2})})\in\bigwedge^{w}F=0.$$

Thus,

$$\left(\varphi^{(n-\frac{t+j}{2})}(b)\right)(\varphi^{(\frac{i'+j}{2})})=\sum_{\substack{0\le w\le j\\ j-w\text{ even}}}\binom{n-\frac{i'+t}{2}}{\frac{j-w}{2}}b(\varphi^{(n+\frac{w-t}{2})})\wedge\varphi^{(\frac{i'-w}{2})}.$$

In light of (5.2), we see that

$$\left(\varphi^{(n-\frac{t+j}{2})}(b)\right)(\varphi^{([[\frac{i+j}{2}]])})=\sum_{\substack{0\le w\le j\\ j-w\text{ even}}}\binom{n-[[\frac{i+t}{2}]]}{\frac{j-w}{2}}b(\varphi^{(n-\frac{t-w}{2})})\wedge\varphi^{([[\frac{i-w}{2}]])}.$$

To complete the proof, replace b with the element $a(\xi)$ of $\bigwedge^{2n-t}F^*$. The module action of $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$ gives

$$\varphi^{(r)}(b)=\varphi^{(r)}(a(\xi))=\left(\varphi^{(r)}\wedge a\right)(\xi)$$

and Proposition 2.2 (b) gives

$$b(\varphi^{(r)})=(a(\xi))\left(\varphi^{(r)}\right)=\left(\varphi^{(r)}(\xi)\right)(a).\quad\square$$

Proposition 5.3. *Adopt Data 2.1. Let $\widetilde{\varphi}\colon F^*\to F$ be the homomorphism which is given by $\widetilde{\varphi}(b_1)=b_1(\varphi)$ for all $b_1\in F^*$, and let i and d be integers. If $b\in\bigwedge^d F^*$, then*

$$b(\varphi^{(i)})=\sum_{k=1}^{[[i-\frac{d}{2}]]}(-1)^{k+1}b(\varphi^{(i-k)})\wedge\varphi^{(k)}+\left(\bigwedge^{2i-d}\widetilde{\varphi}\right)\left(\varphi^{(d-i)}(b)\right).$$

Proof. In the notation of (3.9), observe that

$$\left(\bigwedge^{2i-d}\widetilde{\varphi}\right)\left(\varphi^{(d-i)}(b)\right)=\Psi_{2i-d}(1,\varphi^{(d-i)}(b));$$

therefore, Theorem 3.7 and Proposition 2.3 yield

$$\begin{aligned}\left(\bigwedge^{2i-d}\widetilde{\varphi}\right)\left(\varphi^{(d-i)}(b)\right)&=\sum_{k\in\mathbb{Z}}(-1)^k\left(\varphi^{(k)}(\varphi^{(d-i)}(b))\right)(\varphi^{(2i-d-k)})\\&=\sum_{k\in\mathbb{Z}}(-1)^k\binom{k+d-i}{d-i}\left(\varphi^{(k+d-i)}(b)\right)(\varphi^{(2i-d-k)})\\&=(-1)^i\sum_{\ell\in\mathbb{Z}}(-1)^{d+\ell}\binom{\ell}{d-i}\left(\varphi^{(\ell)}(b)\right)(\varphi^{(i-\ell)}).\end{aligned}$$

Apply Corollary 3.3, with $L = 0$ and $B = A = i$, in order to see that

$$\begin{aligned}\left(\bigwedge^{2i-d}\widetilde{\varphi}\right)\left(\varphi^{(d-i)}(b)\right) &= (-1)^i\sum_{\ell\in\mathbb{Z}}(-1)^\ell b(\varphi^{(\ell)})\wedge\varphi^{(i-\ell)}\\ &= \sum_{k\in\mathbb{Z}}(-1)^k b(\varphi^{(i-k)})\wedge\varphi^{(k)}\\ &= \sum_{k=0}^{[[i-\frac{d}{2}]]}(-1)^k b(\varphi^{(i-k)})\wedge\varphi^{(k)}.\end{aligned}$$

The last equality holds because if $k < 0$, then $\varphi^{(k)} = 0$; and if $[[i - \frac{d}{2}]] < k$, then $b(\varphi^{(i-k)}) = 0$. $\square$

Recall, from section 2, that $X(I_2; J)$ is a minor of X.

Proposition 5.4. *Let X be an alternating matrix with entries from the commutative noetherian ring R. Let I and J be index sets of size i and j, respectively. If $i + j$ is even, then*

$$\sum_{\substack{t\in\mathbb{Z}\\ t+i \text{ even}}}(-1)^{\frac{t+j}{2}}\sum_{\substack{J_1\cup J_2=J\\ |J_1|=t}}\sigma(J_1J_2)X_{IJ_1}\cdot X_{J_2} = (-1)^{\frac{j(j+1)}{2}}\sum_{\substack{I_1\cup I_2=I\\ |I_1|=i-j}}\sigma(I_1I_2)X_{I_1}\cdot X(I_2;J).$$

Proof. Evaluate Theorem 3.7 at $b' = \varepsilon_I$, $b = \varepsilon_J$, and $A = \frac{i+j}{2}$, in order to see that $L = R$, where

$$L = \sum_{k\in\mathbb{Z}}(-1)^k\left(\left(\varepsilon_I(\varphi^{(k)})\right)(\varepsilon_J)\right)\left(\varphi^{(\frac{i+j}{2}-k)}\right) \qquad \text{and}$$

$$R = \sum_{k\in\mathbb{Z}}(-1)^k\varphi^{(\frac{i-j}{2}-k)}\wedge\left(\varphi^{(k)}(\varepsilon_I)\right)\left((\bigwedge^{j}\widetilde{\varphi})(\varepsilon_J)\right).$$

Apply (2.6) to see that

$$L = \sum_{k\in\mathbb{Z}}(-1)^k\sum_{\substack{J_1\cup J_2=J\\ |J_1|=2k-i}}\sigma(J_1J_2)\left(\varepsilon_I(\varphi^{(k)})\right)(\varepsilon_{J_1})\cdot\varepsilon_{J_2}\left(\varphi^{(\frac{i+j}{2}-k)}\right).$$

We know, from (2.8), that

$$\left(\varepsilon_I(\varphi^{(k)})\right)(\varepsilon_{J_1}) = (-1)^k X_{J_1I} = (-1)^{k+i}X_{IJ_1}.$$

It follows that

$$L = \sum_{k\in\mathbb{Z}}(-1)^{k+i+\frac{i+j}{2}}\sum_{\substack{J_1\cup J_2=J\\ |J_1|=2k-i}}\sigma(J_1J_2)X_{IJ_1}\cdot X_{J_2}.$$

Let $t = 2k - i$ in order to see that L is equal to the left side of the announced identity.

Notice that

$$\frac{i-j}{2} < k \implies \varphi^{(\frac{i-j}{2}-k)} = 0 \text{ and } k < \frac{i-j}{2} \implies \left(\varphi^{(k)}(\varepsilon_I)\right)\left((\bigwedge^j \widetilde{\varphi})(\varepsilon_J)\right) = 0.$$

Thus, there is only one non-zero term in R. If we write k for $\frac{i-j}{2}$, then R is equal to

$$\begin{aligned}&(-1)^k \left(\varphi^{(k)}(\varepsilon_I)\right)\left((\bigwedge^j \widetilde{\varphi})(\varepsilon_J)\right)\\ &= (-1)^k \sum_{\substack{I_1\cup I_2=I\\ |I_1|=2k}} \sigma(I_1I_2)\left(\varphi^{(k)}(\varepsilon_{I_1})\right) \wedge \varepsilon_{I_2}\left((\bigwedge^j \widetilde{\varphi})(\varepsilon_J)\right).\end{aligned}$$

Apply (2.8) to see that the

$$R = \sum_{\substack{I_1\cup I_2=I\\ |I_1|=2k}} \sigma(I_1I_2)X_{I_1}\cdot \varepsilon_{I_2}\left((\bigwedge^j \widetilde{\varphi})(\varepsilon_J)\right).$$

We know that $\widetilde{\varphi}\colon F^* \to F$ is the map

$$\widetilde{\varphi}(\varepsilon_j) = \varepsilon_j(\varphi) = \varepsilon_j\left(\sum_{p<q} x_{pq}e_p\wedge e_q\right) = \sum_i -x_{ij}e_i.$$

It follows, from (2.7), that, if A and B are index sets with $|A| = |B| = r$, then

$$\varepsilon_A\left((\bigwedge^r \widetilde{\varphi})(\varepsilon_B)\right) = (-1)^{\frac{r(r+1)}{2}} X(A;B).$$

Thus,

$$R = (-1)^{\frac{j(j+1)}{2}} \sum_{\substack{I_1\cup I_2=I\\ |I_1|=i-j}} \sigma(I_1I_2)X_{I_1}\cdot X(I_2;J). \quad \square$$

If X has size $N\times N$ and $\{1,\dots,N\}$ is the disjoint union of A, B, C, and D, then one obtains Srinivasan's version [15, page 447] of the following identity by letting $J = B\cup C$ and $I = C\cup D$.

Proposition 5.5. *Let X be an alternating matrix with entries from the commutative noetherian ring R, t be an integer, and A, B, and C be index sets of size a, b, and c, respectively. If $t+b$ is even and $a+c$ is even, then*

$$\sum_{\substack{A_1\cup A_2=A\\ |A_1|=t}} \sigma(A_1A_2)X_{A_1B}\cdot X_{A_2BC} = \sum_{\substack{s\in\mathbb{Z}\\ s+a+b \text{ even}}} \binom{\frac{a-c}{2}}{\frac{s+t-c}{2}} \sum_{\substack{C_1\cup C_2=C\\ |C_1|=s}} \sigma(C_1C_2)X_{ABC_1}\cdot X_{BC_2}.$$

Proof. Apply Corollary 3.2 (a) to the element $\varepsilon_B \wedge \varepsilon_C$. Replace the element b of $\bigwedge^d F^*$ with $\varepsilon_A \wedge \varepsilon_B$. If p and q are integers which satisfy $2q + 2p = a + 2b + c$, then

$$\left(\left(\varphi^{(q)}(\varepsilon_A \wedge \varepsilon_B)\right)(\varphi^{(p)})\right)(\varepsilon_B \wedge \varepsilon_C) \quad \text{and}$$

$$\sum_{k\in\mathbb{Z}} (-1)^{k+p} \binom{q-a-b-1+k}{k-p} \left((\varepsilon_A \wedge \varepsilon_B)(\varphi^{(k)}) \wedge \varphi^{(q+p-k)}\right)(\varepsilon_B \wedge \varepsilon_C)$$

are equal elements of R. The techniques which are employed in the proof of Proposition 4.1 yield that

$$\sum_{\substack{A_1 \cup A_2 = A \\ |A_1| + b = 2q}} \sigma(A_1 A_2) X_{A_1 B} \cdot X_{A_2 BC}$$

is equal to

$$\sum_{k\in\mathbb{Z}} (-1)^{k+p} \binom{q-a-b-1+k}{k-p} \sum_{\substack{C_1 \cup C_2 = C \\ |C_1| + a + b = 2k}} \sigma(C_1 C_2) X_{ABC_1} \cdot X_{BC_2}.$$

Replace k with $\frac{s+a+b}{2}$, q with $\frac{t+b}{2}$, and p with $\frac{a+b+c-t}{2}$. Use Observation 1.2 (e) to complete the proof. □

6. An application to Algebras with Straightening Law

Let X be an $N \times N$ alternating matrix with indeterminate entries. Consider the poset $(P, \leq)$ of pfaffians of X. The order on P is given as follows. If r and s are even integers, $a_1 < \cdots < a_r$, and $b_1 < \cdots < b_s$, then

$$X_{a_1 \ldots a_r} \leq X_{b_1 \ldots b_s} \text{ provided } s \leq r \text{ and } a_i \leq b_i \text{ for } 1 \leq i \leq s.$$

It is well known (see, for example, [7]) that the polynomial ring $\mathbb{Z}[X]$ is an algebra with straightening law on P over $\mathbb{Z}$. (We use the language of [3].) In particular, for each pair of index sets A and B, there exist integers r_{CD} such that

$$X_A X_B = \sum_{C,D} r_{CD} X_C X_D, \tag{6.1}$$

where the sum is taken over all pairs of index sets C and D with

$$X_C \leq X_D \quad \text{and} \quad X_C \leq X_A. \tag{6.2}$$

Most, but not all, of the above "straightening formula" is proved in [8, Lemma 6.2]. Indeed, [8, Lemma 6.2] establishes (6.1), where C and D vary over all pairs of index sets with $X_C \leq X_D$. One must modify the argument of [8] in order to obtain (6.1), where C and D fulfill both of the requirements of (6.2). One version of this modification occurs in [6]. Suppose that $|A|$ and $|B|$ are even integers with $|B| \leq |A|$. Let $i(A, B)$ be the largest index i for which $a_k \leq b_k$ for all k with $1 \leq k \leq i$. If $|A| = N$ or $i(A, B) = |B|$, then $X_A \leq X_B$ and $X_A X_B = X_A X_B$ satisfies both (6.1) and (6.2). Day's argument proceeds by induction on $|A|$ and

$i(A,B)$. Let $i=i(A,B)$. It is convenient to write $A=A'\cup C_1'$ and $B=C_2'\cup B'$, where A' is $a_1<\cdots<a_i$, and C_1' is $c_1<\cdots<c_\lambda$, with $a_i<c_1$; and C_2' is $c_{\lambda+1}<\cdots<c_{\lambda+i+1}$, and B' is $b_1<\cdots<b_t$, with $c_{\lambda+i+1}<b_1$. The information about the indices is summarized as

$$\begin{array}{lccccccccccccc} B: & c_{\lambda+1} & < & \dots & < & c_{\lambda+i} & < & c_{\lambda+i+1} & < & b_1 & < & \dots & < & b_t \\ & \vee\! \mid & & & & \vee\! \mid & & \wedge & & & & & & \\ A: & a_1 & < & \dots & < & a_i & < & c_1 & < & \dots & & \dots & < & c_\lambda. \end{array}$$

Let C represent the index set $c_{\lambda+1}<\cdots<c_{\lambda+i+1}<c_1\cdots<c_\lambda$. Day proves that there exist integers r_{C_1} and $r_{B_1C_1}$ such that

$$X_{A'C_1'}X_{C_2'B'}=\sum_{\substack{C_1\cup C_2=C\\|C_1|=\lambda\\C_1\neq C_1'}} r_{C_1}X_{A'C_1}X_{C_2B'}+\sum_{\substack{B_1\cup B_2=B'\\0\le|B_1|\le t}}\ \sum_{\substack{C_1\cup C_2=C\\\lambda+1\le|C_1|}} r_{C_1B_1}X_{A'C_1B_1}X_{C_2B_2}. \tag{6.3}$$

It is clear that $X_{A'C_1}\le X_A$ and $X_{A'C_1B_1}\le X_A$. Furthermore,

$$i(A,B)<i(A'C_1,C_2B')\quad\text{and}\quad |A|<|A'C_2B_2|;$$

so, (6.1) with hypotheses (6.2) follows by induction.

In Proposition 6.6 we generalize (6.3) and produce explicit values for the integers r_{C_1} and $r_{B_1C_1}$; indeed, (6.3) is the special case $M=\lambda$, $Q=t-1$. Corollary 6.5, which is the base step in our proof of Proposition 6.6, is a translation of Corollary 3.5 from the language of multilinear algebra to the language of pfaffians. The following conventions are in effect throughout the rest of this section.

Data 6.4. *Let X be an alternating matrix with entries from the commutative noetherian ring R, and let A, B, and C be index sets of size a, b, and c, respectively. Assume that $a+b+c$ is even. For fixed values of ℓ and r, let*

$$X(\ell,r)=\sum_{\substack{B_1\cup B_2=B\\|B_1|=r}}\ \sum_{\substack{C_1\cup C_2=C\\|C_1|=\ell}}\sigma(B_1B_2)\sigma(C_1C_2)X_{AB_1C_1}X_{B_2C_2}.$$

Corollary 6.5. *Adopt Data 6.4. If M and Q are integers with $Q+M$ even, $M+a$ even, $-1\le Q$, and $Q+M\le b+c-a-2$, then*

$$\sum_{\substack{r,\ell\in\mathbb{Z}\\\ell+r+a\text{ even}}}(-1)^{\ell c+\frac{r+a-\ell}{2}}\binom{\frac{\ell+r+Q}{2}}{\frac{\ell+r-M}{2}}X(\ell,r)=0.$$

Proof. Apply Corollary 3.5 with A replaced by $\frac{a+b+c}{2}$, B replaced by $\frac{Q-a}{2}$, C replaced by $-\frac{M+a}{2}$, d replaced by a, and b replaced by ε_A. Since the hypothesis of Corollary 3.5 is equivalent to the present hypothesis, we conclude that

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{\frac{Q-a}{2}+k}{\frac{-M-a}{2}+k}\left[\varepsilon_A(\varphi^{(k)})\wedge\varphi^{(\frac{c+a+b}{2}-k)}\right](\varepsilon_B\wedge\varepsilon_C)$$

is the zero element of R. Calculations similar to those in section 4 yield

$$\left[\varepsilon_A(\varphi^{(k)}) \wedge \varphi^{(\frac{c+a+b}{2}-k)}\right](\varepsilon_B \wedge \varepsilon_C) = \sum_{\substack{r,\,\ell\in\mathbb{Z}\\ r+\ell+a=2k}} (-1)^{\ell c+\ell+a+\frac{c+a+b}{2}} X(\ell,r).$$

Replace k by $\frac{r+\ell+a}{2}$ to see that

$$(-1)^{a+\frac{c+a+b}{2}} \sum_{\substack{r,\,\ell\in\mathbb{Z}\\ r+\ell+a \text{ even}}} (-1)^{\ell c+\frac{r+a-\ell}{2}} \binom{\frac{r+\ell+Q}{2}}{\frac{r+\ell-M}{2}} X(\ell,r) = 0. \quad \square$$

Proposition 6.6. *Adopt Data 6.4. If Q and M are integers with $Q+M$ even, $M+a$ even, $b-1 \le Q$, and $Q+M \le b+c-a-2$, then*

$$\sum_{\substack{r,\ell\in\mathbb{Z}\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{r+a-\ell}{2}} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell-r-M}{2}} X(\ell,r) = 0.$$

Remark. We note, for the purpose of induction, that Proposition 6.6 says that $X(M,0)$ is equal to

$$\sum_{\substack{M+1\le\ell\\ \ell+a \text{ even}}} (-1)^{1+\frac{M-\ell}{2}} \binom{\frac{\ell+Q}{2}}{\frac{\ell-M}{2}} X(\ell,0) + \sum_{r=1}^{b} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell+r+a \text{ even}}} (-1)^{rc+1+\frac{M+r-\ell}{2}} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell-r-M}{2}} X(\ell,r).$$

Proof. For each integer r, with $0 \le r \le b$, let

$$T_r = \sum_{\substack{\ell\in\mathbb{Z}\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{r+a-\ell}{2}} \binom{\frac{\ell+r+Q}{2}}{\frac{\ell+r-M}{2}} X(\ell,r).$$

Recall, from Corollary 6.5, that $\sum_{r=0}^{b} T_r = 0$. If $b=0$, then the result holds. The proof proceeds by induction on b; henceforth, we assume that $0<b$. Write $T_r = T_r' + T_r''$, with

$$T_r' = (-1)^{Mc+rc+r+\frac{a-M}{2}} X(M-r,r)$$

and

$$T_r'' = \sum_{\substack{M+1-r\le\ell\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{r+a-\ell}{2}} \binom{\frac{\ell+r+Q}{2}}{\frac{\ell+r-M}{2}} X(\ell,r).$$

For each r with $1 \le r \le b$, we apply the induction hypothesis with A replaced by AB_1, a replaced by $a+r$, B replaced by B_2, b replaced by $b-r$, M replaced by $M-r$, and Q replaced by $Q-r$. Notice that all of the hypotheses are still satisfied. When the induction hypothesis is applied to

$$T_r' = (-1)^{Mc+rc+r+\frac{a-M}{2}} \sum_{\substack{B_1\cup B_2=B\\ |B_1|=r}} \sigma(B_1B_2) \sum_{\substack{C_1\cup C_2=C\\ |C_1|=M-r}} \sigma(C_1C_2) X_{AB_1C_1} X_{B_2C_2},$$

we see that $T'_r = V_r + V'_r$, where

$$V_r = \sum_{\substack{M-r+1\le \ell\\ \ell+a+r \text{ even}}} (-1)^{\ell c+1+\frac{a+r-\ell}{2}} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell+r-M}{2}} X(\ell,r)$$

and

$$V'_r = \sum_{r'=1}^{b-r} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell+r'+a+r \text{ even}}} (-1)^{\ell c+1+\frac{a+r+r'-\ell}{2}} \binom{\frac{\ell-r'-r+Q}{2}}{\frac{\ell-r'+r-M}{2}} X(\ell;r;r'),$$

for $X(\ell;r;r')$ equal to

$$\sum_{\substack{C_1\cup C_2=C\\ |C_1|=\ell}} \sigma(C_1C_2) \sum_{\substack{B_1\cup B_2=B\\ |B_1|=r}} \sum_{\substack{B'_1\cup B'_2=B_2\\ |B'_1|=r'}} \sigma(B'_1B'_2)\sigma(B_1B_2) X_{AB_1B'_1C_1}X_{B'_2C_2}.$$

Co-associativity in the co-algebra $\bigwedge^\bullet F^*$, together with the fact that the composition

$$\bigwedge^{r+r'} F^* \xrightarrow{\Delta} \bigwedge^{r} F^* \otimes \bigwedge^{r'} F^* \xrightarrow{\wedge} \bigwedge^{r+r'} F^*$$

is equal to multiplication by $\binom{r+r'}{r'}$, yields that

$$X(\ell;r;r') = \binom{r+r'}{r'} X(\ell, r+r').$$

Notice that

$$V'_r = \sum_{r'=1}^{b-r} \sum_{\substack{M-r-r'+1\le \ell\\ \ell+r'+a+r \text{ even}}} (-1)^{\ell c+1+\frac{a+r+r'-\ell}{2}} \binom{\frac{\ell-r'-r+Q}{2}}{\frac{\ell-r'+r-M}{2}} \binom{r+r'}{r'} X(\ell, r+r');$$

because, if $\ell < M-r'-r+1$, then $\ell - r' + r - M < 1-2r' < 0$, and the first binomial coefficient is zero. It follows that

$$T'_r = V_r + V'_r = \sum_{r'=0}^{b-r} \sum_{\substack{M-r-r'+1\le \ell\\ \ell+r'+a+r \text{ even}}} (-1)^{\ell c+1+\frac{a+r+r'-\ell}{2}} \binom{\frac{\ell-r'-r+Q}{2}}{\frac{\ell-r'+r-M}{2}} \binom{r+r'}{r'} X(\ell, r+r').$$

Replace r with $r-r'$ to see that

$$\sum_{r=1}^{b} T'_r = \sum_{r=1}^{b} \sum_{\substack{M-r+1\le \ell\\ \ell+r+a \text{ even}}} (-1)^{\ell c+1+\frac{a+r-\ell}{2}} \sum_{r'=0}^{r-1} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell+r-M}{2}-r'} \binom{r}{r'} X(\ell,r).$$

Recall that $T_r = T'_r + T''_r$. We now see that is equal to

$$\sum_{r=1}^{b} T_r = \sum_{r=1}^{b} \sum_{\substack{M-r+1\le \ell\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{a+r-\ell}{2}} \left[\binom{\frac{\ell+r+Q}{2}}{\frac{\ell+r-M}{2}} - \sum_{r'=0}^{r-1} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell+r-M}{2}-r'} \binom{r}{r'} \right] X(\ell,r).$$

Use Corollary 1.4 to see that

$$\sum_{r=1}^{b} T_r = \sum_{r=1}^{b} \sum_{\substack{M-r+1\le \ell\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{a+r-\ell}{2}} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell-r-M}{2}} X(\ell,r).$$

The constraint $M-r+1\le \ell$ is irrelevant; thus,

$$\sum_{r=1}^{b} T_r = \sum_{r=1}^{b} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell+r+a \text{ even}}} (-1)^{\ell c+\frac{a+r-\ell}{2}} \binom{\frac{\ell-r+Q}{2}}{\frac{\ell-r-M}{2}} X(\ell,r).$$

The proof is complete because $\sum\limits_{r=0}^{b} T_r = 0$. □

7. Other Applications

In this section we compare our results to a few familiar pfaffian identities. We begin with the "Laplace expansion" of a pfaffian.

Proposition 7.1. *Let X be an alternating matrix with entries from the commutative noetherian ring R. If N is an even integer, then*

$$\sum_{k=2}^{N} (-1)^k X_{1k} X_{2\cdots\widehat{k}\cdots N} = X_{12\cdots N}.$$

Proof. Apply Proposition 5.5 with $C=\{1\}$, $A=\{2,\dots,N\}$, and B empty. □

The next two examples are Lemmas 2.3 and 2.4 of [4].

Proposition 7.2. *Adopt Data 2.1.*

(a) *Let $\widetilde{\varphi}\colon F^*\to F$ be the homomorphism which is given by $\widetilde{\varphi}(b_1)=b_1(\varphi)$ for all $b_1\in F^*$. If $b\in\bigwedge^d F^*$, then*

$$(\bigwedge^d \widetilde{\varphi})(b) = \sum_{k\in\mathbb{Z}} (-1)^k \left(\varphi^{(k)}(b)\right)(\varphi^{(d-k)}).$$

(b) *If $b\in\bigwedge^{2m+1} F^*$, then $b(\varphi^{(m+1)}) = \left(\varphi^{(m)}(b)\right)(\varphi)$.*

Proof. Apply Theorem 3.7 with $A=d$ and $b'=1$ to prove (a). Assertion (b) is Corollary 3.3 (a) with $L=0$ and $A=B=m+1$. □

The following identities played a crucial role in preliminary versions of [2]; see, for example, [1]. Other proofs may be found in [5] and [9, Relations 2.23 and 2.24].

Proposition 7.3. *Let X be an alternating matrix with entries from the commutative noetherian ring R. If $p,q,p_1,\dots,p_\ell$ are integers, then*

$$X(p,p_1,\dots,p_\ell;q,p_1,\dots,p_\ell) = \begin{cases} X_{p_1,\dots,p_\ell} X_{p,q,p_1,\dots,p_\ell} & \text{if } \ell \text{ is even, and}\\ X_{p,p_1,\dots,p_\ell} X_{q,p_1,\dots,p_\ell} & \text{if } \ell \text{ is odd.}\end{cases}$$

Proof. Apply Proposition 5.4 with $I=\{p,p_1,\dots,p_\ell\}$ and $J=\{q,p_1,\dots,p_\ell\}$. □

References

1. G. Boffi and R. Sánchez, *Some classical formulas and a determinantal ideal*, Geometry Seminars, 1988–1991, Univ. Stud. Bologna, Bologna, 1991, pp. 27–35.
2. G. Boffi and R. Sánchez, *On the resolutions of the powers of the pfaffian ideal*, preprint (1990).
3. W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer Verlag, Berlin Heidelberg New York, 1988.
4. D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.
5. A. Cayley, *Sur les déterminants gauche*, J. reine angew. Math. **38** (1849), 93–96.
6. C. Day, *Ring around the poset: an introduction to Algebras with Straightening Law*, M. A. thesis, University of South Carolina, 1989.
7. C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque **91** (1982), 1–87.
8. C. De Concini and C. Procesi, *A characteristic free approach to invariant theory*, Advances Math. **21** (1976), 330–354.
9. P. Heymans, *Pfaffians and skew-symmetric matrices*, Proc. Lond. Math. Soc. **19** (1969), 730–768.
10. C. Huneke and B. Ulrich, *Divisor class groups and deformations*, Amer. J. Math. **107** (1985), 1265–1303.
11. A. Kustin, *The minimal free resolutions of the Huneke-Ulrich deviation two Gorenstein ideals*, J. Alg. **100** (1986), 265–304.
12. A. Kustin, *The deviation two Gorenstein rings of Huneke and Ulrich*, in preparation.
13. A. Kustin and B. Ulrich, *If the socle fits*, J. Alg. **147** (1992), 63–80.
14. A. Kustin and B. Ulrich, *A family of complexes associated to an almost alternating map, with applications to residual intersections*, Mem. Amer. Math. Soc. **95** (1992), 1–94.
15. H. Srinivasan, *Minimal algebra resolutions for cyclic modules defined by Huneke-Ulrich ideals*, J. Alg. **137** (1991), 433–472.
16. H. Srinivasan, *Decomposition formulas for pfaffians*, J. Alg. (to appear).

Mathematics Department, University of South Carolina, Columbia, SC 29208
E-mail address: kustin@milo.math.scarolina.edu