THE SYZYGIES OF THE IDEAL $(x_1^N, x_2^N, x_3^N, x_4^N)$ IN THE HYPERSURFACE RING DEFINED BY $x_1^n + x_2^n + x_3^n + x_4^n$ OVER ANY FIELD.

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ABSTRACT. Let $\boldsymbol{k}$ be a field of characteristic zero, and $\boldsymbol{n}$, $\boldsymbol{d}$, and $\boldsymbol{r}$ be non-negative integers with $1 \le \boldsymbol{r} \le \boldsymbol{n}-1$. Let N be the integer $\boldsymbol{dn}+\boldsymbol{r}$, P be the polynomial ring $\boldsymbol{k}[x_1,x_2,x_3,x_4]$, $f_{\boldsymbol{n}}$ be the polynomial $x_1^{\boldsymbol{n}}+x_2^{\boldsymbol{n}}+x_3^{\boldsymbol{n}}+x_4^{\boldsymbol{n}}$ in P, $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the ideal $(x_1^N,x_2^N,x_3^N,x_4^N)$ of P, $\bar{P}_{\boldsymbol{n}}$ be the hypersurface ring $P/(f_{\boldsymbol{n}})$, $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the quotient ring $\bar{P}_{\boldsymbol{n}}/C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\bar{P}_{\boldsymbol{n}}$ and $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the third syzygy module of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ as a $\bar{P}_{\boldsymbol{n}}$-module. We prove that $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is isomorphic to the direct sum of $2\boldsymbol{d}+1$ copies of $\Omega^3_{0,\boldsymbol{n},\boldsymbol{r}}$.

1. INTRODUCTION

One often studies an ideal $\boldsymbol{I}$ in a commutative ring $\boldsymbol{R}$ by computing invariants of the quotient ring which is defined by some sort of power of $\boldsymbol{I}$. Sometimes one studies ordinary powers of $\boldsymbol{I}$, see, for example, [8, 18]; sometimes one studies symbolic powers [9, 16], sometimes one studies Frobenius powers [19, 26]. We consider bracket powers; indeed we have found that projects which begin as projects about Frobenius powers often end up being projects about bracket powers [24, 21].

For an arbitrary graded algebra $\boldsymbol{R}$, over an arbitrary field $\boldsymbol{k}$, with maximal homogeneous ideal $\mathfrak{m} = (x_1,\dots,x_m)$, it is very natural to ask how the bracket powers, $\mathfrak{m}^{[N]} = (x_1^N,\dots,x_m^N)$, of $\mathfrak{m}$ are related. In particular, how is the resolution of $\boldsymbol{R}/\mathfrak{m}^{[N]}$ by free $\boldsymbol{R}$-modules related to the resolution of $\boldsymbol{R}/\mathfrak{m}^{[qN]}$ for various exponents N and qN?

We focus on hypersurfaces of the form $\boldsymbol{R} = P/(f)$, where P is a polynomial ring over a field, and f is a homogeneous polynomial in P. The most interesting feature of the $\boldsymbol{R}$-resolution of $\boldsymbol{R}/\boldsymbol{I}^{[N]}\boldsymbol{R}$ is the infinite tail of the resolution, which is a matrix factorization of f, see [10].

The situation has been fairly seriously studied when $P = \boldsymbol{k}[x,y,z]$, $\mathfrak{m}$ is the maximal ideal (x,y,z), $\boldsymbol{k}$ is a field of characteristic p, f is a homogeneous polynomial of P, and $\boldsymbol{R}$ is the hypersurface ring $\bar{P} = P/(f)$. If $f = x^{\boldsymbol{n}}+y^{\boldsymbol{n}}+z^{\boldsymbol{n}}$, then the Betti numbers of $\bar{P}/\mathfrak{m}^{[q]}\bar{P}$ are calculated in [24] and the resolution of $\bar{P}/\mathfrak{m}^{[q]}\bar{P}$ is given in [21]. If f is a general homogeneous form of P, then the Betti numbers of $\bar{P}/\mathfrak{m}^{[q]}\bar{P}$ are calculated in [25]. In the present paper, P is a polynomial ring with four variables.

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Data 1.1. Let $\boldsymbol{k}$ be a field, and $\boldsymbol{n}$, $\boldsymbol{d}$, and $\boldsymbol{r}$ be non-negative integers with

$$1 \le \boldsymbol{r} \le \boldsymbol{n}-1. \tag{1.1.1}$$

(Usually $\boldsymbol{d}$ is also positive.) Once $\boldsymbol{d}$, $\boldsymbol{n}$, and $\boldsymbol{r}$ have been chosen, then

$$N = \boldsymbol{dn}+\boldsymbol{r}.$$

Let P be the polynomial ring $P=\boldsymbol{k}[x_1,x_2,x_3,x_4]$, $f_{\boldsymbol{n}}$ be the polynomial

$$f_{\boldsymbol{n}} = x_1^{\boldsymbol{n}}+x_2^{\boldsymbol{n}}+x_3^{\boldsymbol{n}}+x_4^{\boldsymbol{n}}$$

in P, $\bar{P}_{\boldsymbol{n}}$ be the hypersurface ring $P/(f_{\boldsymbol{n}})$, $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the ideal $(x_1^N,x_2^N,x_3^N,x_4^N)$ of P, and $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the quotient ring

$$Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}} = \bar{P}_{\boldsymbol{n}}/C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\bar{P}_{\boldsymbol{n}}. \tag{1.1.2}$$

Let $\mathbf{M}_{\boldsymbol{n}}$ be the Abelian group $\mathbb{Z}\oplus(\frac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}})^4$. The polynomial ring P is $\mathbf{M}_{\boldsymbol{n}}$-graded, where the degree of the monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}$ is

$$(a_1+a_2+a_3+a_4,\bar{a}_1,\bar{a}_2,\bar{a}_3,\bar{a}_4),$$

with $\bar{a}_i$ equal to the image of a_i in $\frac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}}$. The polynomial $f_{\boldsymbol{n}}$ and the ideal $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ are both homogeneous with respect to the $\mathbf{M}_{\boldsymbol{n}}$-grading on P. Let $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules.

If $\boldsymbol{d}=0$, then

$$(f_{\boldsymbol{n}}) \subseteq (x_1^{\boldsymbol{r}},x_2^{\boldsymbol{r}},x_3^{\boldsymbol{r}},x_4^{\boldsymbol{r}})$$

are nested complete intersection ideals, $C_{0,\boldsymbol{n},\boldsymbol{r}}\bar{P}_{\boldsymbol{n}}$ is a quasi-complete intersection ideal of $\bar{P}_{\boldsymbol{n}}$, in the sense of [1], and $G_{0,\boldsymbol{n},\boldsymbol{r}}$ is the two-step Tate complex [23, 11]. Indeed, $G_{0,\boldsymbol{n},\boldsymbol{r}}$ looks like

$$\cdots \xrightarrow{\bar{B}} \bar{P}_{\boldsymbol{n}}^8 \xrightarrow{\bar{A}} \bar{P}_{\boldsymbol{n}}^8 \xrightarrow{\bar{B}} \bar{P}_{\boldsymbol{n}}^8 \xrightarrow{\bar{A}} \bar{P}_{\boldsymbol{n}}^8 \longrightarrow \bar{P}_{\boldsymbol{n}}^7 \longrightarrow \bar{P}_{\boldsymbol{n}}^4 \longrightarrow \bar{P}_{\boldsymbol{n}}, \tag{1.1.3}$$

where matrices A and B with entries in P are given in Table 1, and $\bar{A}$ and $\bar{B}$ are the images of A and B in $\bar{P}_{\boldsymbol{n}}$. The matrices A and B form a matrix factorization of $f_{\boldsymbol{n}}$ in the sense that AB and BA both equal $f_{\boldsymbol{n}}$ times the 8×8 identity matrix over P. Some further discussion about the case $\boldsymbol{d}=0$ may be found in Section 11.

If $\boldsymbol{d}$ is positive, then $f_{\boldsymbol{n}}$ is not in the ideal $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ and there is no two-step Tate complex associated to the $\bar{P}_{\boldsymbol{n}}$-module $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$. Nonetheless, when the characteristic of $\boldsymbol{k}$ is zero, the multi-graded Betti numbers in the minimal resolution of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}$-modules have been calculated in [22]. (The calculation is summarized in Section 4.) This calculation shows that the resolution $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ looks like

$$\cdots \longrightarrow \bar{P}_{\boldsymbol{n}}^{16\boldsymbol{d}+8} \xrightarrow{\bar{g}_5} \bar{P}_{\boldsymbol{n}}^{16\boldsymbol{d}+8} \xrightarrow{\bar{g}_4} \bar{P}_{\boldsymbol{n}}^{16\boldsymbol{d}+8} \xrightarrow{\bar{g}_3} \bar{P}_{\boldsymbol{n}}^{8\boldsymbol{d}+7} \xrightarrow{\bar{g}_2} \bar{P}_{\boldsymbol{n}}^{4} \xrightarrow{\bar{g}_1} \bar{P}_{\boldsymbol{n}}, \tag{1.1.4}$$

where the form of the matrices g_4 and g_5 is given in Table 2. (As before g_i is a matrix with entries from P and $\bar{g}_i$ is the image of g_i in $\bar{P}_{\boldsymbol{n}}$.) The matrices M_{ij} and N_{ij} are $(2\boldsymbol{d}+1)\times(2\boldsymbol{d}+1)$ invertible matrices of constants.

The similarity between the form of the matrices in Table 1 and the matrices in Table 2 lead us to make the following conjecture.

Conjecture 1.2. If the characteristic of $\boldsymbol{k}$ is zero, then the infinite tail of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is isomorphic to the direct sum of $2\boldsymbol{d}+1$ copies of the infinite tail of $G_{0,\boldsymbol{n},\boldsymbol{r}}$.

$$A = \begin{bmatrix}
x_1^r & 0 & 0 & 0 & x_2^{n-r} & -x_3^{n-r} & x_4^{n-r} & 0 \\
-x_2^r & 0 & x_4^{n-r} & -x_3^{n-r} & x_1^{n-r} & 0 & 0 & 0 \\
x_3^r & x_4^{n-r} & 0 & -x_2^{n-r} & 0 & x_1^{n-r} & 0 & 0 \\
-x_4^r & x_3^{n-r} & -x_2^{n-r} & 0 & 0 & 0 & x_1^{n-r} & 0 \\
0 & -x_2^r & -x_3^r & -x_4^r & 0 & 0 & 0 & x_1^{n-r} \\
0 & x_1^r & 0 & 0 & 0 & -x_4^r & -x_3^r & x_2^{n-r} \\
0 & 0 & x_1^r & 0 & -x_4^r & 0 & x_2^r & x_3^{n-r} \\
0 & 0 & 0 & x_1^r & x_3^r & x_2^r & 0 & x_4^{n-r}
\end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix}
x_1^{n-r} & -x_2^{n-r} & x_3^{n-r} & -x_4^{n-r} & 0 & 0 & 0 & 0 \\
0 & 0 & x_4^r & x_3^r & -x_2^{n-r} & x_1^{n-r} & 0 & 0 \\
0 & x_4^r & 0 & -x_2^r & -x_3^{n-r} & 0 & x_1^{n-r} & 0 \\
0 & -x_3^r & -x_2^r & 0 & -x_4^{n-r} & 0 & 0 & x_1^{n-r} \\
x_2^r & x_1^r & 0 & 0 & 0 & 0 & -x_4^{n-r} & x_3^{n-r} \\
-x_3^r & 0 & x_1^r & 0 & 0 & -x_4^{n-r} & 0 & x_2^{n-r} \\
x_4^r & 0 & 0 & x_1^r & 0 & -x_3^{n-r} & x_2^{n-r} & 0 \\
0 & 0 & 0 & 0 & x_1^r & x_2^r & x_3^r & x_4^r
\end{bmatrix}$$

TABLE 1. The matrices A and B for the resolution of (1.1.3).

The matrix g_4 has the form

$x_1^rM_{11}$	0	0	0	$x_2^{n-r}M_{15}$	$-x_3^{n-r}M_{16}$	$x_4^{n-r}M_{17}$	0
$-x_2^rM_{21}$	0	$x_4^{n-r}M_{23}$	$-x_3^{n-r}M_{24}$	$x_1^{n-r}M_{25}$	0	0	0
$x_3^rM_{31}$	$x_4^{n-r}M_{32}$	0	$-x_2^{n-r}M_{34}$	0	$x_1^{n-r}M_{36}$	0	0
$-x_4^rM_{41}$	$x_3^{n-r}M_{42}$	$-x_2^{n-r}M_{43}$	0	0	0	$x_1^{n-r}M_{47}$	0
0	$-x_2^rM_{52}$	$-x_3^rM_{53}$	$-x_4^rM_{54}$	0	0	0	$x_1^{n-r}M_{58}$
0	$x_1^rM_{62}$	0	0	0	$-x_4^rM_{66}$	$-x_3^rM_{67}$	$x_2^{n-r}M_{68}$
0	0	$x_1^rM_{73}$	0	$-x_4^rM_{75}$	0	$x_2^rM_{77}$	$x_3^{n-r}M_{78}$
0	0	0	$x_1^rM_{84}$	$x_3^rM_{85}$	$x_2^rM_{86}$	0	$x_4^{n-r}M_{88}$

and the matrix g_5 has the form

$x_1^{n-r}N_{11}$	$-x_2^{n-r}N_{12}$	$x_3^{n-r}N_{13}$	$-x_4^{n-r}N_{14}$	0	0	0	0
0	0	$x_4^rN_{23}$	$x_3^rN_{24}$	$-x_2^{n-r}N_{25}$	$x_1^{n-r}N_{26}$	0	0
0	$x_4^rN_{32}$	0	$-x_2^rN_{34}$	$-x_3^{n-r}N_{35}$	0	$x_1^{n-r}N_{37}$	0
0	$-x_3^rN_{42}$	$-x_2^rN_{43}$	0	$-x_4^{n-r}N_{45}$	0	0	$x_1^{n-r}N_{48}$
$x_2^rN_{51}$	$x_1^rN_{52}$	0	0	0	0	$-x_4^{n-r}N_{57}$	$x_3^{n-r}N_{58}$
$-x_3^rN_{61}$	0	$x_1^rN_{63}$	0	0	$-x_4^{n-r}N_{66}$	0	$x_2^{n-r}N_{68}$
$x_4^rN_{71}$	0	0	$x_1^rN_{74}$	0	$-x_3^{n-r}N_{76}$	$x_2^{n-r}N_{77}$	0
0	0	0	0	$x_1^rN_{85}$	$x_2^rN_{86}$	$x_3^rN_{87}$	$x_4^rN_{88}$

TABLE 2. The matrices g_4 and g_5 for the resolution of (1.1.4). Each matrix M_{ij} and N_{ij} is a $(2\boldsymbol{d}+1)\times(2\boldsymbol{d}+1)$ invertible matrix of constants.

Possibly it is helpful to observe that Conjecture 1.2 is equivalent to conjecturing that the matrix factorization of $f_{\boldsymbol{n}}$ associated to $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is given in Table 3, where there are $2\boldsymbol{d}+1$ copies of A and B on the main diagonal. Another alternate phrasing of Conjecture 1.2 involves the third syzygy module, $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$, of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ as a $\bar{P}_{\boldsymbol{n}}$-module.

$$\begin{bmatrix} A & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B \end{bmatrix}$$

TABLE 3. An alternate version of Conjecture 1.2 is that the matrix factorization of $f_{\boldsymbol{n}}$ associated to $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is given above, where A and B are the matrices of Table 1 and there are $2\boldsymbol{d}+1$ copies of A and B on the main diagonal.

Conjecture 1.2 is equivalent to the assertion that $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is isomorphic to the direct sum of $2\boldsymbol{d}+1$ copies of $\Omega^3_{0,\boldsymbol{n},\boldsymbol{r}}$,.

We establish Conjecture 1.2 in Corollary 10.2.

In order to prove Conjecture 1.2, one "need only" find bases so that, in the new bases, all of the invertible matrices M_{ij} and N_{ij} from g_4 and g_5 in Table 2 simultaneously become the identity matrix. We carried out this calculation when $1 \le \boldsymbol{d} \le 2$ (with $\boldsymbol{n}$ and $\boldsymbol{r}$ arbitrary satisfying (1.1.1)). That is, we applied the procedure of [20]; found explicit matrices M_{ij} and N_{ij}; and simultaneously inverted all M_{ij} and N_{ij}. We had hoped that the explicit calculations would show us the "special bases" for the free modules in $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ that give rise to

$$M_{ij} = N_{ij} = I_{2\boldsymbol{d}+1}, \tag{1.2.1}$$

where I_m is the $m \times m$ identity matrix. Alas, we learned that there are no special bases. One can choose any basis for any one of the 16 indicated summands of the free modules in position 4 or 5 of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ and then there is a unique choice of basis for each of the other 15 indicated summands which give rise to (1.2.1).

In other words, Conjecture 1.2 is a conceptual problem and not a calculational problem.We move to a "Universal resolution" which maps to <u>all</u> of the $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ and which is known to have (approximately) the desired form. We adopt the following setting for this "universal resolution".

Data 1.3. Let $\boldsymbol{k}$ be a field, $\mathfrak{P}$ be the polynomial ring

$$\mathfrak{P} = \boldsymbol{k}[y_1, y_2, y_3, y_4, w_1, w_2, w_3, w_4],$$

$\mathfrak{f}$ be the polynomial

$$\mathfrak{f} = y_1w_1 + y_2w_2 + y_3w_3 + y_4w_4$$

in $\mathfrak{P}$, and $\bar{\mathfrak{P}}$ be the hypersurface ring $\mathfrak{P}/(\mathfrak{f})$. For each non-negative integer $\boldsymbol{d}$, let $\mathfrak{C}_{\boldsymbol{d}}$ be the ideal $(y_1^{\boldsymbol{d}+1}w_1^{\boldsymbol{d}}, y_2^{\boldsymbol{d}+1}w_2^{\boldsymbol{d}}, y_3^{\boldsymbol{d}+1}w_3^{\boldsymbol{d}}, y_4^{\boldsymbol{d}+1}w_4^{\boldsymbol{d}})$ of $\mathfrak{P}$, and $\mathfrak{Q}_{\boldsymbol{d}}$ be the quotient ring

$$\mathfrak{Q}_{\boldsymbol{d}} = \bar{\mathfrak{P}}/\mathfrak{C}_{\boldsymbol{d}}\bar{\mathfrak{P}}.$$

Let $\mathfrak{M}$ be the Abelian group $\mathbb{Z}^6$. The polynomial ring $\mathfrak{P}$ is $\mathfrak{M}$-graded, where the degree of the monomial $y_1^{a_1}y_2^{a_2}y_3^{a_3}y_4^{a_4}w_1^{b_1}w_2^{b_2}w_3^{b_3}w_4^{b_4}$ is

$$(a_1+a_2+a_3+a_4, b_1+b_2+b_3+b_3+b_4, a_1-b_1, a_2-b_2, a_3-b_3, a_4-b_4).$$

The polynomial $\mathfrak{f}$ and the ideal $\mathfrak{C}_{\boldsymbol{d}}$ are both homogeneous with respect to the $\mathfrak{M}$-grading on $\mathfrak{P}$. Let $\mathfrak{G}_{\boldsymbol{d}}$ be the minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\bar{\mathfrak{P}}$-modules.

The set up of Data 1.3 is relevant for three reasons. First of all, for each pair $(\boldsymbol{n},\boldsymbol{r})$, which satisfies (1.1.1), there is a ring homomorphism $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}:\bar{\mathfrak{P}}\to\bar{P}_{\boldsymbol{n}}$ and a homomorphism of Abelian groups $\alpha_{\boldsymbol{n},\boldsymbol{r}}:\mathfrak{M}\to\mathbf{M}_{\boldsymbol{n}}$ such that

1.4. $\mathfrak{G}_{\boldsymbol{d}}\otimes_{\bar{\mathfrak{P}}}\bar{P}_{\boldsymbol{n}}$ is a resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules (see Theorem 7.5), and

1.5. if θ is an $\mathfrak{M}$-homogeneous element of $\bar{\mathfrak{P}}$ of multi-degree m, then $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}(\theta)$ is a homogeneous element of $\bar{P}_{\boldsymbol{n}}$, in the $\mathbf{M}_{\boldsymbol{n}}$-grading, of multi-degree $\alpha_{\boldsymbol{n},\boldsymbol{r}}(m)$ (see Remark 5.3.(b)).

Also, the polynomial $\mathfrak{f}$ is a quadratic form in the polynomial ring $\mathfrak{P}$; consequently,

1.6. there are at most two isomorphism classes of non-free indecomposable maximal Cohen-Macaulay (MCM) $\bar{\mathfrak{P}}$-modules.

The result 1.6 is explicitly established, over any field, in [6, Prop. 3.1]. (The MCM modules over the particular ring $\bar{\mathfrak{P}}$ are also discussed in [7, Remark 2.5.4]. Indeed, the classification of MCM-modules over $\bar{\mathfrak{P}}$ may be deduced from Knörrer periodicity [17].) At any rate,

$$\mathfrak{f}I_8=\mathfrak{A}\mathfrak{B}\quad\text{and}\quad\mathfrak{f}I_8=\mathfrak{B}\mathfrak{A},$$

for $\mathfrak{A}$ and $\mathfrak{B}$ as given in Table 4, and every non-free indecomposable MCM $\bar{\mathfrak{P}}$-module is isomorphic to $\operatorname{Im}\bar{\mathfrak{A}}$ or $\operatorname{Im}\bar{\mathfrak{B}}$. The fourth syzygy, $\mathfrak{S}_{\boldsymbol{d}}^4$, of the $\bar{\mathfrak{P}}$-module $\mathfrak{Q}_d$ is a MCM $\bar{\mathfrak{P}}$-module; consequently,

$$\mathfrak{S}_{\boldsymbol{d}}^4\cong(\operatorname{Im}\bar{\mathfrak{A}})^a\oplus(\operatorname{Im}\bar{\mathfrak{B}})^b\oplus\bar{\mathfrak{P}}^c, \tag{1.6.1}$$

for some non-negative integers a, b, and c.

Apply 1.4 to see that $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{G}_{\boldsymbol{d}})=G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$. The resolution $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is homogeneous with respect to the $\mathbf{M}_{\boldsymbol{n}}$-grading; the multi-homogeneous Betti numbers are given in [22]. The $\mathbf{M}_{\boldsymbol{n}}$-grading on $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ may be pulled back along $\alpha_{\boldsymbol{n},\boldsymbol{r}}$ to obtain the $\mathfrak{M}$-grading on $\mathfrak{G}_{\boldsymbol{d}}$. (It is clear that any given $\alpha_{\boldsymbol{n},\boldsymbol{r}}$ has a large kernel; however it is equally clear that $\bigcap\ker\alpha_{\boldsymbol{n},\boldsymbol{r}}=0$.)

(1.6.1) is that

$$\mathfrak{S}_{\boldsymbol{d}}^4\cong(\operatorname{Im}\bar{\mathfrak{A}})^{2\boldsymbol{d}+1}.$$

The proof of Conjecture 1.2 is then complete because $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}(\bar{\mathfrak{A}})$ is equal to the matrix A of Table 1. (A few tricks involving MCM modules establishes that the third syzygy, $\mathfrak{S}_{\boldsymbol{d}}^3$, of $\mathfrak{Q}_d$ satisfies $\mathfrak{S}_{\boldsymbol{d}}^3\cong(\operatorname{Im}\bar{\mathfrak{B}})^{2\boldsymbol{d}+1}$.)

The $\bar{\mathfrak{P}}$-modules $\operatorname{Im}\bar{\mathfrak{A}}$ and $\operatorname{Im}\bar{\mathfrak{B}}$ were called "maximally generated maximal Cohen Macaulay" modules by Ulrich in [30]. They were called "linear maximal Cohen-Macaulay modules" by Backelin, Herzog, and Sanders in [2]. At present they are called Ulrich modules.

The main results in this paper take place in characteristic zero. Indeed, we use the multi-graded Betti numbers in the minimal resolution of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}$-modules which were calculated in [22]. The calculation in [22] is largely a Hilbert series calculation and this calculation often appeals to Stanley's theorem [28, Thm. 2.4] that every Artinian monomial complete intersection over a polynomial ring $\boldsymbol{k}[x_1,\dots,x_n]$, where $\boldsymbol{k}$ is a field of characteristic zero, has the strong Lefschetz property.

$$\mathfrak{A}=\begin{bmatrix} y_1 & 0 & 0 & 0 & w_2 & -w_3 & w_4 & 0\\ -y_2 & 0 & w_4 & -w_3 & w_1 & 0 & 0 & 0\\ y_3 & w_4 & 0 & -w_2 & 0 & w_1 & 0 & 0\\ -y_4 & w_3 & -w_2 & 0 & 0 & 0 & w_1 & 0\\ 0 & -y_2 & -y_3 & -y_4 & 0 & 0 & 0 & w_1\\ 0 & y_1 & 0 & 0 & 0 & -y_4 & -y_3 & w_2\\ 0 & 0 & y_1 & 0 & -y_4 & 0 & y_2 & w_3\\ 0 & 0 & 0 & y_1 & y_3 & y_2 & 0 & w_4 \end{bmatrix}$$

$$\mathfrak{B}=\begin{bmatrix} w_1 & -w_2 & w_3 & -w_4 & 0 & 0 & 0 & 0\\ 0 & 0 & y_4 & y_3 & -w_2 & w_1 & 0 & 0\\ 0 & y_4 & 0 & -y_2 & -w_3 & 0 & w_1 & 0\\ 0 & -y_3 & -y_2 & 0 & -w_4 & 0 & 0 & w_1\\ y_2 & y_1 & 0 & 0 & 0 & 0 & -w_4 & w_3\\ -y_3 & 0 & y_1 & 0 & 0 & -w_4 & 0 & w_2\\ y_4 & 0 & 0 & y_1 & 0 & -w_3 & w_2 & 0\\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

TABLE 4. The matrices $\mathfrak{A}$ and $\mathfrak{B}$ give a matrix factorization of $\mathfrak{f}I_8$.

CONTENTS

2. NOTATION, CONVENTIONS, AND PRELIMINARY RESULTS.

2.1. Let R be a Noetherian ring, I be a proper ideal of R, and M be a non-zero finitely generated R-module.

(a) The *grade* of I is the length of a maximal regular sequence on R which is contained in I. (If R is Cohen-Macaulay, then the grade of I is equal to the height of I.)

(b) The R-module M is called *perfect* if the grade of the annihilator of M (denoted $\operatorname{ann}_R M$) is equal to the projective dimension of M (denoted $\operatorname{pd}_R M$). The inequality
$$\operatorname{grade}(\operatorname{ann}_R M) \le \operatorname{pd}_R M$$
holds automatically.

(c) Perfect modules are grade unmixed; see for example [4, Prop. 16.17]. In particular, if R is Cohen-Macaulay and M is perfect, then the set of associated prime ideals of M is equal to the set of prime ideals which are minimal in the support of M.

(d) If R is a polynomial ring over a field and M is a finitely generated graded R-module, then M is a perfect R-module if and only if M is a Cohen-Macaulay R-module. (This is not the full story. For more information, see, for example, [4, Prop. 16.19] or [3, Thm. 2.1.5].)

(e) The ideal I in R is called a *perfect ideal* if R/I is a perfect R-module.

2.2. A complex $\mathscr{C} : \cdots \to C_2 \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \to 0$ of R-modules is called *acyclic* if $\mathrm{H}_j(\mathscr{C}) = 0$ for $1 \le j$. If $\mathscr{C}$ is an acyclic complex, then $\mathscr{C}$ resolves $\mathrm{H}_0(\mathscr{C})$. If R is $\mathfrak{M}$-homogeneous for some Abelian group $\mathfrak{M}$, with R_0 a field, and $\mathscr{C}$ is a minimal $\mathscr{M}$-homogeneous resolution of $\mathrm{H}_0(\mathscr{C})$, then the image of c_i is the i^{th} syzygy of the R-module $\mathrm{H}_0(\mathscr{C})$.

2.3. If ϕ is a homomorphism, then we write $\operatorname{Im}\phi$ for the image of ϕ.

Notation 2.4. If $j \in \{1,2,3,4\}$, then let
$$z_j = (\#_1, \#_2, \#_3, \#_4) \in \mathbb{Z}^4,$$
with
$$\#_i = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j. \end{cases}$$
If a is an integer, let $\underline{a}$ represent the four tuple $\underline{a} = (a,a,a,a)$; in particular $\underline{0} = (0,0,0,0)$, $\underline{1} = (1,1,1,1)$, and $\underline{\boldsymbol{r}} = (\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$.

Once $\boldsymbol{n}$ is chosen, we use the following notation to compactify the $\mathbf{M}_{\boldsymbol{n}}$-degree of a monomial from P.

Notation 2.5. Adopt the data of 1.1. Consider the homomorphism of Abelian groups
$$\mathbb{Z}^5 \to \mathbf{M}_{\boldsymbol{n}},$$
which is given by
$$(k, \rho_1, \rho_2, \rho_3, \rho_4) \mapsto m_{(k,\rho_1,\rho_2,\rho_3,\rho_4)},$$
where
$$m_{(k,\rho_1,\rho_2,\rho_3,\rho_4)} = (k\boldsymbol{n} + \rho_1 + \rho_2 + \rho_3 + \rho_4, \bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4) \quad \text{in } \mathbf{M}_{\boldsymbol{n}}.$$

3. THE MINIMAL $\mathbf{M}_{\boldsymbol{n}}$-HOMOGENEOUS RESOLUTION OF $\boldsymbol{r}$-RESTRICTED IDEALS.

Recall the data of 1.1. In this section we observe that if an $\mathbf{M}_{\boldsymbol{n}}$-homogeneous P-module X is "$\boldsymbol{r}$-restricted" (see Definition 3.1), then every module in the minimal $\mathbf{M}_n$-homogeneous resolution of X by free P-modules is also "$\boldsymbol{r}$-restricted". We use this idea in Section 6 when we deform the P-module $P/(x_1^N,x_2^N,x_3^N,x_4^N,f_{\boldsymbol{n}})$ into the $\mathfrak{P}$-module

$$\mathfrak{P}/(y_1^{\boldsymbol{d}+1}w_1^{\boldsymbol{d}},y_2^{\boldsymbol{d}+1}w_2^{\boldsymbol{d}},y_3^{\boldsymbol{d}+1}w_3^{\boldsymbol{d}},y_4^{\boldsymbol{d}+1}w_4^{\boldsymbol{d}},\mathfrak{f}),$$

in the language of Data 1.3.

Definition 3.1. Retain Data 1.1 and Notation 2.5. If ε is an integer, then let $\bar{\varepsilon}$ represent the image of ε in $\frac{\mathbb{Z}}{(\boldsymbol{n})}$. Let X be a $\mathbf{M}_{\boldsymbol{n}}$-homogeneous P-module. The element x of X is called <u>$\boldsymbol{r}$-restricted</u> if the $\mathbf{M}_{\boldsymbol{n}}$-degree of x is

(3.1.1) $\qquad m_{(A,\boldsymbol{r}\varepsilon_1,\boldsymbol{r}\varepsilon_2,\boldsymbol{r}\varepsilon_3,\boldsymbol{r}\varepsilon_4)}$ for A and ε_h in $\mathbb{Z}$, with $\bar{\varepsilon}_h\in\{\bar{0},\bar{1}\}$, for $1\le h\le 4$.

The module X is called <u>$\boldsymbol{r}$-restricted</u> if there exists a minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous presentation

$$F_1\xrightarrow{d_1}F_0\xrightarrow{\text{aug}}X\to 0$$

in which the generators of F_0 and F_1 all are $\boldsymbol{r}$-restricted.

Example 3.2. Adopt Data 1.1. The P-module $P/(x_1^N,x_2^N,x_3^N,x_4^N,f_{\boldsymbol{n}})$ is $\boldsymbol{r}$-restricted.

Remark 3.3. If m_1 and m_2 are in $\mathbf{M}_{\boldsymbol{n}}$, with $P(-m_1)$ and $P(-m_2)$ both $\boldsymbol{r}$-restricted free P-modules of rank one, then every $\mathbf{M}_{\boldsymbol{n}}$-homogeneous P-module homomorphism $\phi:P(-m_2)\to P(-m_1)$ is given by multiplication by an element of P of the form

$$x_1^{e_1}x_2^{e_2}x_3^{e_3}x_4^{e_4}g(x_1^{\boldsymbol{n}},x_2^{\boldsymbol{n}},x_3^{\boldsymbol{n}},x_4^{\boldsymbol{n}}) \tag{3.3.1}$$

for some homogeneous polynomial g of degree A and some integers A,e_1,e_2,e_3,e_4, with each $e_h\in\{\boldsymbol{n}-\boldsymbol{r},0,\boldsymbol{r}\}$ and $m_{(A,e_1,e_2,e_3,e_4)}=m_2-m_1$. Indeed, ϕ is given by multiplication by an $\mathbf{M}_{\boldsymbol{n}}$-homogeneous element of P of degree m_2-m_1 and every such element has the form (3.3.1).

Lemma 3.4. *Retain Data* 1.1. *Let X be a finitely generated $\mathbf{M}_{\boldsymbol{n}}$-homogeneous P-module which is $\boldsymbol{r}$-restricted in the sense of Definition* 3.1. *Then the following statements hold.*

(a) *Every free module which appears in the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution*

$$F:\quad \cdots\xrightarrow{d_3}F_2\xrightarrow{d_2}F_1\xrightarrow{d_1}F_0$$

of X by free P-modules is $\boldsymbol{r}$-restricted.

(b) *Every entry in every matrix d_i from F has the form* (3.3.1).

(c) *If $P(-m_2)$ and $P(-m_1)$ are $\mathbf{M}_n$-homogeneous summands of F_{i+1} and F_{i-1}, respectively, for some integer i, then the composition*

$$P(-m_2)\hookrightarrow F_{i+1}\xrightarrow{d_{i+1}}F_i\xrightarrow{d_i}F_{i-1}\twoheadrightarrow P(-m_1)$$

is multiplication by an element of the form (3.3.1).

Proof. Assertions (b) and (c) follow from (a) combined with Remark 3.3. We prove (a). Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} X \to 0$$

be a minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous exact sequence of P-module homomorphisms with

$$F_2 = \bigoplus_{s=1}^{b_2} P(-m_{2,s}), \quad F_1 = \bigoplus_{q=1}^{b_1} P(-m_{1,q}), \quad \text{and} \quad F_0 = \bigoplus_{p=1}^{b_0} P(-m_{0,p}),$$

where the shifts

$$\begin{aligned} m_{0,p} &= m_{(A_{0,p}, \boldsymbol{r}\varepsilon_{1,p}, \boldsymbol{r}\varepsilon_{2,p}, \boldsymbol{r}\varepsilon_{3,p}, \boldsymbol{r}\varepsilon_{4,p})}, &&\text{with } \bar{\varepsilon}_{h,p} \in \{\bar{0}, \bar{1}\}, \text{ and} \\ m_{1,q} &= m_{(A_{1,q}, \boldsymbol{r}\varepsilon'_{1,q}, \boldsymbol{r}\varepsilon'_{2,q}, \boldsymbol{r}\varepsilon'_{3,q}, \boldsymbol{r}\varepsilon'_{4,q})}, &&\text{with } \bar{\varepsilon}'_{h,q} \in \{\bar{0}, \bar{1}\}, \end{aligned}$$

have the form of (3.1.1). We prove that each $m_{2,s} = m_{(A_{2,s}, \lambda_{1,s}, \lambda_{2,s}, \lambda_{3,s}, \lambda_{4,s})}$ has the form of (3.1.1). At that point $\operatorname{Im} d_1$ satisfies the hypotheses that are satisfied by X; hence one can iterate the procedure to construct

$$F_{j+1} \xrightarrow{d_j} F_j,$$

for $2 \le j$.

Let

$$\begin{bmatrix} H_{1,1} & \cdots & H_{1,b_2} \\ \vdots & & \vdots \\ H_{b_1,1} & \cdots & H_{b_1,b_2} \end{bmatrix}$$

be the $\mathbf{M}_{\boldsymbol{n}}$-homogeneous matrix for d_2. Observe that

$$m_{2,s} = m_{1,q} + \deg H_{q,s}, \tag{3.4.1}$$

for all q and s with $1 \le q \le b_1$ and $1 \le s \le b_2$, where "deg" represents the degree in the $\mathbf{M}_{\boldsymbol{n}}$-grading. Let

$$\deg H_{p,q} = m_{(B_{p,q}, w_{1,p,q}, w_{2,p,q}, w_{3,p,q}, w_{4,p,q})},$$

with $B_{p,q}$ and $w_{h,p,q}$ in $\mathbb{Z}$, for each ordered pair (p,q) which appears in the matrix d_2. It follows, from (3.4.1), that

$$\bar{\lambda}_{h,s} = \boldsymbol{r}\bar{\varepsilon}'_{h,q} + \bar{w}_{h,q,s}. \tag{3.4.2}$$

for all h, q, s with $1 \le h \le 4$, $1 \le q \le b_1$, and $1 \le s \le b_2$.

Fix (h,s) with $1 \le h \le 4$ and $1 \le s \le b_2$. The relation

$$\begin{bmatrix} H_{1,s} \\ \vdots \\ H_{b_1,s} \end{bmatrix}$$

on d_1 is a minimal relation. The target of d_1 is the free P-module F_0 and P is a domain. Consequently, the variable x_h does not divide all of the polynomials $\{H_{q,s} \mid 1 \le q \le b_1\}$. Indeed, when h and s are fixed there exits a parameter q with $\bar{w}_{h,q,s} = \bar{0}$. Apply (3.4.2). Keep in mind that $\bar{\varepsilon}'_{h,q} \in \{\bar{0}, \bar{1}\}$. Conclude $\bar{\lambda}_{h,s} \in \{\bar{0}, \bar{\boldsymbol{r}}\}$. This process holds for all fixed (h,s). We conclude that all $\bar{\lambda}_{h,s}$ are in $\{\bar{0}, \bar{\boldsymbol{r}}\}$. Thus, all $m_{2,s}$ have the desired form. □

4. THE MULTI-GRADED RESOLUTION $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ FROM [22].

The following result is [22, Cor. 7.1].

Theorem 4.1. *Adopt the setup of Data* 1.1*, Notation* 2.4*, and Notation* 2.5 *with* $\boldsymbol{k}$ *a field of characteristic zero and* $\boldsymbol{d}$ *positive, then the minimal* $\mathbf{M}_{\boldsymbol{n}}$*-homogeneous resolution* $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *of* $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *by free* $\bar{P}_{\boldsymbol{n}}$*-modules has the form*

$$G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}: \quad \dots \xrightarrow{G}{}_5 \xrightarrow{\bar{g}_5} G_4 \xrightarrow{\bar{g}_4} G_3 \xrightarrow{\bar{g}_3} G_2 \xrightarrow{\bar{g}_2} G_1 \xrightarrow{\bar{g}_1} G_0,$$

with

$$G_0 = \bar{P}_{\boldsymbol{n}},$$

$$G_1 = \bigoplus_{j=1}^{4} \bar{P}_{\boldsymbol{n}}[-m_{(\boldsymbol{d},\boldsymbol{r}z_j)}],$$

$$G_2 = \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}-1,\underline{\boldsymbol{r}})}]^{\boldsymbol{d}} \oplus \bigoplus_{1\le j<k\le 4} \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d},\boldsymbol{r}z_j+\boldsymbol{r}z_k)}]^{\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,\underline{0})}]^{\boldsymbol{d}+1},$$

$$G_i = \bigoplus_{1\le j<k<\ell\le 4} \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+\frac{i-3}{2},\boldsymbol{r}z_j+\boldsymbol{r}z_k+\boldsymbol{r}z_\ell)}]^{2\boldsymbol{d}+1} \oplus \bigoplus_{j=1}^{4} \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+\frac{i-1}{2},\boldsymbol{r}z_j)}]^{2\boldsymbol{d}+1},$$

for i odd with $3 \le i$*, and*

$$G_i = \begin{cases} \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+\frac{i-4}{2},\underline{\boldsymbol{r}})}]^{2\boldsymbol{d}+1} \oplus \bigoplus_{1\le j<k\le 4} \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+\frac{i-2}{2},\boldsymbol{r}z_j+\boldsymbol{r}z_k)}]^{2\boldsymbol{d}+1} \\ \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+\frac{i}{2},\underline{0})}]^{2\boldsymbol{d}+1}, \end{cases}$$

for i even with $4 \le i$*.*

The paper [22] does not explicitly give the form of the matrices g_i from $G_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{d}}$; however, this is an easy exercise. (As always, the matrix g_i has entries from P and $\bar{g}_i$ is the image of g_i with entries in $\bar{P}_{\boldsymbol{n}}$.)

Corollary 4.2. *Retain the notation of Theorem* 4.1*. The form of the matrices* g_4 *and* g_5 *is given in Table* 2 *with each* $(2\boldsymbol{d}+1)\times(2\boldsymbol{d}+1)$ *matrix* M_{ij} *and* N_{ij} *an invertible matrix of constants.*

Proof. Decompose the free modules G_k, with $3 \le k \le 5$ as follows:

$$\begin{aligned} G_3 = \quad & \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d},0,\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d},\boldsymbol{r},0,\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},0,\boldsymbol{r})}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,0,0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},0,0)}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,0,0,\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,0,0,0,\boldsymbol{r})}]^{2\boldsymbol{d}+1}, \\ G_4 = \quad & \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,\boldsymbol{r},\boldsymbol{r},0,0)}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,0,\boldsymbol{r})}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,0,0,\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},0,\boldsymbol{r})}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+2,0,0,0,0)}]^{2\boldsymbol{d}+1}, \text{ and} \end{aligned}$$

$$\begin{aligned} G_5 = \quad & \bar{P}_{\boldsymbol{n}}[-m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,\boldsymbol{r},\boldsymbol{r})}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,\boldsymbol{r},\boldsymbol{r},0,\boldsymbol{r})}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+1,\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+2,\boldsymbol{r},0,0,0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+2,0,\boldsymbol{r},0,0)}]^{2\boldsymbol{d}+1} \\ & \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+2,0,0,\boldsymbol{r},0)}]^{2\boldsymbol{d}+1} \oplus \bar{P}_{\boldsymbol{n}}[m_{(2\boldsymbol{d}+2,0,0,0,\boldsymbol{r})}]^{2\boldsymbol{d}+1}. \end{aligned}$$

The only possible non-zero $\mathbf{M}_{\boldsymbol{n}}$-homogeneous maps $G_5 \to G_4$ and $G_4 \to G_3$ have the form of the matrices in Table 2, for some matrices of constants M_{ij} and N_{ij}. It remains to explain why all of the M_{ij} and N_{ij} must be invertible. According to Eisenbud's results on matrix factorization [10], $f_{\boldsymbol{n}}$ is in the radical of the ideal generated by determinant of g_k for each k, with $3 \le k$. If some M_{ij} or N_{ij} is singular, then the determinant of the corresponding g_k is contained in the ideal generated by three of the variables x_1, x_2, x_3, x_4; hence $f_{\boldsymbol{n}} \notin \sqrt{\det g_k}$ and this is a contradiction. □

5. THE MAPS FROM THE "UNIVERSAL RESOLUTION" $\mathfrak{G}_{\boldsymbol{d}}$ TO EACH $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$.

5.1. Fix a non-negative integer $\boldsymbol{d}$. Recall $\mathfrak{P}$, $\mathfrak{f}$, $\bar{\mathfrak{P}}$, $\mathfrak{C}_{\boldsymbol{d}}$, $\mathfrak{Q}_{\boldsymbol{d}}$, $\mathfrak{M}$, and $\mathfrak{G}_{\boldsymbol{d}}$ from Data 1.3. Let $\mathfrak{I}_{\boldsymbol{d}}$ be the ideal $(\mathfrak{C}_d, \mathfrak{f})$ of $\mathfrak{P}$ and let $\mathfrak{L}_{\boldsymbol{d}}$ be the minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$ by free $\mathfrak{P}$-modules. Let $\boldsymbol{n}$ and $\boldsymbol{r}$ be arbitrary integers which satisfy (1.1.1). Recall P, $f_{\boldsymbol{n}}$, $\bar{P}_{\boldsymbol{n}}$, $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$, $Q_{\boldsymbol{d}}$, $\mathbf{M}_{\boldsymbol{n}}$, and $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ from Data 1.1. Let $\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the ideal $(C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}, f_{\boldsymbol{n}})$ of P and let $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $P/\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free P-modules. Let $S_{\boldsymbol{n},\boldsymbol{r}}$ be the subring

$$\boldsymbol{k}[x_1^{\boldsymbol{r}},\ x_1^{\boldsymbol{n}-\boldsymbol{r}},\ x_2^{\boldsymbol{r}},\ x_2^{\boldsymbol{n}-\boldsymbol{r}},\ x_3^{\boldsymbol{r}},\ x_3^{\boldsymbol{n}-\boldsymbol{r}},\ x_4^{\boldsymbol{r}},\ x_4^{\boldsymbol{n}-\boldsymbol{r}}]$$

of P.

Definition 5.2. Retain the data of 5.1. Define the $\boldsymbol{k}$-algebra homomorphism

$$\Delta_{\boldsymbol{n},\boldsymbol{r}} : \mathfrak{P} \to P$$

with

$$\Delta_{\boldsymbol{n},\boldsymbol{r}}(y_i) = x_i^{\boldsymbol{r}} \quad \text{and} \quad \Delta_{\boldsymbol{n},\boldsymbol{r}}(w_i) = x_i^{\boldsymbol{n}-\boldsymbol{r}}$$

and define the group homomorphism $\alpha_{\boldsymbol{n},\boldsymbol{r}} : \mathfrak{M} \to \mathbf{M}_{\boldsymbol{n}}$ by

$$\alpha_{\boldsymbol{n},\boldsymbol{r}}(a_1,a_2,a_3,a_4,a_5,a_6) = (\boldsymbol{r}a_1 + (\boldsymbol{n}-\boldsymbol{r})a_2, \boldsymbol{r}\bar{a}_3, \boldsymbol{r}\bar{a}_4, \boldsymbol{r}\bar{a}_5, \boldsymbol{r}\bar{a}_6)$$

for

$$\begin{aligned} &(a_1,a_2,a_3,a_4,a_5,a_6) \in \mathfrak{M} = \mathbb{Z}^6 \text{ and} \\ &(\boldsymbol{r}a_1 + (\boldsymbol{n}-\boldsymbol{r})a_2, \boldsymbol{r}\bar{a}_3, \boldsymbol{r}\bar{a}_4, \boldsymbol{r}\bar{a}_5, \boldsymbol{r}\bar{a}_6) \in \mathbf{M}_{\boldsymbol{n}} = \mathbb{Z} \oplus (\tfrac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}})^4, \end{aligned}$$

where $\bar{a}_i$ is the image of the integer a_i in $\frac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}}$.

Remarks 5.3. Retain the notation of Definition 5.2.

(a) Observe that

$$\Delta_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{f}) = f_{\boldsymbol{n}}, \quad \Delta_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{C}_{\boldsymbol{d}}) = C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}, \quad \text{and} \quad \Delta_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{I}_{\boldsymbol{d}}) = \mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}.$$

It follows that the ring homomorphism $\Delta_{\boldsymbol{n},\boldsymbol{r}} : \mathfrak{P} \to P$ induces a ring homomorphism $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}} : \bar{\mathfrak{P}} \to \bar{P}_{\boldsymbol{n}}$.

(b) If θ is an $\mathfrak{M}$-homogeneous element of $\mathfrak{P}$, then $\Delta_{\boldsymbol{n},\boldsymbol{r}}(\theta)$ is an $\mathbf{M}_{\boldsymbol{n}}$-homogeneous element of P and

$$\alpha_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{M}\text{-degree}(\theta)) = \mathbf{M}_{\boldsymbol{n}}\text{-degree}(\Delta_{\boldsymbol{n},\boldsymbol{r}}(\theta)). \tag{5.3.1}$$

Indeed, if $\theta = y_1^{a_1}w_1^{b_1}y_2^{a_2}w_2^{b_2}y_3^{a_3}w_3^{b_3}y_4^{a_4}w_4^{b_4}$, then both sides of (5.3.1) are equal to

$$\left(\boldsymbol{r}\sum_{i=1}^{4} a_i + (\boldsymbol{n}-\boldsymbol{r})\sum_{i=1}^{4} b_i, \boldsymbol{r}(\bar{a}_1-\bar{b}_1), \boldsymbol{r}(\bar{a}_2-\bar{b}_2), \boldsymbol{r}(\bar{a}_3-\bar{b}_3), \boldsymbol{r}(\bar{a}_4-\bar{b}_4)\right)$$

in $\mathbf{M}_{\boldsymbol{n}} = \mathbb{Z}\oplus(\frac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}})^4$.

(c) Let

$$\gamma = \gcd(\boldsymbol{r}, \boldsymbol{n}-\boldsymbol{r}), \quad \boldsymbol{r}' = \boldsymbol{r}/\gamma, \quad \text{and} \quad \boldsymbol{n}' = \boldsymbol{n}/\gamma,$$

where "gcd" means greatest common divisor. It is well-known, and easy to prove, that the $\boldsymbol{k}$-algebra homomorphism

$$\boldsymbol{k}[y,w] \to \boldsymbol{k}[x],$$

given by $y \mapsto x^{\boldsymbol{r}}$ and $w \mapsto x^{\boldsymbol{n}-\boldsymbol{r}}$, has kernel $(y^{\boldsymbol{n}'-\boldsymbol{r}'} - w^{\boldsymbol{r}'})$. Thus, the $\boldsymbol{k}$-algebra homomorphism

$$\Delta_{\boldsymbol{n},\boldsymbol{r}} : \mathfrak{P} \to P$$

induces a ring isomorphism from

$$\frac{\mathfrak{P}}{(y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'})} \to S_{\boldsymbol{n},\boldsymbol{r}}.$$

View $\bar{P}_{\boldsymbol{n}}$ as a $\bar{\mathfrak{P}}$-module by way of the ring homomorphism $\Delta_{\boldsymbol{n},\boldsymbol{r}} : \bar{\mathfrak{P}} \to \bar{P}_{\boldsymbol{n}}$ of Remark 5.3.(a). The heart of the paper is Theorem 7.5, where we prove that the complex $\mathfrak{G}_d \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is a resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules. The first step in the proof of Theorem 7.5 is to show that the ideal $\mathfrak{I}_{\boldsymbol{d}} = (\mathfrak{C}_{\boldsymbol{d}}, \mathfrak{f})$ in the polynomial ring $\mathfrak{P}$ is perfect of grade four. This step is carried Section 6.

6. THE IDEAL $\mathfrak{I}_{\boldsymbol{d}} = (y_1^{\boldsymbol{d}+1}w_1^{\boldsymbol{d}}, y_2^{\boldsymbol{d}+1}w_2^{\boldsymbol{d}}, y_3^{\boldsymbol{d}+1}w_3^{\boldsymbol{d}}, y_4^{\boldsymbol{d}+1}w_4^{\boldsymbol{d}}, \mathfrak{f})$ IN THE POLYNOMIAL RING $\mathfrak{P}$.

Adopt the notation of 5.1. In Theorem 6.2 we prove that the ideal $\mathfrak{I}_{\boldsymbol{d}} = (\mathfrak{C}_{\boldsymbol{d}}, \mathfrak{f})$ in the polynomial ring $\mathfrak{P}$ is perfect of grade four and we give the structure of the minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$ by free $\mathfrak{P}$-modules. This result is significant to us because the rings

$$\frac{\mathfrak{P}}{\mathfrak{I}_{\boldsymbol{d}}} \quad \text{and} \quad \mathfrak{Q}_{\boldsymbol{d}}$$

are equal.

To prove Theorem 6.2, we first consider the ideal $\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}} = (C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}, f_{\boldsymbol{n}})$ in P. The numerical information about the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution, $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$, of $P/\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free P-modules is calculated in [22]. This numerical information may be read to give the precise form of each entry of each differential in $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$. We carefully "lift" the differentials of $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ to homomorphisms of $\mathfrak{P}$-modules. We prove Theorem 6.2 by showing that the "lifted" homomorphisms form a resolution of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$ by free $\mathfrak{P}$-modules.

Theorem 6.1. *Adopt Data* 1.1 *with* $\boldsymbol{d}$ *positive and* $\boldsymbol{k}$ *a field of characteristic zero. Let* $\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *be the ideal* $(C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}, f_{\boldsymbol{n}})$ *of* P *and* $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *be the minimal* $\mathbf{M}_{\boldsymbol{n}}$*-homogeneous resolution of* $P/\mathscr{I}_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *by free* P*-modules. Then the following statements hold.*

(a) *The resolution* $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *has the form*

$$0 \to (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4 \xrightarrow{(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4} (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3 \xrightarrow{(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3} (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2 \xrightarrow{(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2} (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2 \xrightarrow{(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_1} (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_0,$$

where $(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_0 = P$,

$$(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_1 = \begin{cases} P(-m_{(1,\underline{0})}) \\ \oplus \bigoplus_{i=1}^{4} P(-m_{(\boldsymbol{d},\boldsymbol{r}z_i)}), \end{cases}$$

$$(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2 = \begin{cases} \bigoplus_{i=1}^{4} P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_i)}) \\ \oplus P(-m_{(2\boldsymbol{d}-1,\underline{r})})^{\boldsymbol{d}} \\ \oplus \bigoplus_{1\le i<j\le 4} P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_i+\boldsymbol{r}z_j)})^{\boldsymbol{d}+1} \\ \oplus P(-m_{(2\boldsymbol{d}+1,\underline{0})})^{\boldsymbol{d}+1}, \end{cases}$$

$$(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3 = \begin{cases} \bigoplus_{i=1}^{4} P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_i)})^{2\boldsymbol{d}+1} \\ \oplus \bigoplus_{1\le i<j<k\le 4} P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_i+\boldsymbol{r}z_j+\boldsymbol{r}z_k)})^{2\boldsymbol{d}+1}, \quad \textit{and} \end{cases}$$

$$(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4 = \begin{cases} P(-m_{(2\boldsymbol{d}+2,\underline{0})})^{\boldsymbol{d}} \\ \oplus \bigoplus_{1\le i<j\le 4} P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_i+\boldsymbol{r}z_j)})^{\boldsymbol{d}} \\ \oplus P(-m_{(2\boldsymbol{d},\underline{r})})^{\boldsymbol{d}+1}. \end{cases}$$

(b) *If* θ *is an entry in one of the matrices* $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i$*, then*

$$\theta = x_1^{e_1} x_2^{e_2} x_3^{e_3} x_4^{e_4} g(x_1^{\boldsymbol{n}}, x_2^{\boldsymbol{n}}, x_3^{\boldsymbol{n}}, x_4^{\boldsymbol{n}}),$$

where each e_i *is in the set* $\{\boldsymbol{r}, \boldsymbol{n}-\boldsymbol{r}, 0\}$ *and* g *is a homogeneous polynomial in the polynomial ring* $\boldsymbol{k}[\xi_1,\xi_2,\xi_3,\xi_4]$.

(c) *The form of each differential of* $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ *is given in Section* 12*. The notation is explained in* 12.1*. The form of the differential* $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_1$ *is given in Table* 6*; the form of* $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2$ *is given in Tables* 7 *and* 8*; the form of* $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3$ *is given in Tables* 9 *and* 10*; and the form of* $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4$ *is given in Tables* 11 *and* 12.

Proof. Assertion (a) is [22, Cor. 7.2]; (b) is an immediate consequence of (a); and (c) is a rephrasing of (b). □

We carefully lift the complex $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ of free P-modules to maps and modules over $\mathfrak{P}$. It does not matter what we take for $\boldsymbol{n}$ and $\boldsymbol{r}$ as long as $\boldsymbol{n} \neq 2\boldsymbol{r}$.

Theorem 6.2. *Adopt the notation of* 1.3 *and let* $\mathfrak{I}_{\boldsymbol{d}}$ *be the ideal* $(\mathfrak{C}_{\boldsymbol{d}}, \mathfrak{f})$ *of* $\mathfrak{P}$*. Then the following statements hold.*

(a) *The ideal* $\mathfrak{I}_{\boldsymbol{d}}$ *in the polynomial ring* $\mathfrak{P}$ *is perfect of grade four.*

(b) *The minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$ by free $\mathfrak{P}$-modules has the form*

$$0 \to (\mathfrak{L}_{\boldsymbol{d}})_4 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_4} (\mathfrak{L}_{\boldsymbol{d}})_3 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_3} (\mathfrak{L}_{\boldsymbol{d}})_2 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_2} (\mathfrak{L}_{\boldsymbol{d}})_1 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_1} (\mathfrak{L}_{\boldsymbol{d}})_0,$$

where

$$(\mathfrak{L}_{\boldsymbol{d}})_0 = \mathfrak{P},$$

$$(\mathfrak{L}_{\boldsymbol{d}})_1 = \mathfrak{P}(-(1,1,\underline{0}))^1 \oplus \bigoplus_{i=1}^{4} \mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_i))^1,$$

$$(\mathfrak{L}_{\boldsymbol{d}})_2 = \begin{cases} \bigoplus_{i=1}^{4} \mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_i))^1 \oplus \mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}-1,\underline{1}))^{\boldsymbol{d}} \\ \oplus \bigoplus_{1\le i<j\le 4} \mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_i+z_j))^{\boldsymbol{d}+1} \\ \oplus \mathfrak{P}(-(2\boldsymbol{d}+1,2\boldsymbol{d}+1,\underline{0}))^{\boldsymbol{d}+1}, \end{cases}$$

$$(\mathfrak{L}_{\boldsymbol{d}})_3 = \begin{cases} \bigoplus_{i=1}^{4} \mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_i))^{2\boldsymbol{d}+1} \\ \oplus \bigoplus_{1\le i<j<k\le 4} \mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},z_i+z_j+z_k))^{2\boldsymbol{d}+1}, \textit{and} \end{cases}$$

$$(\mathfrak{L}_{\boldsymbol{d}})_4 = \begin{cases} \mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+2,\underline{0}))^{\boldsymbol{d}} \oplus \bigoplus_{1\le i<j\le 4} \mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_i+z_j))^{\boldsymbol{d}} \\ \oplus \mathfrak{P}(-(2\boldsymbol{d}+4,2\boldsymbol{d},\underline{1}))^{\boldsymbol{d}+1}. \end{cases}$$

(c) *The form of each differential of $\mathfrak{L}_{\boldsymbol{d}}$ is given in Section* 12. *The notation is explained in* 12.1. *The form of the differential $\mathfrak{l}_1$ is given in Table* 13*; the form of $\mathfrak{l}_2$ is given in Tables* 14, 15, *and* 16*; the form of $\mathfrak{l}_3$ is given in Tables* 17 *and* 18*; and the form of $\mathfrak{l}_4$ is given in Tables* 19 *and* 20.

Proof. Let $\boldsymbol{n}$ and $\boldsymbol{r}$ be arbitrary integers which satisfy (1.1.1) with $\boldsymbol{n} \neq 2\boldsymbol{r}$. We begin by building the matrices $\mathfrak{l}_i$ with entries in $\mathfrak{P}$. Consider an entry θ in one of the matrices $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i$. According to Theorem 6.1.(b),

$$\theta = x_1^{e_1}x_2^{e_2}x_3^{e_3}x_4^{e_4}g(x_1^{\boldsymbol{n}},x_2^{\boldsymbol{n}},x_3^{\boldsymbol{n}},x_4^{\boldsymbol{n}}),$$

where each e_i is in the set $\{\boldsymbol{r},\boldsymbol{n}-\boldsymbol{r},0\}$ and g is a homogeneous form in four variables over $\boldsymbol{k}$. We now define the corresponding entry in $(\mathfrak{l}_{\boldsymbol{d}})_i$ to be

$$\lambda_1\lambda_2\lambda_3\lambda_4 g(y_1w_1,y_2w_2,y_3w_3,y_4w_4),$$

where

$$\lambda_i = \begin{cases} y_i, & \text{if } e_i = \boldsymbol{r}, \\ w_i, & \text{if } e_i = \boldsymbol{n}-\boldsymbol{r}, \text{ and} \\ 1, & \text{if } e_i = 0. \end{cases}$$

(The integers $\boldsymbol{r}$ and $\boldsymbol{n}-\boldsymbol{r}$ are different because we have chosen $\boldsymbol{n}$ and r with $\boldsymbol{n} \neq 2\boldsymbol{r}$.)

We have made maps and modules

$$(6.2.1) \qquad \mathfrak{L}_{\boldsymbol{d}}: \quad 0 \to (\mathfrak{L}_{\boldsymbol{d}})_4 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_4} (\mathfrak{L}_{\boldsymbol{d}})_3 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_3} (\mathfrak{L}_{\boldsymbol{d}})_2 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_2} (\mathfrak{L}_{\boldsymbol{d}})_1 \xrightarrow{(\mathfrak{l}_{\boldsymbol{d}})_1} (\mathfrak{L}_{\boldsymbol{d}})_0.$$

In fact, we have recorded the maps $(\mathfrak{l}_{\boldsymbol{d}})_i$, for $1\le i\le 4$, in Tables 13–20. We have also recorded the $\mathfrak{M}$-homogeneous Betti numbers of these maps in the statement of Theorem 6.2.(b). It is clear that the image of $(\mathfrak{l}_{\boldsymbol{d}})_1$ is $\mathfrak{I}_{\boldsymbol{d}}$. It is clear that the diagram

$$\begin{tikzcd}(\mathfrak{L}_{\boldsymbol{d}})_i \arrow[r, "(\mathfrak{l}_{\boldsymbol{d}})_i"] \arrow[d, "\Delta_{\boldsymbol{n},\boldsymbol{r}}"] & (\mathfrak{L}_{\boldsymbol{d}})_{i-1} \arrow[d, "\Delta_{\boldsymbol{n},\boldsymbol{r}}"'] \\ (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i \arrow[r, "(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i"] & (L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i-1}\end{tikzcd} \tag{6.2.2}$$

commutes for all i. It remains to show that $\mathfrak{L}_{\boldsymbol{d}}$ from (6.2.1) is a complex and is acyclic.

We first show that $\mathfrak{L}_{\boldsymbol{d}}$ is a complex.

Claim. *Each entry of each composition $(\mathfrak{l}_{\boldsymbol{d}})_i\circ(\mathfrak{l}_{\boldsymbol{d}})_{i+1}$ has the form*

$$\lambda_1\lambda_2\lambda_3\lambda_4 g(y_1w_1,y_2w_2,y_3w_3,y_4w_4), \tag{6.2.3}$$

where each λ_h is an element of $\{y_h,w_h,1\}$ and each g is a homogeneous polynomial in four variables over $\boldsymbol{k}$.

Proof of Claim. Fix integers i,p,q with $1\le i\le 3$, $1\le p\le \operatorname{rank}(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i-1}$, and

$$1\le q\le \operatorname{rank}(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}.$$

Let $P(-m_2)$ be the summand of $(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}$ in position q and $P(-m_1)$ be the summand of $(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}$ in position p. The product row p of $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i$ times column q of $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}$ is

$$\sum_{j=1}^{\operatorname{rank}(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i}[(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i])_{p,j}[(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}]_{j,q}.$$

According to Lemma 3.4.(c) there are uniquely determined integers A,e_1,e_2,e_3,e_4 with

$$e_h\in\{\boldsymbol{n}-\boldsymbol{r},0,\boldsymbol{r}\},\quad\text{and}\quad m_{(A,e_1,e_2,e_3,e_4)}=m_2-m_1,$$

such that, for each index j, with $1\le j\le \operatorname{rank}(L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i$, there is a homogeneous polynomial g_j, of degree A, in $\boldsymbol{k}[\xi_1,\xi_2,\xi_3,\xi_4]$, with

$$[(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i])_{p,j}[(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_{i+1}]_{j,q}=x_1^{e_1}x_2^{e_2}x_3^{e_3}x_4^{e_4}g(x_1^{\boldsymbol{n}},x_2^{\boldsymbol{n}},x_3^{\boldsymbol{n}},x_4^{\boldsymbol{n}}).$$

Now lift the calculation from P to $\mathfrak{P}$. Observe that the product row p of $(\mathfrak{l}_{\boldsymbol{d}})_i$ times column q of $(\mathfrak{l}_{\boldsymbol{d}})_{i+1}$ is the sum of $\operatorname{rank}(L_{\boldsymbol{d}})_i$ elements of $\mathfrak{P}$ each of which has the form of (6.2.3). This completes the proof of the claim. $\square$

Resume the proof that $\mathfrak{L}_{\boldsymbol{d}}$ is a complex. We know from (6.2.2) that

$$\Delta_{\boldsymbol{n},\boldsymbol{r}}((\mathfrak{l}_{\boldsymbol{d}})_i\circ(\mathfrak{l}_{\boldsymbol{d}})_{i+1})=0,$$

for each i. Thus, each entry of $(\mathfrak{l}_{\boldsymbol{d}})_i\circ(\mathfrak{l}_{\boldsymbol{d}})_{i+1}$ is in the kernel of $\Delta_{\boldsymbol{n},\boldsymbol{r}}$. The element (6.2.3) is in $\ker\Delta_{\boldsymbol{n},\boldsymbol{r}}$ if and only if $g(y_1w_1,y_2w_2,y_3w_3,y_4w_4)$ is in $\ker\Delta_{\boldsymbol{n},\boldsymbol{r}}$. The only homogeneous polynomial $g\in\boldsymbol{k}[\xi_1,\xi_2,\xi_3,\xi_4]$, with

$$g(y_1w_1,y_2w_2,y_3w_3,y_4w_4)\in\ker\Delta_{\boldsymbol{n},\boldsymbol{r}}$$

is the zero polynomial. We conclude that $(\mathfrak{l}_{\boldsymbol{d}})_i\circ(\mathfrak{l}_{\boldsymbol{d}})_{i+1}=0$, for all i; hence, $\mathfrak{L}_{\boldsymbol{d}}$ is a complex; furthermore $\Delta_{\boldsymbol{n},\boldsymbol{r}}:\mathfrak{L}_{\boldsymbol{d}}\to L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is a map of complexes.

We employ the Buchsbaum-Eisenbud criteria [5] in order to show that $\mathfrak{L}_{\boldsymbol{d}}$ is acyclic. It suffices to show that the matrices $(\mathfrak{l}_{\boldsymbol{d}})_i$ have the expected rank (denoted er_i) and that the ideal $I((\mathfrak{l}_{\boldsymbol{d}})_i)$, generated by the er_i-minors of $(\mathfrak{l}_{\boldsymbol{d}})_i$, has grade at least i, for each i. Let $I((\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i)$ represent the ideal of P which is generated by the er_i-minors of $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i$. The map of complexes $\Delta_{\boldsymbol{n},\boldsymbol{r}}$ carries $I((\mathfrak{l}_{\boldsymbol{d}})_i)$ to $I((\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i)$, for each i. The complex $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ resolves $P/C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$, which is a perfect P-module of projective dimension four. It follows that the ideals $I((\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i)$, with $1 \le i \le 4$, all are primary to the maximal ideal (x_1,x_2,x_3,x_4) of P. (See, for example, [14, Props. 6.8 and 6.3.(c)].) Thus, each ideal $\Delta_{\boldsymbol{n},\boldsymbol{r}}(I((\mathfrak{l}_{\boldsymbol{d}})_i))$ has grade four. We finish the argument by showing that each $I((\mathfrak{l}_{\boldsymbol{d}})_i)$ has grade at least four. The concepts of grade and height coincide in the Cohen-Macaulay ring $\mathfrak{P}$; so we show that each ideal $I((\mathfrak{l}_{\boldsymbol{d}})_i)$ has height at least four.

Let q_1 be a prime ideal of $\mathfrak{P}$ which is minimal over $I((\mathfrak{l}_{\boldsymbol{d}})_i)$ and has the same height. The ideal $I((\mathfrak{l}_{\boldsymbol{d}})_i)$ is $\mathfrak{M}$-homogeneous and q_1 is minimal over $I((\mathfrak{l}_{\boldsymbol{d}})_i)$; so q_1 is also $\mathfrak{M}$-homogeneous. Let q_2 be the kernel of $\Delta_{\boldsymbol{n},\boldsymbol{r}} : \mathfrak{P} \to P$. Recall from Remark 5.3.(c) that the ring $\mathfrak{P}/q_2$ is isomorphic to $S_{\boldsymbol{n},\boldsymbol{r}}$. The ideal $(q_1+q_2)/q_2$ is a proper $\mathbf{M}_{\boldsymbol{n}}$-homogeneous ideal of $\mathfrak{P}/q_2 \cong S_{\boldsymbol{n},\boldsymbol{r}}$; so q_1+q_2 is a proper ideal of $\mathfrak{P}$. Let q_3 be a prime ideal of $\mathfrak{P}$ which is minimal over q_1+q_2. It is well-known that

$$\operatorname{ht} q_3 \le \operatorname{ht} q_1 + \operatorname{ht} q_2.$$

(A proof from Algebraic Geometry (when $\boldsymbol{k}$ is algebraically closed) may be found in [13, Chapt. 1, Prop. 7.1]. A proof in the present generality is given in [27, III, Prop. 17]. A proof which works over an arbitrary regular ring is given in [27, V. Thm. 3].) At any rate, it follows that

$$\operatorname{ht} q_3 - \operatorname{ht} q_2 \le \operatorname{ht} q_1 = \operatorname{ht} I((\mathfrak{l}_{\boldsymbol{d}})_i).$$

Of course, $\operatorname{ht} q_3 - \operatorname{ht} q_2 = \operatorname{ht} \frac{q_3}{q_2}$. We have seen that $\frac{q_3}{q_2}$ is the maximal ideal

$$\operatorname{radical}(I((\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_i)) \cap S_{\boldsymbol{n},\boldsymbol{r}} = (x_1^{\boldsymbol{r}}, x_1^{\boldsymbol{n}-\boldsymbol{r}}, x_2^{\boldsymbol{r}}, x_2^{\boldsymbol{n}-\boldsymbol{r}}, x_3^{\boldsymbol{r}}, x_3^{\boldsymbol{n}-\boldsymbol{r}}, x_4^{\boldsymbol{r}}, x_4^{\boldsymbol{n}-\boldsymbol{r}})$$

of $S_{\boldsymbol{n},\boldsymbol{r}}$. Thus,

$$4 = \operatorname{ht} \frac{q_3}{q_2} = \operatorname{ht} q_3 - \operatorname{ht} q_2 \le \operatorname{ht} q_1 = \operatorname{ht} I((\mathfrak{l}_{\boldsymbol{d}})_i)$$

and the proof is complete. □

7. THE HOMOMORPHISM $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}$ CARRIES THE RESOLUTION OF $\mathfrak{Q}_{\boldsymbol{d}}$ TO A RESOLUTION OF $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$.

Retain the data of 5.1. View $\bar{P}_{\boldsymbol{n}}$ as a $\bar{\mathfrak{P}}$-module by way of the ring homomorphism $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}} : \bar{\mathfrak{P}} \to \bar{P}_{\boldsymbol{n}}$. Recall that $\mathfrak{G}_{\boldsymbol{d}}$ is a resolution of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\bar{\mathfrak{P}}$-modules. In Theorem 7.5 we prove that the complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is a resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules.

The proof of Theorem 7.5 is given at the end of the section. In the meantime we record various intermediate results. Notice first that

$$\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}) \cong Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}. \tag{7.0.1}$$

Indeed, the complex $\mathfrak{G}_{\boldsymbol{d}}$ is a resolution of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\bar{\mathfrak{P}}$-modules; hence, the homology of $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is $\operatorname{Tor}_{\bullet}^{\bar{\mathfrak{P}}}(\mathfrak{Q}_{\boldsymbol{d}}, \bar{P}_{\boldsymbol{n}})$. In particular,

$$\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}) = \operatorname{Tor}_0^{\bar{\mathfrak{P}}}(\mathfrak{Q}_{\boldsymbol{d}}, \bar{P}_{\boldsymbol{n}}) = \frac{\bar{\mathfrak{P}}}{\mathfrak{C}_{\boldsymbol{d}}\bar{\mathfrak{P}}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}} = \frac{\bar{P}_{\boldsymbol{n}}}{\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}(\mathfrak{C}_{\boldsymbol{d}}) \cdot \bar{P}_{\boldsymbol{n}}} = \frac{\bar{P}_{\boldsymbol{n}}}{C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\bar{P}_{\boldsymbol{n}}} = Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}.$$

The complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is clearly a complex of free $\bar{P}_{\boldsymbol{n}}$-modules. It remains to show that the complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is acyclic.

In order to prove Theorem 7.5, we view $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}} : \bar{\mathfrak{P}} \to \bar{P}_{\boldsymbol{n}}$ as the composition of two ring homomorphisms:

$$\bar{\mathfrak{P}} \twoheadrightarrow \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) \hookrightarrow \bar{P}_{\boldsymbol{n}},$$

where

$$\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) = \frac{S_{\boldsymbol{n},\boldsymbol{r}}}{(f_{\boldsymbol{n}})}. \tag{7.0.2}$$

The ultimate complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\mathfrak{P}} \bar{P}_{\boldsymbol{n}}$ from Theorem 7.5 is equal to

$$\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) \otimes_{\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}.$$

We prove that $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ is acyclic in Lemma 7.2. Then we apply Lemma 7.4 to conclude that the complex

$$\Big(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})\Big) \otimes_{\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$$

is acyclic.

Lemma 7.1. *Retain the data of* 5.1 *and Remark* 5.3.(c). *Then the following statements hold.*

(a) *The elements*

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'}$$

of $\mathfrak{P}$ form a regular sequence on $\mathfrak{Q}_{\boldsymbol{d}}$.

(b) *The elements*

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'},\ \mathfrak{f}$$

of $\mathfrak{P}$ form a regular sequence on $\mathfrak{P}$.

Proof. We first prove (a). According to Theorem 6.2, the ideal

$$\mathfrak{I}_{\boldsymbol{d}} = (y_1^{\boldsymbol{d}+1}w_1^{\boldsymbol{d}},\ y_2^{\boldsymbol{d}+1}w_2^{\boldsymbol{d}},\ y_3^{\boldsymbol{d}+1}w_3^{\boldsymbol{d}},\ y_4^{\boldsymbol{d}+1}w_4^{\boldsymbol{d}}, y_1w_1 + y_2w_2 + y_3w_3 + y_4w_4)$$

of the polynomial ring $\mathfrak{P}$ is perfect of grade four. It follows that the associated prime ideals of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$ are the prime ideals of $\mathfrak{P}$ which are minimal over $\mathfrak{I}_{\boldsymbol{d}}$. It is clear that these ideals all have the form (v_1, v_2, v_3, v_4), where v_i is equal to either y_i or w_i. It is also clear that $y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'}$ is not in any of the associated prime ideals of $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$. Thus, the element $y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'}$ of $\mathfrak{P}$ is regular on $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$. It follows that $(\mathfrak{I}_{\boldsymbol{d}}, y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'})$ is a perfect ideal of $\mathfrak{P}$ of grade five. The associated prime ideals of

$$\mathfrak{P}/(\mathfrak{I}_{\boldsymbol{d}}, y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'})$$

are the prime ideals of $\mathfrak{P}$ which are minimal over $(\mathfrak{I}_{\boldsymbol{d}}, y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'})$. It is clear that these ideals all have the form $(y_1, w_1, v_2, v_3, v_4)$, where v_i is equal to either y_i or w_i. It is also clear that $y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'}$ is not in any of the associated prime ideals of $\mathfrak{P}/(\mathfrak{I}_{\boldsymbol{d}}, y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'})$. Continue in this manner to conclude that the elements

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'}$$

of $\mathfrak{P}$ form a regular sequence on $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}} = \mathfrak{Q}_{\boldsymbol{d}}$.

Now we prove (b). We have seen that each of the ideals

$$\begin{aligned}(y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'}) &\subseteq (y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'})\\ &\subseteq (y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'})\\ &\subseteq (y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'})\end{aligned}$$

of $\mathfrak{P}$ is prime; indeed, these ideals define the domains

$$\begin{aligned}\boldsymbol{k}[x_1^{\boldsymbol{r}},\ x_1^{\boldsymbol{n}-\boldsymbol{r}}] &\subseteq \boldsymbol{k}[x_1^{\boldsymbol{r}},\ x_1^{\boldsymbol{n}-\boldsymbol{r}},\ x_1^{\boldsymbol{r}},\ x_2^{\boldsymbol{n}-\boldsymbol{r}}] \subseteq \boldsymbol{k}[x_1^{\boldsymbol{r}},\ x_1^{\boldsymbol{n}-\boldsymbol{r}},\ x_1^{\boldsymbol{r}},\ x_2^{\boldsymbol{n}-\boldsymbol{r}},\ x_3^{\boldsymbol{r}},\ x_3^{\boldsymbol{n}-\boldsymbol{r}}]\\ &\subseteq \boldsymbol{k}[x_1^{\boldsymbol{r}},\ x_1^{\boldsymbol{n}-\boldsymbol{r}},\ x_1^{\boldsymbol{r}},\ x_2^{\boldsymbol{n}-\boldsymbol{r}},\ x_3^{\boldsymbol{r}},\ x_3^{\boldsymbol{n}-\boldsymbol{r}},\ x_4^{\boldsymbol{r}},\ x_4^{\boldsymbol{n}-\boldsymbol{r}}] = S_{\boldsymbol{n},\boldsymbol{r}}.\end{aligned}$$

Thus, the elements $y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'}$ of $\mathfrak{P}$ form a regular sequence in $\mathfrak{P}$. On the other hand, $y_1w_1 + y_2w_2 + y_3w_3 + y_4w_4$ is not an element of the prime ideal

$$(y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'})$$

since $y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'}, y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'}, y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'}$, and $y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'}$ all vanish at the points

$$\mathfrak{p} = (1,1,1,1,1,1,1,1) \quad \text{and} \quad \mathfrak{p}' = (1,1,1,1,1,1,0,0),$$

but $y_1w_1 + y_2w_2 + y_3w_3 + y_4w_4$ does not vanish at both $\mathfrak{p}$ and $\mathfrak{p}'$. We conclude that the elements

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'},\ y_1w_1 + y_2w_2 + y_3w_3 + y_4w_4$$

of $\mathfrak{P}$ form a regular sequence on $\mathfrak{P}$. $\square$

Lemma 7.2. *Retain the data of* 5.1 *and* (7.0.2)*. Then the complex* $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ *is acyclic.*

Proof. We show that the complex

$$\mathfrak{G}_d \otimes_{\bar{\mathfrak{P}}} \frac{\bar{\mathfrak{P}}}{(y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'})\bar{\mathfrak{P}}} \tag{7.2.1}$$

is acyclic. The homology of (7.2.1) is

$$\operatorname{Tor}_\bullet^{\bar{\mathfrak{P}}}\left(\mathfrak{Q}_{\boldsymbol{d}}\ ,\ \frac{\bar{\mathfrak{P}}}{(y_1^{\boldsymbol{n}'-\boldsymbol{r}'} - w_1^{\boldsymbol{r}'},\ y_2^{\boldsymbol{n}'-\boldsymbol{r}'} - w_2^{\boldsymbol{r}'},\ y_3^{\boldsymbol{n}'-\boldsymbol{r}'} - w_3^{\boldsymbol{r}'},\ y_4^{\boldsymbol{n}'-\boldsymbol{r}'} - w_4^{\boldsymbol{r}'})\bar{\mathfrak{P}}}\right).$$

Thus, the homology of (7.2.1) is equal to

$$\mathrm{H}_\bullet\left(\mathfrak{Q}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathbb{K}\right),$$

where $\mathbb{K}$ is a resolution of

$$\frac{\bar{\mathfrak{P}}}{(y_1^{\boldsymbol{n}'-\boldsymbol{r}'}-w_1^{\boldsymbol{r}'},\, y_2^{\boldsymbol{n}'-\boldsymbol{r}'}-w_2^{\boldsymbol{r}'},\, y_3^{\boldsymbol{n}'-\boldsymbol{r}'}-w_3^{\boldsymbol{r}'},\, y_4^{\boldsymbol{n}'-\boldsymbol{r}'}-w_4^{\boldsymbol{r}'})\bar{\mathfrak{P}}}$$

by free $\bar{\mathfrak{P}}$-modules. According to Lemma 7.1.(b), the elements

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'}-w_1^{\boldsymbol{r}'},\, y_2^{\boldsymbol{n}'-\boldsymbol{r}'}-w_2^{\boldsymbol{r}'},\, y_3^{\boldsymbol{n}'-\boldsymbol{r}'}-w_3^{\boldsymbol{r}'},\, y_4^{\boldsymbol{n}'-\boldsymbol{r}'}-w_4^{\boldsymbol{r}'},\, \mathfrak{f}$$

of $\mathfrak{P}$ form a regular sequence on $\mathfrak{P}$. The ring $\mathfrak{P}$ is a domain and $\mathfrak{f}$ is not zero; so, $\mathfrak{f}$ is a regular element on $\mathfrak{P}$ and the ideal

$$(y_1^{\boldsymbol{n}'-\boldsymbol{r}'}-w_1^{\boldsymbol{r}'},\, y_2^{\boldsymbol{n}'-\boldsymbol{r}'}-w_2^{\boldsymbol{r}'},\, y_3^{\boldsymbol{n}'-\boldsymbol{r}'}-w_3^{\boldsymbol{r}'},\, y_4^{\boldsymbol{n}'-\boldsymbol{r}'}-w_4^{\boldsymbol{r}'})\bar{\mathfrak{P}}$$

has grade four. Consequently, we may take $\mathbb{K}$ to be the Koszul complex of free $\bar{\mathfrak{P}}$-modules which is associated to the elements

$$y_1^{\boldsymbol{n}'-\boldsymbol{r}'}-w_1^{\boldsymbol{r}'},\, y_2^{\boldsymbol{n}'-\boldsymbol{r}'}-w_2^{\boldsymbol{r}'},\, y_3^{\boldsymbol{n}'-\boldsymbol{r}'}-w_3^{\boldsymbol{r}'},\, y_4^{\boldsymbol{n}'-\boldsymbol{r}'}-w_4^{\boldsymbol{r}'} \tag{7.2.2}$$

of $\bar{\mathfrak{P}}$. Apply Lemma 7.1.(a) in order to see that the elements (7.2.2) of $\mathfrak{P}$ form a regular sequence on $\mathfrak{Q}_{\boldsymbol{d}}$ It follows that $\mathfrak{Q}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathbb{K}$ is acyclic; and therefore, the complex (7.2.1) is also acyclic. □

Corollary 7.3. *The complex* $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ *of Lemma* 7.2 *is a minimal* $\mathbf{M}_n$ *homogeneous resolution of* $\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}))$.

Proof. The complex $\mathfrak{G}_{\boldsymbol{d}}$ is the minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\mathfrak{P}$-modules. No units appear in the differentials of $\mathfrak{G}_{\boldsymbol{d}}$. Every non-zero entry of every differential matrix from $\mathfrak{G}_{\boldsymbol{d}}$ has $\mathfrak{M}$-degree $(a_1,a_2,a_3,a_4,a_5,a_6)$ with a_1 and a_2 both non-negative and at least one of the integers a_1 or a_2 positive. It follows, from (5.3.1), that every non-zero entry of every differential matrix from $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ has $\mathbf{M}_{\boldsymbol{n}}$-degree

$$\alpha_{\boldsymbol{n},\boldsymbol{r}}(a_1,a_2,a_3,a_4,a_5,a_6) = (\boldsymbol{r}a_1 + (\boldsymbol{n}-\boldsymbol{r})a_2, \boldsymbol{r}\bar{a}_3, \boldsymbol{r}\bar{a}_4, \boldsymbol{r}\bar{a}_5, \boldsymbol{r}\bar{a}_6),$$

with $\boldsymbol{r}a_1 + (\boldsymbol{n}-\boldsymbol{r})a_2$ positive. Thus, $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ is an $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}))$ and no units appear in the differential matrices of this resolution. We conclude that $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ is the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}))$ by free $\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$-modules. □

The final step in the proof of Theorem 7.5 is to show that $-\otimes_{\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$ carries a particular acyclic complex of free $\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$-modules to an acyclic complex. The proof is somewhat delicate because the inclusion map $\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) \hookrightarrow \bar{P}_{\boldsymbol{n}}$ is not a flat ring homomorphism. Indeed, it is clear that the inclusion map

$$R' = \boldsymbol{k}[x^2,x^3] \hookrightarrow \boldsymbol{k}[x] = R$$

is not a flat ring homomorphism; for example,

$$\operatorname{Tor}_+^{\boldsymbol{k}[x^2,x^3]}\left(\frac{\boldsymbol{k}[x^2,x^3]}{(x^2,x^3)\boldsymbol{k}[x^2,x^3]}\ ,\ \boldsymbol{k}[x]\right)$$

is far from zero. On the other hand, there do exist $\boldsymbol{k}[x^2,x^3]$-modules M for which

$$\operatorname{Tor}_+^{\boldsymbol{k}[x^2,x^3]}(M\ ,\ \boldsymbol{k}[x]) = 0. \tag{7.3.1}$$

In particular, $M = \frac{\boldsymbol{k}[x^2,x^3]}{(x^2)\boldsymbol{k}[x^2,x^3]}$ has property (7.3.1). In Lemma 7.4 we identify many modules M with property (7.3.1).

Lemma 7.4 has many hypotheses, but almost no proof. We refer to the technique as "lifting a resolution over the ring R to a resolution over the subring R'". When we apply Lemma 7.4 in the proof of Theorem 7.5,

$$\begin{array}{ll}
\operatorname{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) & \text{will play the role of } R', \\
\bar{P}_{\boldsymbol{n}} & \text{will play the role of } R, \\
\dfrac{\operatorname{Im}\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}}{C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}} & \text{will play the role of } \mathrm{H}_0(\mathbb{F}'), \\
\mathbf{M}_{\boldsymbol{n}} & \text{will play the role of } \mathscr{G},
\end{array}$$

and the subgroup of $\mathbf{M}_{\boldsymbol{n}}$ generated by

$$\left\{\begin{array}{l} m_{(1,\underline{0})},\ m_{(0,\boldsymbol{r},0,0,0)},\ m_{(0,0,\boldsymbol{r},0,0)},\ m_{(0,0,0,\boldsymbol{r},0)},\ m_{(0,0,0,0,\boldsymbol{r})},\ m_{(0,\boldsymbol{n}-\boldsymbol{r},0,0,0)}, \\ \mathfrak{m}_{(0,0,\boldsymbol{n}-\boldsymbol{r},0,0)},\ m_{(0,0,0,\boldsymbol{n}-\boldsymbol{r},0)},\ m_{(0,0,0,0,\boldsymbol{n}-\boldsymbol{r})} \end{array}\right\} \tag{7.3.2}$$

will play the role of $\mathscr{G}'$.

Lemma 7.4. *Let R be a commutative ring which is graded by an Abelian group $\mathscr{G}$. Suppose that $\mathscr{G}'$ is a subgroup of $\mathscr{G}$ and that R' is the subring of R defined by the following rule. If r is a homogeneous element of R, then*

$$r \text{ is in } R' \text{ if and only if the degree of } r \text{ is in } \mathscr{G}'. \tag{7.4.1}$$

Let

$$\mathbb{F}: \quad \cdots \to \bigoplus_{j=1}^{\beta_i} R[-m_{i,j}] \xrightarrow{d_i} \bigoplus_{j=1}^{\beta_{i-1}} R[-m_{i-1,j}] \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} \bigoplus_{j=1}^{\beta_0} R[-m_{0,j}] \to 0$$

be an acyclic, $\mathscr{G}$-homogeneous, complex of finitely generated free R-modules. Suppose that every twist $m_{i,j}$ is actually an element of $\mathscr{G}'$. Then

$$\mathbb{F}': \quad \cdots \to \bigoplus_{j=1}^{\beta_i} R'[-m_{i,j}] \xrightarrow{d_i} \bigoplus_{j=1}^{\beta_{i-1}} R'[-m_{i-1,j}] \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} \bigoplus_{j=1}^{\beta_0} R'[-m_{0,j}] \to 0$$

is an acyclic, $\mathscr{G}'$-homogeneous, complex of finitely generated R'-modules and

$$\mathbb{F}' \otimes_{R'} R$$

is a $\mathscr{G}$-homogeneous resolution of $\mathrm{H}_0(\mathbb{F}') \otimes_{R'} R$ by free R-modules.

Proof. Each map d_i of $\mathbb{F}$ may be represented by a matrix. Each entry in each matrix d_i is in R' (because of hypothesis (7.4.1) and the hypothesis that each $m_{i,j}$ is in $\mathscr{G}'$). The product $d_i d_{i+1}$ is zero in R; so the product is also zero in R'. Thus, F' is a complex of free R'-modules. If $\xi \in (\mathbb{F}')_i$ is a homogeneous cycle in $\mathbb{F}'$, for some positive i, then ξ is a homogeneous i-cycle in $\mathbb{F}$. The complex $\mathbb{F}$ is acyclic (and i is positive); so, ξ is a homogeneous boundary in $\mathbb{F}_i$. In other words, there is a homogeneous element Ξ in $\mathbb{F}_i$ with

$$d_{i+1}(\Xi) = \xi. \tag{7.4.2}$$

View equation (7.4.2) as matrix multiplication: d_{i+1} is a matrix and Ξ and ξ are column vectors. Each entry of d_{i+1} and each entry of ξ is homogeneous, is in R', and has degree in the group $\mathscr{G}'$; furthermore each entry of Ξ is homogeneous and is in R. It follows that each entry of Ξ also has degree in $\mathscr{G}'$. Thus, according to (7.4.1), each entry of Ξ is in R' and $\Xi \in (\mathbb{F}')_{i+1}$.

We have shown that $\mathbb{F}'$ is a resolution of $\mathrm{H}_0(\mathbb{F}')$ by free R'-modules. Apply $-\otimes_{R'} R$ in order to conclude that $\mathbb{F}' \otimes_{R'} R$ (which is equal to $\mathbb{F}$) is a resolution of $\mathrm{H}_0(\mathbb{F}') \otimes_{R'} R$ by free R-modules. □

Theorem 7.5. *Retain the data of* 5.1. *View $\bar{P}_{\boldsymbol{n}}$ as a $\bar{\mathfrak{P}}$-module by way of the ring homomorphism $\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}} : \bar{\mathfrak{P}} \to \bar{P}_{\boldsymbol{n}}$ of Remark* 5.3.(a)*. Then the complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}$ is a resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules.*

Proof. In light of (7.0.1) and Lemma 7.2, it suffices to show that $-\otimes_{\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$ carries the acyclic complex $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ to an acyclic complex.

We know from Corollary 7.3 that $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ is the minimal $\mathbf{M}_n$ homogeneous resolution of $\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}))$. We know from Theorem 4.1 that $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules. We apply Lemma 7.4 to "lift" $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ from from an acyclic complex of free $\bar{P}_{\boldsymbol{n}}$-modules to an acyclic complex $G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ of free modules over the subring $\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$ of $\bar{P}_{\boldsymbol{n}}$. According to Lemma 7.4 the "lift" $G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ will be another minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules; hence $G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ will be isomorphic to $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$. Lemma 7.4 guarantees that $G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}} \otimes_{\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$ is acyclic; so, $\Big(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})\Big) \otimes_{\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$ is also acyclic.

It remains to show that all of the hypotheses of Lemma 7.4 are satisfied. Let $R = \bar{P}_{\boldsymbol{n}}$ and F be the resolution $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ from Theorem 4.1. The resolution $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is $\mathscr{G}$-homogeneous for $\mathscr{G} = \mathbf{M}_{\boldsymbol{n}}$. Observe that every twist that actually appears in $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is in the subgroup $\mathscr{G}'$ of (7.3.2). Observe, also, that $\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$, which is recorded in (7.0.2), is equal to the subring of $\bar{P}_{\boldsymbol{n}}$ which is generated by the set of homogeneous elements θ in $\bar{P}_{\boldsymbol{n}}$ such that the $\mathbf{M}_{\boldsymbol{n}}$-degree of θ is an element of $\mathscr{G}'$. The hypotheses of Lemma 7.4 are satisfied and $G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is minimal $\mathbf{M}_n$-homogeneous resolution of $\mathrm{H}_0(G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})$ by free $\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})$-modules.

It should be observed that the zeroth homologies all behave correctly:

$$\begin{aligned}
\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}}) &= \mathfrak{P}/\mathfrak{C}_{\boldsymbol{d}}\mathfrak{P}, \\
\mathrm{H}_0(\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) &= \frac{\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})}{C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\,\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} = \mathrm{H}_0(G'_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}), \quad \text{and} \\
\mathrm{H}_0(G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}) &= Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}.
\end{aligned}$$

We conclude that

$$\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}} = \mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} \mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}}) \otimes_{\mathrm{Im}(\bar{\Delta}_{\boldsymbol{n},\boldsymbol{r}})} \bar{P}_{\boldsymbol{n}}$$

is the minimal $\mathbf{M}_{\boldsymbol{n}}$-homogeneous resolution of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules. □

8. The structure of the “universal resolution” $\mathfrak{G}_{\boldsymbol{d}}$.

We proved in Theorem 7.5 that for each non-negative integer $\boldsymbol{d}$ there exists a single “universal resolution” $\mathfrak{G}_{\boldsymbol{d}}$ such that for each pair $\boldsymbol{n},\boldsymbol{r}$, which satisfy (1.1.1), every resolution $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ can be obtained from $\mathfrak{G}_{\boldsymbol{d}}$ by way of a base change. The resolutions $\{G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\}$ are the main object of study in the present paper as well as in the paper [22]. Thus, the resolutions $\{\mathfrak{G}_{\boldsymbol{d}}\}$ become resolutions of significant interest. In the present section, we give the explicit description of the infinite tail of each $\mathfrak{G}_{\boldsymbol{d}}$; we also give the multi-degrees of all of the Betti numbers in each $\mathfrak{G}_{\boldsymbol{d}}$. In particular, we give the multi-graded Betti numbers for $(\mathfrak{G}_{\boldsymbol{d}})_2$. The differentials entering and leaving $(\mathfrak{G}_{\boldsymbol{d}})_2$ involve polynomials of high degree and a large number of terms.

The results of Lemma 8.1 are interesting in their own right and they are enormously important in the proof of Corollary 10.2, which is the the main theorem of the paper.

Lemma 8.1. *Fix a non-negative integer $\boldsymbol{d}$ and a field $\boldsymbol{k}$ of characteristic zero. Adopt the setup of Data 1.3 and Notation 2.4.*

Then the minimal $\mathfrak{M}$-homogeneous resolution $\mathfrak{G}_{\boldsymbol{d}}$ of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\bar{\mathfrak{P}}$-modules has the form

$$\mathfrak{G}_{\boldsymbol{d}}: \quad \dots \to (\mathfrak{G}_{\boldsymbol{d}})_5 \xrightarrow{\mathfrak{g}_5} (\mathfrak{G}_{\boldsymbol{d}})_4 \xrightarrow{\mathfrak{g}_4} (\mathfrak{G}_{\boldsymbol{d}})_3 \xrightarrow{\mathfrak{g}_3} (\mathfrak{G}_{\boldsymbol{d}})_2 \xrightarrow{\mathfrak{g}_2} (\mathfrak{G}_{\boldsymbol{d}})_1 \xrightarrow{\mathfrak{g}_1} (\mathfrak{G}_{\boldsymbol{d}})_0,$$

with

$$(\mathfrak{G}_{\boldsymbol{d}})_0 = \bar{\mathfrak{P}},$$

$$(\mathfrak{G}_{\boldsymbol{d}})_1 = \bigoplus_{j=1}^{4} \bar{\mathfrak{P}}[-(\boldsymbol{d}+1,\boldsymbol{d},z_j)],$$

$$(\mathfrak{G}_{\boldsymbol{d}})_2 = \begin{cases} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}-1,\underline{1})]^{\boldsymbol{d}} \\ \oplus \bigoplus\limits_{1\le j<k\le 4} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d},z_j+z_k)]^{\boldsymbol{d}+1} \\ \oplus \bar{\mathfrak{P}}[-(2\boldsymbol{d}+1,2\boldsymbol{d}+1,\underline{0})]^{\boldsymbol{d}+1}, \end{cases}$$

$$(\mathfrak{G}_{\boldsymbol{d}})_i = \begin{cases} \bigoplus\limits_{1\le j<k<\ell\le 4} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+\frac{i-3}{2}+3,2\boldsymbol{d}+\frac{i-3}{2},z_j+z_k+z_\ell)]^{2\boldsymbol{d}+1} \\ \oplus \bigoplus_{1\le j\le 4} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+\frac{i-1}{2}+1,2\boldsymbol{d}+\frac{i-1}{2},z_j)]^{2\boldsymbol{d}+1}, \end{cases}$$

for i odd with $3\le i$, and

$$(\mathfrak{G}_{\boldsymbol{d}})_i = \begin{cases} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+\frac{i-4}{2}+4,2\boldsymbol{d}+\frac{i-4}{2},\underline{1})]^{2\boldsymbol{d}+1} \\ \oplus \bigoplus\limits_{1\le j<k\le 4} \bar{\mathfrak{P}}[-(2\boldsymbol{d}+\frac{i-2}{2}+2,2\boldsymbol{d}+\frac{i-2}{2},z_j+z_k)]^{2\boldsymbol{d}+1} \\ \oplus \bar{\mathfrak{P}}[-(2\boldsymbol{d}+\frac{i}{2},2\boldsymbol{d}+\frac{i}{2},\underline{0})]^{2\boldsymbol{d}+1}, \end{cases}$$

for i even with $4\le i$.

Proof. We saw in (5.3.1) and in the proof of Corollary 7.3 that the group homomorphism $\alpha_{\boldsymbol{n},\boldsymbol{r}}: \mathfrak{M} \to \mathbf{M}_{\boldsymbol{n}}$ carries the $\mathfrak{M}$-homogeneous twists in $\mathfrak{G}_{\boldsymbol{d}}$ to the $\mathbf{M}_{\boldsymbol{n}}$-homogeneous twists in $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ for all $(\boldsymbol{n},\boldsymbol{r})$, as described in (1.1.1). This happens even though in the proof of Corollary 7.3, we know the $\mathbf{M}_{\boldsymbol{n}}$-twists of $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$; but we do not yet know the $\mathfrak{M}$-homogeneous twists of $\mathfrak{G}_{\boldsymbol{d}}$. Nonetheless, it is an easy exercise to verify that the $\mathfrak{M}$-homogeneous twists for $\mathfrak{G}_d$ that are listed in the statement of Lemma 8.1 are carried by $\alpha_{\boldsymbol{n},\boldsymbol{r}}$ to the $\mathbf{M}_{\boldsymbol{n}}$-homogeneous twists for $G_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ which are listed in Theorem 4.1. It is clear that any given $\alpha_{\boldsymbol{n},\boldsymbol{r}}$ has a large kernel; but it is also clear that the only twist from any $\mathfrak{G}_{\boldsymbol{d}}$ which is in $\ker_{\boldsymbol{n},\boldsymbol{r}}$ for all pairs $(\boldsymbol{n},\boldsymbol{r})$ is zero.

Indeed, suppose m is a twist from some $\mathfrak{G}_{\boldsymbol{d}}$, with $m \in \ker_{\boldsymbol{n},\boldsymbol{r}}$ for all pairs $(\boldsymbol{n},\boldsymbol{r})$ which satisfy (1.1.1). Let $\mathfrak{M}\text{-}\deg m = (c_1,c_2,c_3,c_4,c_5,c_6)$. Apply Data 1.3 to see that

$$\bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = 0 \quad \text{in } \tfrac{\mathbb{Z}}{\boldsymbol{n}\mathbb{Z}}, \quad \text{for all } \boldsymbol{n},$$

and

$$\boldsymbol{r}c_1 + (\boldsymbol{n}-\boldsymbol{r})c_2 = 0 \text{ for all } (\boldsymbol{n},\boldsymbol{r}) \text{ satisfying (1.1.1).} \tag{8.1.1}$$

It is immediately clear that $c_3 = c_4 = c_5 = c_6 = 0$ in $\mathbb{Z}$. Now apply Lemma 8.2 to see that $c_1 = c_2$. It follows from (8.1.1) that $c_1 = c_2 = 0$ in $\mathbb{Z}$. $\square$

Lemma 8.2. *Adopt Data* 1.3. *Let $m \in \mathfrak{M}$ be a homogeneous twist which appears in $\mathfrak{G}_{\boldsymbol{d}}$, for some non-negative integer $\boldsymbol{d}$. The following statements hold.*

(a) *Either m is zero or there exists a non-zero homogeneous element $\theta \in \mathfrak{P}$ such that the $\mathfrak{M}$-degree of θ is equal to m.*
(b) *If the $\mathfrak{M}$-degree of m is equal $(c_1,c_2,c_3,c_4,c_5,c_6)$, then*

$$c_1 = c_2 + c_3 + c_4 + c_5 + c_6.$$

Proof. We prove (a) by induction on the position of the twist m in the resolution $\mathfrak{G}_{\boldsymbol{d}}$. The only twist that appears in $\mathfrak{G}_0 = \mathfrak{P}$ is zero.

If m appears in $(\mathfrak{G}_{\boldsymbol{d}})_1$, then the degree of m is equal to the degree of a minimal generator of the ideal $\mathfrak{C}_{\boldsymbol{d}}$.

Suppose m is a twist from $(\mathfrak{G}_{\boldsymbol{d}})_i$, for some i with $2 \le i$. In this case, m is the $\mathfrak{M}$-degree of some basis element e in $(\mathfrak{G}_{\boldsymbol{d}})_i$. The resolution $\mathfrak{G}_{\boldsymbol{d}}$ is a minimal $\mathfrak{M}$-homogeneous resolution; so e is not sent to zero. Thus, $\deg e = \deg p + m'$, for some non-zero homogeneous element $p \in \mathfrak{P}$ and some m' which appears as an $\mathfrak{M}$-homogeneous twist in $(\mathfrak{G}_{\boldsymbol{d}})_{i-1}$. By induction $\deg m' = \deg p'$ for some non-zero $\mathfrak{M}$-homogeneous element p' of $\mathfrak{P}$. Thus, pp' is a non-zero $\mathfrak{M}$-homogeneous element of $\mathfrak{P}$ with $\deg m = \deg pp'$. This completes the proof of (a).

We prove (b) by showing that each monomial in $\mathfrak{P}$ satisfies the equation. If $p = y_1^{a_1} y_2^{a_2} y_3^{a_3} y_4^{a_4} w_1^{b_1} w_2^{b_2} w_3^{b_3} w_4^{b_4}$ is a monomial in $\mathfrak{P}$ and $\deg p = (c_1,c_2,c_3,c_4,c_5,c_6)$, then according to Data 1.3,

$$c_1 = a_1+a_2+a_3+a_4,\ c_2 = b_1+b_2+b_3+b_3+b_4,\ c_3 = a_1-b_1,\ c_4 = a_2-b_2,$$
$$c_5 = a_3-b_3,\ c_6 = a_4-b_4,$$

and it is clear that $c_2+c_3+c_4+c_5+c_6 = c_1$. $\square$

Lemma 8.3. *Adopt the data of* 1.3. *Then the differentials*

$$(\mathfrak{G}_{\boldsymbol{d}})_5 \xrightarrow{\mathfrak{g}_5} (\mathfrak{G}_{\boldsymbol{d}})_4 \xrightarrow{\mathfrak{g}_4} (\mathfrak{G}_{\boldsymbol{d}})_3$$

have the form given in Table 5, *where each M_{ij} and each N_{ij} is an invertible*

$$(2\boldsymbol{d}+1)\times(2\boldsymbol{d}+1)$$

matrix of constants.

Proof. Decompose $(\mathfrak{G}_{\boldsymbol{d}})_3$ as

$$\begin{aligned}
&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d},0,1,1,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d},1,0,1,1)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,0,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,1,0)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,1,0,0,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,0,1,0,0)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,0,0,1,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,0,0,0,1)]^{2\boldsymbol{d}+1},
\end{aligned}$$

$(\mathfrak{G}_{\boldsymbol{d}})_4$ as

$$\begin{aligned}
&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+4,2\boldsymbol{d},1,1,1,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,1,1,0,0)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,1,0,1,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,1,0,0,1)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,0,0,1,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,0,1,0,1)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,0,1,1,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+2,2\boldsymbol{d}+2,0,0,0,0)]^{2\boldsymbol{d}+1},
\end{aligned}$$

and $(\mathfrak{G}_{\boldsymbol{d}})_5$ as

$$\begin{aligned}
&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+4,2\boldsymbol{d}+1,0,1,1,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+4,2\boldsymbol{d}+1,1,0,1,1)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+4,2\boldsymbol{d}+1,1,1,0,1)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+4,2\boldsymbol{d}+1,1,1,1,0)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+2,1,0,0,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+2,0,1,0,0)]^{2\boldsymbol{d}+1}\\
\oplus&\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+2,0,0,1,0)]^{2\boldsymbol{d}+1}\oplus\bar{\mathfrak{P}}[-(2\boldsymbol{d}+3,2\boldsymbol{d}+2,0,0,0,1)]^{2\boldsymbol{d}+1}.
\end{aligned}$$

The only possible non-zero $\mathfrak{M}$-homogeneous maps

$$\mathfrak{g}_5:(\mathfrak{G}_{\boldsymbol{d}})_5\to(\mathfrak{G}_{\boldsymbol{d}})_4 \quad\text{and}\quad \mathfrak{g}_4:(\mathfrak{G}_{\boldsymbol{d}})_4\to(\mathfrak{G}_{\boldsymbol{d}})_3$$

have the form of the matrices in Table 5, for some matrices of constants M_{ij} and N_{ij}. The matrices M_{ij} and N_{ij} are invertible for the same reason that the matrices of Corollary 4.2 are invertible. If any of the M_{ij} or N_{ij} were singular, then $\mathfrak{f}$ would be an element of the ideal generated by three of the eight variables $y_1,\dots,y_4,w_1,\dots,w_4$ and this is not possible. □

Remark 8.4. In light of Theorem 7.5, the matrix M_{ij} from Table 2 is equal to the matrix M_{ij} from Table 5 and the matrix N_{ij} from Table 2 is equal to the matrix N_{ij} from Table 5.

9. ORDER IDEALS OF SYZYGIES.

We use the theory of order ideals (see, for example, [12, page 397]) to distinguish the non-free indecomposable Maximal Cohen-Macaulay modules of the ring $\bar{\mathfrak{P}}$ of 1.3.

If E is a module over the commutative ring R, then $E^*=\operatorname{Hom}_R(E,R)$ represents the R-dual of E.

Definition 9.1. Let R be a ring, and let E be an R-module. For $e\in E$, the ideal

$$\mathscr{O}_E(e)=\{\phi(e)\mid\phi\in E^*\}$$

is the *order ideal* of e in the module E.

The differential $\mathfrak{g}_4 : (\mathfrak{G}_{\boldsymbol{d}})_4 \to (\mathfrak{G}_{\boldsymbol{d}})_3$ has the form

y_1M_{11}	0	0	0	w_2M_{15}	$-w_3M_{16}$	w_4M_{17}	0
$-y_2M_{21}$	0	w_4M_{23}	$-w_3M_{24}$	w_1M_{25}	0	0	0
y_3M_{31}	w_4M_{32}	0	$-w_2M_{34}$	0	w_1M_{36}	0	0
$-y_4M_{41}$	w_3M_{42}	$-w_2M_{43}$	0	0	0	w_1M_{47}	0
0	$-y_2M_{52}$	$-y_3M_{53}$	$-y_4M_{54}$	0	0	0	w_1M_{58}
0	y_1M_{62}	0	0	0	$-y_4M_{66}$	$-y_3M_{67}$	w_2M_{68}
0	0	y_1M_{73}	0	$-y_4M_{75}$	0	y_2M_{77}	w_3M_{78}
0	0	0	y_1M_{84}	y_3M_{85}	y_2M_{86}	0	w_4M_{88}

and the differential $\mathfrak{g}_5 : (\mathfrak{G}_{\boldsymbol{d}})_5 \to (\mathfrak{G}_{\boldsymbol{d}})_4$ has the form

w_1N_{11}	$-w_2N_{12}$	w_3N_{13}	$-w_4N_{14}$	0	0	0	0
0	0	y_4N_{23}	y_3N_{24}	$-w_2N_{25}$	w_1N_{26}	0	0
0	y_4N_{32}	0	$-y_2N_{34}$	$-w_3N_{35}$	0	w_1N_{37}	0
0	$-y_3N_{42}$	$-y_2N_{43}$	0	$-w_4N_{45}$	0	0	w_1N_{48}
y_2N_{51}	y_1N_{52}	0	0	0	0	$-w_4N_{57}$	w_3N_{58}
$-y_3N_{61}$	0	y_1N_{63}	0	0	$-w_4N_{66}$	0	w_2N_{68}
y_4N_{71}	0	0	y_1N_{74}	0	$-w_3N_{76}$	w_2N_{77}	0
0	0	0	0	y_1N_{85}	y_2N_{86}	y_3N_{87}	y_4N_{88}

TABLE 5. The differentials $(\mathfrak{G}_{\boldsymbol{d}})_5 \xrightarrow{\mathfrak{g}_5} (\mathfrak{G}_{\boldsymbol{d}})_4 \xrightarrow{\mathfrak{g}_4} (\mathfrak{G}_{\boldsymbol{d}})_3$, as described in Lemma 8.3. The matrices we have recorded have entries in $\mathfrak{P}$; the matrices $\mathfrak{g}_4$ and $\mathfrak{g}_5$ are the images of these matrices with entries in $\bar{\mathfrak{P}}$.

Notice that the notion of order ideal is intrinsic to the data $e \in E$. This notion has nothing to do with how E is presented. On the other hand, when possible, it is convenient to use information about the presentation of E to calculate order ideals.

Lemma 9.2. *If R is a ring and $\alpha : E \to F$ is an injective R-module homomorphism of finitely generated R-modules, with F free and $\alpha^* : F^* \to E^*$ surjective, then, for each element e of E, the order ideal $\mathscr{O}_E(e)$ is generated by the coordinates of $\alpha(e)$ with respect to any basis for F.*

Proof. Let $u_1, \dots, u_n$ be an arbitrary basis for F. Define $u_1^*, \dots, u_n^*$ in F^* by

$$u_i^*(u_j) = \begin{cases} 1, & \text{if } i = j, \text{ and} \\ 0, & \text{if } i \neq j. \end{cases}$$

Observe that $u_1^*, \dots, u_n^*$ generate F^*. The hypothesis that α^* is surjective ensures that if ϕ is an element in E^*, then there exists $\Phi \in F^*$ such that $\phi = \Phi \circ \alpha$. It follows that $u_1^* \circ \alpha, \dots, u_n^* \circ \alpha$ is a generating set for E^*. Consequently, $\mathscr{O}_E(e)$ is generated by $(u_1^* \circ \alpha)(e), \dots, (u_n^* \circ \alpha)(e)$; and therefore, $\mathscr{O}_E(e)$ is generated by the coordinates of $\alpha(e)$ with respect to the basis $u_1, \dots, u_n$ for E^*. $\square$

Observation 9.3. *Let R be a Gorenstein ring and*

$$\cdots \xrightarrow{f_0} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} \cdots$$

be a periodic exact sequence of finitely generated free R-modules. Then, for each element $e \in F_0$, the order ideal $\mathscr{O}_{\operatorname{Im} f_0} f_0(e)$ is generated by the coordinates of $f_0(e)$ with respect to any basis for F_1. In particular, if $F_0 = \oplus_j R e_{0,j}$, $F_1 = \oplus_j R e_{1,j}$, and the maps f_0 and f_1 are the corresponding matrices, then $\mathscr{O}_{\operatorname{Im}(f_0)} f_0(e_{0,j})$ is the ideal of R generated by the entries of column j of the matrix f_0 and $\mathscr{O}_{\operatorname{Im}(f_1)} f_1(e_{1,j})$ is the ideal of R generated by the entries of column j of the matrix f_1.

Remark. We view an element of F_1 as a column vector v. The homomorphism

$$f_1 : F_1 \to F_0$$

sends the column vector v to the column vector $f_1 v$, which is the product of the matrix f_1 and the column vector v.

Proof. The R-module $f_1(F_1)$ is an i^{th} syzygy module for all non-negative integers i; hence, $f_1(F_1)$ is an MCM R-module. It follows from local duality (and the fact that the canonical module of R is R) that $\operatorname{Ext}_R^j(f_1(F_1), R) = 0$ for all positive integers j; see, for example, [3, Cor. 3.5.11]. Apply $\operatorname{Hom}_R(-, R)$ to the short exact sequence

$$0 \to f_0(F_0) \xrightarrow{\text{incl}} F_1 \to f_1(F_1) \to 0$$

to obtain the exact sequence

$$F_1^* \xrightarrow{\text{incl}^*} (f_0(F_0))^* \to \operatorname{Ext}_R^1(f_1(F_1), R) = 0.$$

The assertion is now a special case of Lemma 9.2. $\square$

Definition 9.4. Retain the usual $\mathfrak{M}$-homogeneous $\boldsymbol{k}$-algebra $\bar{\mathfrak{P}}$ from 1.3. Let $\mathfrak{m}$ be the maximal $\mathfrak{M}$-homogeneous ideal of $\bar{\mathfrak{P}}$; so that $\bar{\mathfrak{P}}/\mathfrak{m} = \boldsymbol{k}$. If E is a finitely generated $\mathfrak{M}$-homogeneous $\bar{\mathfrak{P}}$-module, then define

$$\lambda(E) = \max\left\{ \dim_{\boldsymbol{k}}\left(\frac{E' + \mathfrak{m}E}{\mathfrak{m}E}\right) \,\middle|\, \begin{array}{l} E' \text{ is a } \bar{\mathfrak{P}}\text{-submodule of } E \text{ and} \\ e' \in E' \implies \mathscr{O}_E(e') \subseteq (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) \end{array} \right\}.$$

Observation 9.5. *Recall the matrices $\mathfrak{A}$ and $\mathfrak{B}$ from Table 4 on page 6 and the matrix $\mathfrak{g}_4$ from Table 5 on page 25. The following statements hold:*

(a) $\lambda(\operatorname{Im}\bar{\mathfrak{A}}) = 1$,
(b) $\lambda(\operatorname{Im}\bar{\mathfrak{B}}) = 0$,
(c) $\lambda(\bar{\mathfrak{P}}) = 0$,
(d) $\lambda((\operatorname{Im}\bar{\mathfrak{A}})^a \oplus (\operatorname{Im}\bar{\mathfrak{B}})^b \oplus \bar{\mathfrak{P}}^c) = a$, *and*
(e) $\lambda(\operatorname{Im}\mathfrak{g}_4) = 2\boldsymbol{d} + 1$.

Proof. We prove (a). Let the domain of $\bar{\mathfrak{A}}$ be called $\bigoplus_{i=1}^8 \bar{\mathfrak{P}} e_{0,i}$. Apply Observation 9.3 to see that $\mathscr{O}_{\operatorname{Im}\bar{\mathfrak{A}}}(\bar{\mathfrak{A}}(e_{0,1})) = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$; but

$$\mathscr{O}_{\operatorname{Im}\bar{\mathfrak{A}}}(\bar{\mathfrak{A}}(e)) \not\subseteq (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4),$$

for any minimal generator e of $\bigoplus_{i=2}^8 \bar{\mathfrak{P}} e_{0,i}$.

(b) In a similar manner, we see that

$$\mathscr{O}_{\operatorname{Im}\bar{\mathfrak{B}}}(\bar{\mathfrak{B}}(e)) \not\subseteq (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4),$$

for any $e \in \bar{\mathfrak{P}}^8 \setminus \mathfrak{m}(\bar{\mathfrak{P}}^8)$.

(c) If $e\in\bar{\mathfrak{P}}\setminus\mathfrak{m}\bar{\mathfrak{P}}$, then $\mathscr{O}_{\bar{\mathfrak{P}}}(e)=\bar{\mathfrak{P}}$, which is not contained in $(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4)$; hence, $\lambda(\bar{\mathfrak{P}})=0$.

(d) This assertion is a consequence of (a), (b), and (c).

(e) Let the domain of $\mathfrak{g}_4$ be $\bigoplus\limits_{i=1}^{16\boldsymbol{d}+8}\bar{\mathfrak{P}}e_i$ and let $W=\bigoplus\limits_{i=2d+2}^{16\boldsymbol{d}+8}\bar{\mathfrak{P}}e_i$. Apply Observation 9.3 to see that

$$\begin{aligned}&\mathscr{O}_{\mathrm{Im}(\mathfrak{g}_4)}(\mathfrak{g}_4(e_i))=(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4), &&\text{for } 1\le i\le 2\boldsymbol{d}+1\text{; but}\\ &\mathscr{O}_{\mathrm{Im}(\mathfrak{g}_4)}(\mathfrak{g}_4(e))\not\subseteq(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4), &&\text{for any } e\in W\setminus\mathfrak{m}W.\end{aligned}$$

(Keep in mind that each matrix M_{ij} from Table 5 is invertible.) We conclude that $\lambda(\mathrm{Im}\,\mathfrak{g}_4)=2\boldsymbol{d}+1$. $\square$

10. THE MAIN RESULT.

Corollary 10.2 is the main result of the paper. It establishes Conjecture 1.2.

Theorem 10.1. *Let $\boldsymbol{k}$ be a field of characteristic zero, $\boldsymbol{d}$ be a positive integer, $\mathfrak{P}$ be the polynomial ring*

$$\mathfrak{P}=\boldsymbol{k}[y_1,y_2,y_3,y_4,w_1,w_2,w_3,w_4],$$

$\mathfrak{f}$ be the polynomial

$$\mathfrak{f}=y_1w_1+y_2w_2+y_3w_3+y_4w_4$$

in $\mathfrak{P}$, $\bar{\mathfrak{P}}$ be the hypersurface ring $\mathfrak{P}/(\mathfrak{f})$, $\mathfrak{C}_{\boldsymbol{d}}$ be the ideal

$$\mathfrak{C}_{\boldsymbol{d}}=(y_1^{\boldsymbol{d}+1}w_1^{\boldsymbol{d}},\,y_2^{\boldsymbol{d}+1}w_2^{\boldsymbol{d}},\,y_3^{\boldsymbol{d}+1}w_3^{\boldsymbol{d}},\,y_4^{\boldsymbol{d}+1}w_4^{\boldsymbol{d}})$$

of $\mathfrak{P}$, $\mathfrak{Q}_{\boldsymbol{d}}$ be the quotient ring

$$\mathfrak{Q}_{\boldsymbol{d}}=\bar{\mathfrak{P}}/\mathfrak{C}_{\boldsymbol{d}}\bar{\mathfrak{P}},$$

and $\mathfrak{S}_{\boldsymbol{d}}^3$ be the third syzygy of $\mathfrak{Q}_{\boldsymbol{d}}$ as a $\bar{\mathfrak{P}}$-module. Then

$$\mathfrak{S}_{\boldsymbol{d}}^3\cong(\mathrm{Im}\,\mathfrak{B})^{2\boldsymbol{d}+1},$$

where $\mathfrak{B}$ is given in Table 4 *on page* 6.

Proof. Recall from Theorem 6.2 that the ideal $\mathfrak{I}_{\boldsymbol{d}}=(\mathfrak{C}_{\boldsymbol{d}},\mathfrak{f})$ of $\mathfrak{P}$ is perfect ideal of grade 4; hence $\mathfrak{P}/\mathfrak{I}_{\boldsymbol{d}}$, which is equal to $\mathfrak{Q}_{\boldsymbol{d}}$, is a Cohen-Macaulay ring of dimension 4. In particular, if $3\le i$, then the i^{th} syzygy of $\mathfrak{Q}_{\boldsymbol{d}}$ as a $\bar{\mathfrak{P}}$-module (denoted $\mathfrak{S}_{\boldsymbol{d}}^i$) has depth 7 and therefore is a maximal Cohen-Macaulay $\bar{\mathfrak{P}}$-module.

It is shown in [6, Prop. 3.1] that there are at most two isomorphism classes of non-free indecomposable maximal Cohen-Macaulay (MCM) $\bar{\mathfrak{P}}$-modules. Indeed, it is observed in [7, Remark 2.5.4] that there are exactly two non-isomorphic, indecomposable, non-free MCM $\bar{\mathfrak{P}}$-modules and these modules have rank 4 as $\bar{\mathfrak{P}}$-modules. (The ring $\bar{\mathfrak{P}}$ is a domain; see for example, [15, Prop. 22]. The rank of a $\bar{\mathfrak{P}}$-module M is the vector space dimension of $K\otimes_{\bar{\mathfrak{P}}}M$, where K is the quotient field of $\bar{\mathfrak{P}}$.) In particular, these MCM $\bar{\mathfrak{P}}$-modules are $\mathrm{Im}\,\bar{\mathfrak{B}}$ and $\mathrm{Im}\,\bar{\mathfrak{A}}$, where $\mathfrak{B}$ and $\mathfrak{A}$ are 8×8 matrices with entries from $\mathfrak{P}$ with $\mathfrak{A}\mathfrak{B}=\mathfrak{f}I_8$.

At any rate,

$$\mathfrak{f}I_8=\mathfrak{A}\mathfrak{B}\quad\text{and}\quad\mathfrak{f}I_8=\mathfrak{B}\mathfrak{A},$$

for $\mathfrak{A}$ and $\mathfrak{B}$ as given in Table 4 on page 6, and every non-free indecomposable MCM $\bar{\mathfrak{P}}$-module is isomorphic to $\operatorname{Im}\bar{\mathfrak{A}}$ or $\operatorname{Im}\bar{\mathfrak{B}}$. The fourth syzygy, $\mathfrak{S}^4_{\boldsymbol{d}}$, of the $\bar{\mathfrak{P}}$-module $\mathfrak{Q}_d$ is a MCM $\bar{\mathfrak{P}}$-module; consequently,

$$\mathfrak{S}^4_{\boldsymbol{d}} \cong (\operatorname{Im}\bar{\mathfrak{A}})^a \oplus (\operatorname{Im}\bar{\mathfrak{B}})^b \oplus \bar{\mathfrak{P}}^c,$$

for some non-negative integers a, b, and c. Notice that the minimal number of generators of $\mathfrak{S}^4_{\boldsymbol{d}}$ is $8a+8b+c$.

On the other hand, it is shown in Lemma 8.3 that $\mathfrak{S}^4_{\boldsymbol{d}}$ is equal to $\operatorname{Im}\mathfrak{g}_4$. In particular, the minimal number of generators of $\mathfrak{S}^4_{\boldsymbol{d}}$ is $8(2\boldsymbol{d}+1)$. Apply Observation 9.5.(d) and (e) to see that

$$a = \lambda((\operatorname{Im}\bar{\mathfrak{A}})^a \oplus (\operatorname{Im}\bar{\mathfrak{B}})^b \oplus \bar{\mathfrak{P}}^c) = \lambda(\mathfrak{S}^4_{\boldsymbol{d}}) = \lambda(\operatorname{Im}\mathfrak{g}_4) = 2\boldsymbol{d}+1,$$

with

$$8a+8b+c = 8(2\boldsymbol{d}+1)$$

to conclude that $a = 2\boldsymbol{d}+1$ and $b = c = 0$. Thus,

$$\mathfrak{S}^4_{\boldsymbol{d}} = (\operatorname{Im}\bar{\mathfrak{A}})^{2\boldsymbol{d}+1}. \tag{10.1.1}$$

We apply standard tricks involving MCM modules to show that in fact

$$\mathfrak{S}^3_{\boldsymbol{d}} \cong (\operatorname{Im}\bar{\mathfrak{B}})^{2\boldsymbol{d}+1}.$$

Indeed, the $\bar{\mathfrak{P}}$-dual of the short exact sequence

$$0 \to \mathfrak{S}^4_{\boldsymbol{d}} \to \mathfrak{G}_4 \to \mathfrak{S}^3_{\boldsymbol{d}} \to 0$$

is

$$0 \to (\mathfrak{S}^3_{\boldsymbol{d}})^* \to (\mathfrak{G}_4)^* \to (\mathfrak{S}^4_{\boldsymbol{d}})^* \to 0 \tag{10.1.2}$$

because $\operatorname{Ext}^1_{\bar{\mathfrak{P}}}(\mathfrak{S}^3_{\boldsymbol{d}}, \bar{\mathfrak{P}}) = 0$ by local duality. But,

$$(\mathfrak{S}^4_{\boldsymbol{d}})^* \cong ((\operatorname{Im}\bar{\mathfrak{A}})^{2\boldsymbol{d}+1})^* \cong (\operatorname{Im}\bar{\mathfrak{B}})^{2\boldsymbol{d}+1}.$$

(The equality on the left is (10.1.1) and the equality on the right is obvious.) The complex (10.1.2) is $\mathfrak{M}$-homogeneous and the minimal first syzygy of $\operatorname{Im}\bar{\mathfrak{B}}$ is $\operatorname{Im}\bar{\mathfrak{A}}$. Thus, $(\mathfrak{S}^3_{\boldsymbol{d}})^* \cong (\operatorname{Im}\bar{\mathfrak{A}})^{2\boldsymbol{d}+1}$. The MCM $\bar{\mathfrak{P}}$-module $\mathfrak{S}^3_{\boldsymbol{d}}$ is reflexive; thus:

$$\mathfrak{S}^3_{\boldsymbol{d}} \cong (\mathfrak{S}^3_{\boldsymbol{d}})^{**} \cong ((\operatorname{Im}\bar{\mathfrak{A}})^{2\boldsymbol{d}+1})^* \cong ((\operatorname{Im}\bar{\mathfrak{A}})^*)^{2\boldsymbol{d}+1} = \operatorname{Im}(\bar{\mathfrak{B}})^{2\boldsymbol{d}+1}.$$

□

Corollary 10.2. *Let $\boldsymbol{k}$ be a field of characteristic zero, and $\boldsymbol{n}$, $\boldsymbol{d}$, and $\boldsymbol{r}$ be non-negative integers with $1 \le \boldsymbol{r} \le \boldsymbol{n}-1$. Let N be the integer $\boldsymbol{dn}+\boldsymbol{r}$, P be the polynomial ring $\boldsymbol{k}[x_1,x_2,x_3,x_4]$, $f_{\boldsymbol{n}}$ be the polynomial $x_1^{\boldsymbol{n}}+x_2^{\boldsymbol{n}}+x_3^{\boldsymbol{n}}+x_4^{\boldsymbol{n}}$ in P, $C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the ideal $(x_1^N,x_2^N,x_3^N,x_4^N)$ of P, $\bar{P}_{\boldsymbol{n}}$ be the hypersurface ring $P/(f_{\boldsymbol{n}})$, $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the quotient ring $\bar{P}_{\boldsymbol{n}}/C_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}\bar{P}_{\boldsymbol{n}}$ and $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ be the third syzygy module of $Q_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ as a $\bar{P}_{\boldsymbol{n}}$-module. Then $\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ is isomorphic to the direct sum of $2\boldsymbol{d}+1$ copies of $\Omega^3_{0,\boldsymbol{n},\boldsymbol{r}}$.*

Proof. Consider the rings $\bar{\mathfrak{P}}$ and $\mathfrak{Q}_{\boldsymbol{d}} = \bar{\mathfrak{P}}/\mathfrak{C}_{\boldsymbol{d}}\bar{\mathfrak{P}}$ of Theorem 10.1. Let $(\mathfrak{G}_{\boldsymbol{d}}, \mathfrak{g})$ be the minimal $\mathfrak{M}$-homogeneous resolution of $\mathfrak{Q}_{\boldsymbol{d}}$ by free $\bar{\mathfrak{P}}$-modules. We learned in Theorem 10.1 that $\operatorname{Im}(\mathfrak{g}_3) \cong (\operatorname{Im}\bar{\mathfrak{B}})^{2\boldsymbol{d}+1}$. We learned in Theorem 7.5 that $\mathfrak{G}_{\boldsymbol{d}} \otimes_{\bar{\mathfrak{P}}} P$ is the minimal $\mathbf{M}_{\boldsymbol{n}}$ homogeneous resolution of $P_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ by free $\bar{P}_{\boldsymbol{n}}$-modules. Thus,

$$\Omega^3_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}} \cong \operatorname{Im}(\mathfrak{g}_3 \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}) \cong (\operatorname{Im}(\bar{\mathfrak{B}} \otimes_{\bar{\mathfrak{P}}} \bar{P}_{\boldsymbol{n}}))^{2\boldsymbol{d}+1} = (\operatorname{Im}\bar{B})^{2\boldsymbol{d}+1}.$$

The final equality holds because $\Delta_{\boldsymbol{n},\boldsymbol{r}}(\bar{\mathfrak{B}}) = \bar{B}$. □

11. THE CASE $\boldsymbol{d} = 0$.

This brief section was promised in the introduction. We include enough information to demonstrate that if A is the matrix of Table 1, on page 3, then $\bar{A}$ presents the third syzygy of the $\bar{P}_{\boldsymbol{n}}$-module $Q_{0,\boldsymbol{n},\boldsymbol{r}}$, in the language of (1.1.2).

The two-step Tate resolution which appears in Observation 11.1 is well-known; see, for example, [29, Thm. 4], [23], or [11].

Observation 11.1. *Adopt Data* 1.1 *with* $\boldsymbol{d} = 0$ *and* $\boldsymbol{k}$ *an arbitrary field. Let E and T be vector spaces over* $\boldsymbol{k}$ *of dimension* 4 *and* 1*, respectively. Let* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *be a basis for E and t be a basis for T. Define* $\boldsymbol{k}$*-module maps*

$$\partial : E \to P \quad \textit{and} \quad \partial : T \to P \otimes_{\boldsymbol{k}} E$$

by $\partial(\varepsilon_i) = x_i^{\boldsymbol{r}}$ *and* $\partial(t) = \sum_{i=1}^{4} x_i^{\boldsymbol{n}-\boldsymbol{r}}\varepsilon_i$*. Then the minimal homogeneous resolution of Q by free $\bar{P}$-modules is given by* $G_{0,\boldsymbol{n},\boldsymbol{r}} = \bar{P}\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, t; \partial\rangle$*, where* $G_{0,\boldsymbol{n},\boldsymbol{r}}$ *is the free Differential Graded $\bar{P}$-algebra with variables* $\{\varepsilon_i\}$ *of degree one and t of degree two and ∂ is the differential on* $G_{0,\boldsymbol{n},\boldsymbol{r}}$*. In other words,* $(G_{0,\boldsymbol{n},\boldsymbol{r}})_j$ *is*

$$\begin{array}{ll}
\bar{P}, & \textit{if } j = 0,\\
\bar{P} \otimes E, & \textit{if } j = 1,\\
(\bar{P} \otimes_{\boldsymbol{k}} D_0 T \otimes_{\boldsymbol{k}} \textstyle\bigwedge^2 E) \oplus (\bar{P} \otimes_{\boldsymbol{k}} D_1 T \otimes_{\boldsymbol{k}} \textstyle\bigwedge^0 E), & \textit{if } j = 2,\\
(\bar{P} \otimes_{\boldsymbol{k}} D_{i-1} T \otimes_{\boldsymbol{k}} \textstyle\bigwedge^3 E) \oplus (\bar{P} \otimes_{\boldsymbol{k}} D_i T \otimes_{\boldsymbol{k}} \textstyle\bigwedge^1 E), & \textit{if } 3 \le j \textit{ and } j = 2i+1,
\end{array}$$

and

$$\begin{cases} (\bar{P} \otimes_{\boldsymbol{k}} D_{i-2} T \otimes_{\boldsymbol{k}} \bigwedge^4 E) \oplus (\bar{P} \otimes_{\boldsymbol{k}} D_{i-1} T \otimes_{\boldsymbol{k}} \bigwedge^2 E) \\ \oplus (\bar{P} \otimes_{\boldsymbol{k}} D_i T \otimes_{\boldsymbol{k}} \bigwedge^0 E), \end{cases} \qquad \textit{if } 4 \le j \textit{ and } j = 2i.$$

Use the ordered bases

$$t^{(i-1)}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4,\ t^{(i)}\varepsilon_1\varepsilon_2,\ t^{(i)}\varepsilon_1\varepsilon_3,\ t^{(i)}\varepsilon_1\varepsilon_4,\ t^{(i)}\varepsilon_3\varepsilon_4,$$
$$t^{(i)}\varepsilon_2\varepsilon_4,\ t^{(i)}\varepsilon_2\varepsilon_3,\ t^{(i+1)}$$

for $(G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i+2}$*;*

$$t^{(i-1)}\varepsilon_2\varepsilon_3\varepsilon_4,\ t^{(i-1)}\varepsilon_1\varepsilon_3\varepsilon_4,\ t^{(i-1)}\varepsilon_1\varepsilon_2\varepsilon_4,\ t^{(i-1)}\varepsilon_1\varepsilon_2\varepsilon_3,\ t^{(i)}\varepsilon_1,$$
$$t^{(i)}\varepsilon_2,\ t^{(i)}\varepsilon_3,\ t^{(i)}\varepsilon_4$$

for $(G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i+1}$*; and*

$$t^{(i-2)}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4,\ t^{(i-1)}\varepsilon_1\varepsilon_2,\ t^{(i-1)}\varepsilon_1\varepsilon_3,\ t^{(i-1)}\varepsilon_1\varepsilon_4,\ t^{(i-1)}\varepsilon_3\varepsilon_4,$$
$$t^{(i-1)}\varepsilon_2\varepsilon_4,\ t^{(i-1)}\varepsilon_2\varepsilon_3,\ t^{(i)}$$

for $(G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i}$*. The matrix for* $(G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i+2} \to (G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i+1}$*, when* $1 \le i$*, is* $\bar{A}$*, for* A *as given in Table* 1*; and the matrix* $(G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i+1} \to (G_{0,\boldsymbol{n},\boldsymbol{r}})_{2i}$*, for* $2 \le i$*, is* $\bar{B}$*, for* B *as given in Table* 1*.*

Remark. The matrix for the differential $\bar{g}_3 : (G_{0,\boldsymbol{n},\boldsymbol{r}})_3 \to (G_{0,\boldsymbol{n},\boldsymbol{r}})_2$ is obtained from B by deleting row 1; nonetheless, the $\bar{P}_{\boldsymbol{n}}$-modules $\operatorname{Im}\bar{B}$ and $\operatorname{Im}\bar{g}_3$ are isomorphic as the commutative diagram with exact rows

$$\begin{tikzcd}
\bar{P}_{\boldsymbol{n}}^8 \arrow[r, "\bar{A}"] \arrow[d, equal] & \bar{P}_{\boldsymbol{n}}^8 \arrow[r, "\bar{B}"] \arrow[d, equal] & \operatorname{Im}\bar{B} \arrow[r] \arrow[d, "\pi"] & 0 \\
\bar{P}_{\boldsymbol{n}}^8 \arrow[r, "\bar{A}"] & \bar{P}_{\boldsymbol{n}}^8 \arrow[r, "\bar{g}_3"] & \operatorname{Im}\bar{g}_3 \arrow[r] & 0
\end{tikzcd}$$

demonstrates, where π is induced by the homomorphism $\bar{P}_{\boldsymbol{n}}^8 \to \bar{P}_{\boldsymbol{n}}^7$ which deletes the top entry.

12. THE MATRICES FROM $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ AND $\mathfrak{L}_{\boldsymbol{d}}$ FROM THEOREMS 6.1 AND 6.2.

The complexes $L_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}}$ and $\mathfrak{L}_{\boldsymbol{d}}$ are described in Theorems 6.1 and 6.2. In this section, we give the precise form of the differentials which appear in these complexes.

12.1. The key to the notation in Tables 6–20. For each three tuple of parameters (a,b,c), let $t^{a,b,c} = (f_{i,j})$ be a $b \times c$ matrix. The entry f_{ij} in row i and column j of $t^{a,b,c}$ is a homogeneous polynomial of degree a in the polynomial ring $\boldsymbol{k}[\xi_1,\xi_2,\xi_3,\xi_4]$. We use $T^{a,b,c}$ to denote the matrix $(f_{i,j}(x_1^{\boldsymbol{n}},x_2^{\boldsymbol{n}},x_3^{\boldsymbol{n}},x_4^{\boldsymbol{n}}))$ and $\mathfrak{T}^{a,b,c}$ to denote the matrix $(f_{i,j}(y_1w_1,y_2w_2,y_3w_3,y_4w_4))$. (In other words, the entry of $t^{a,b,c}$ in position i,j is the polynomial $f_{i,j}$ evaluated at the (place-holder) variables ξ_1, ξ_2, ξ_3, ξ_4; the entry of $T^{a,b,c}$ in position i,j is the polynomial $f_{i,j}$ evaluated at $x_1^{\boldsymbol{n}}$, $x_2^{\boldsymbol{n}}$, $x_3^{\boldsymbol{n}}$, $x_4^{\boldsymbol{n}}$; and the entry of $\mathfrak{T}^{a,b,c}$ in position i,j is the polynomial $f_{i,j}$ evaluated at y_1w_1, y_2w_2, y_3w_3, y_4w_4.) We use subscripts to distinguish the various matrices. In particular, the matrix $T^{a,b,c}_{\pi,\sigma,\tau}$ appears in row σ and column τ of the matrix for the differential in position π.

Notice that we make no claim about the polynomial $f_{i,j}(\xi_1,\xi_2,\xi_3,\xi_4)$ that appears in row i and column j of the matrix $t^{a,b,c}$ other than the fact that it is homogeneous of degree a.

	$P(-m_{(1,0,0,0,0)})^1$	$P(-m_{(\boldsymbol{d},\boldsymbol{r},0,0,0)})^1$	$P(-m_{(\boldsymbol{d},0,\boldsymbol{r},0,0)})^1$	$P(-m_{(\boldsymbol{d},0,0,\boldsymbol{r},0)})^1$	$P(-m_{(\boldsymbol{d},0,0,0,\boldsymbol{r})})^1$
P	$T_{1,1,1}^{1,1,1}$	$x_1^{\boldsymbol{r}} T_{1,1,2}^{\boldsymbol{d},1,1}$	$x_2^{\boldsymbol{r}} T_{1,1,3}^{\boldsymbol{d},1,1}$	$x_3^{\boldsymbol{r}} T_{1,1,4}^{\boldsymbol{d},1,1}$	$x_4^{\boldsymbol{r}} T_{1,1,5}^{\boldsymbol{d},1,1}$

TABLE 6. The matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_1$.

	1 $P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_1)})$	2 $P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_2)})$	3 $P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_3)})$	4 $P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_4)})$	5 $P(-m_{(2\boldsymbol{d}-1,\underline{\boldsymbol{r}})})^{\boldsymbol{d}}$	6 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_2)})^{\boldsymbol{d}+1}$
$P(-m_{(1,\underline{0})})^1$	$x_1^{\boldsymbol{r}} T_{2,1,1}^{\boldsymbol{d},1,1}$	$x_2^{\boldsymbol{r}} T_{2,1,2}^{\boldsymbol{d},1,1}$	$x_3^{\boldsymbol{r}} T_{2,1,3}^{\boldsymbol{d},1,1}$	$x_4^{\boldsymbol{r}} T_{2,1,4}^{\boldsymbol{d},1,1}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} x_3^{\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,1,5}^{2\boldsymbol{d}-2,1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} T_{2,1,6}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$
$P(-m_{(\boldsymbol{d},\boldsymbol{r}z_1)})^1$	$T_{2,2,1}^{1,1,1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}} x_2^{\boldsymbol{r}} T_{2,2,2}^{0,1,1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}} x_3^{\boldsymbol{r}} T_{2,2,3}^{0,1,1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,2,4}^{0,1,1}$	$x_2^{\boldsymbol{r}} x_3^{\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,2,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$x_2^{\boldsymbol{r}} T_{2,2,6}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
$P(-m_{(\boldsymbol{d},\boldsymbol{r}z_2)})^1$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{n}-\boldsymbol{r}} T_{2,3,1}^{0,1,1}$	$T_{2,3,2}^{1,1,1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}} x_3^{\boldsymbol{r}} T_{2,3,3}^{0,1,1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,3,4}^{0,1,1}$	$x_1^{\boldsymbol{r}} x_3^{\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,3,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}} T_{2,3,6}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
$P(-m_{(\boldsymbol{d},\boldsymbol{r}z_3)})^1$	$x_1^{\boldsymbol{r}} x_3^{\boldsymbol{n}-\boldsymbol{r}} T_{2,4,1}^{0,1,1}$	$x_2^{\boldsymbol{r}} x_3^{\boldsymbol{n}-\boldsymbol{r}} T_{2,4,2}^{0,1,1}$	$T_{2,4,3}^{1,1,1}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,4,4}^{0,1,1}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} x_4^{\boldsymbol{r}} T_{2,4,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} x_3^{\boldsymbol{n}-\boldsymbol{r}} T_{2,4,6}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$
$P(-m_{(\boldsymbol{d},\boldsymbol{r}z_4)})^1$	$x_1^{\boldsymbol{r}} x_4^{\boldsymbol{n}-\boldsymbol{r}} T_{2,5,1}^{0,1,1}$	$x_2^{\boldsymbol{r}} x_4^{\boldsymbol{n}-\boldsymbol{r}} T_{2,5,2}^{0,1,1}$	$x_3^{\boldsymbol{r}} x_4^{\boldsymbol{n}-\boldsymbol{r}} T_{2,5,3}^{0,1,1}$	$T_{2,5,4}^{1,1,1}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} x_3^{\boldsymbol{r}} T_{2,5,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}} x_2^{\boldsymbol{r}} x_4^{\boldsymbol{n}-\boldsymbol{r}} T_{2,5,6}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$

TABLE 7. The left side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2$.

	7 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_3)})^{\boldsymbol{d}+1}$	8 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	9 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_2+\boldsymbol{r}z_3)})^{\boldsymbol{d}+1}$	10 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_2+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	11 $P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_3+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	12 $P(-m_{(2\boldsymbol{d}+1,\underline{0})})^{\boldsymbol{d}+1}$
1	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{2,1,7}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,1,8}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{2,1,9}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,1,10}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,1,11}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$T_{2,1,12}^{2\boldsymbol{d},1,\boldsymbol{d}+1}$
2	$x_3^{\boldsymbol{r}}T_{2,2,7}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_4^{\boldsymbol{r}}T_{2,2,8}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{2,2,9}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_2^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,2,10}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,2,11}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{2,2,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
3	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{2,3,7}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,3,8}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_3^{\boldsymbol{r}}T_{2,3,9}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_4^{\boldsymbol{r}}T_{2,3,10}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,3,11}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{2,3,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
4	$x_1^{\boldsymbol{r}}T_{2,4,7}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,4,8}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}T_{2,4,9}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{2,4,10}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_4^{\boldsymbol{r}}T_{2,4,11}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{2,4,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
5	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{2,5,7}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}T_{2,5,8}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{2,5,9}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}T_{2,5,10}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_3^{\boldsymbol{r}}T_{2,5,11}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{2,5,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$

TABLE 8. The right side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_2$.

		1 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_1)})^{2\boldsymbol{d}+1}$	2 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_2)})^{2\boldsymbol{d}+1}$	3 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_3)})^{2\boldsymbol{d}+1}$	4 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_4)})^{2\boldsymbol{d}+1}$
1	$P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_1)})$	$T^{\boldsymbol{d},1,2\boldsymbol{d}+1}_{3,1,1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_2^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,2}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,3}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,4}$
2	$P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_2)})$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,1}$	$T^{\boldsymbol{d},1,2\boldsymbol{d}+1}_{3,2,2}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,3}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,4}$
3	$P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_3)})$	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,1}$	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,2}$	$T^{\boldsymbol{d},1,2\boldsymbol{d}+1}_{3,3,3}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,4}$
4	$P(-m_{(\boldsymbol{d}+1,\boldsymbol{r}z_4)})$	$x_1^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,1}$	$x_2^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,2}$	$x_3^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,3}$	$T^{\boldsymbol{d},1,2\boldsymbol{d}+1}_{3,4,4}$
5	$P(-m_{(2\boldsymbol{d}-1,\underline{\boldsymbol{r}})})^{\boldsymbol{d}}$	0	0	0	0
6	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_2)})^{\boldsymbol{d}+1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,6,1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,6,2}$	0	0
7	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_3)})^{\boldsymbol{d}+1}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,7,1}$	0	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,7,3}$	0
8	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_1+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,8,1}$	0	0	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,8,4}$
9	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_2+\boldsymbol{r}z_3)})^{\boldsymbol{d}+1}$	0	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,9,2}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,9,3}$	0
10	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_2+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	0	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,10,2}$	0	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,10,4}$
11	$P(-m_{(2\boldsymbol{d},\boldsymbol{r}z_3+\boldsymbol{r}z_4)})^{\boldsymbol{d}+1}$	0	0	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,11,3}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,11,4}$
12	$P(-m_{(2\boldsymbol{d}+1,\underline{0})})^{\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,12,1}$	$x_2^{\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,12,2}$	$x_3^{\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,12,3}$	$x_4^{\boldsymbol{r}}T^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,12,4}$

TABLE 9. The left side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3$.

	5 $P(-m_{(2\boldsymbol{d},0,\boldsymbol{r},\boldsymbol{r})})^{2\boldsymbol{d}+1}$	6 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},0,\boldsymbol{r})})^{2\boldsymbol{d}+1}$	7 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},0)})^{2\boldsymbol{d}+1}$	8 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},0)})^{2\boldsymbol{d}+1}$
1	$x_1^{\boldsymbol{n}-\boldsymbol{r}}x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,1,5}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}$	$x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,1,6}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,1,7}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{3,1,8}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
2	$x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,2,5}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{n}-\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,2,6}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,2,7}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{3,2,8}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
3	$x_2^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,3,5}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,3,6}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{r}}x_3^{\boldsymbol{n}-\boldsymbol{r}}x_4^{\boldsymbol{r}}T_{3,3,7}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{r}}T_{3,3,8}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
4	$x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{3,4,5}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_3^{\boldsymbol{r}}T_{3,4,6}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{r}}T_{3,4,7}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}x_2^{\boldsymbol{r}}x_3^{\boldsymbol{r}}x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{3,4,8}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}$
5	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{3,5,5}^{0,\boldsymbol{d},2\boldsymbol{d}+1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{3,5,6}^{0,\boldsymbol{d},2\boldsymbol{d}+1}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{3,5,7}^{0,\boldsymbol{d},2\boldsymbol{d}+1}$	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{3,5,8}^{0,\boldsymbol{d},2\boldsymbol{d}+1}$
6	0	0	$x_4^{\boldsymbol{r}}T_{3,6,7}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$x_3^{\boldsymbol{r}}T_{3,6,8}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
7	0	$x_4^{\boldsymbol{r}}T_{3,7,6}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	$x_2^{\boldsymbol{r}}T_{3,7,8}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
8	0	$x_3^{\boldsymbol{r}}T_{3,8,6}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$x_2^{\boldsymbol{r}}T_{3,8,7}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0
9	$x_4^{\boldsymbol{r}}T_{3,9,5}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	0	$x_1^{\boldsymbol{r}}T_{3,9,8}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
10	$x_3^{\boldsymbol{r}}T_{3,10,5}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	$x_1^{\boldsymbol{r}}T_{3,10,7}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0
11	$x_2^{\boldsymbol{r}}T_{3,11,5}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{r}}T_{3,11,6}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	0
12	0	0	0	0

TABLE 10. The right side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_3$.

	1 $P(-m_{(2\boldsymbol{d}+2,\underline{0})})^{\boldsymbol{d}}$	2 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r},\boldsymbol{r},0,0)})^{\boldsymbol{d}}$	3 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,\boldsymbol{r},0)})^{\boldsymbol{d}}$	4 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r},0,0,\boldsymbol{r})})^{\boldsymbol{d}}$
1 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_1)})^{2\boldsymbol{d}+1}$	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{4,1,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_2^{\boldsymbol{r}}T_{4,1,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_3^{\boldsymbol{r}}T_{4,1,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_4^{\boldsymbol{r}}T_{4,1,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
2 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_2)})^{2\boldsymbol{d}+1}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{4,2,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}}T_{4,2,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0
3 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_3)})^{2\boldsymbol{d}+1}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{4,3,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$x_1^{\boldsymbol{r}}T_{4,3,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
4 $P(-m_{(2\boldsymbol{d}+1,\boldsymbol{r}z_4)})^{2\boldsymbol{d}+1}$	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{4,4,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0	$x_1^{\boldsymbol{r}}T_{4,4,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
5 $P(-m_{(2\boldsymbol{d},0,\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})})^{2\boldsymbol{d}+1}$	0	0	0	0
6 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},0,\boldsymbol{r},\boldsymbol{r})})^{2\boldsymbol{d}+1}$	0	0	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{4,6,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{4,6,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
7 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},0,\boldsymbol{r})})^{2\boldsymbol{d}+1}$	0	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{4,7,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{4,7,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
8 $P(-m_{(2\boldsymbol{d},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},0)})^{2\boldsymbol{d}+1}$	0	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{4,8,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{4,8,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0

TABLE 11. The left side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4$.

	5 $P(-m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},\boldsymbol{r},0)})^{\boldsymbol{d}}$	6 $P(-m_{(2\boldsymbol{d}+1,0,\boldsymbol{r},0,\boldsymbol{r})})^{\boldsymbol{d}}$	7 $P(-m_{(2\boldsymbol{d}+1,0,0,\boldsymbol{r},\boldsymbol{r})})^{\boldsymbol{d}}$	8 $P(-m_{(2\boldsymbol{d},\underline{\boldsymbol{r}})})^{\boldsymbol{d}+1}$
1	0	0	0	0
2	$x_3^{\boldsymbol{r}}T_{4,2,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_4^{\boldsymbol{r}}T_{4,2,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0
3	$x_2^{\boldsymbol{r}}T_{4,3,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$x_4^{\boldsymbol{r}}T_{4,3,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
4	0	$x_2^{\boldsymbol{r}}T_{4,4,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_3^{\boldsymbol{r}}T_{4,4,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
5	$x_4^{\boldsymbol{n}-\boldsymbol{r}}T_{4,5,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_3^{\boldsymbol{n}-\boldsymbol{r}}T_{4,5,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_2^{\boldsymbol{n}-\boldsymbol{r}}T_{4,5,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_1^{\boldsymbol{r}}T_{4,5,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
6	0	0	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{4,6,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$x_2^{\boldsymbol{r}}T_{4,6,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
7	0	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{4,7,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$x_3^{\boldsymbol{r}}T_{4,7,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
8	$x_1^{\boldsymbol{n}-\boldsymbol{r}}T_{4,8,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0	$x_4^{\boldsymbol{r}}T_{4,8,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$

TABLE 12. The right side of the matrix for $(\ell_{\boldsymbol{d},\boldsymbol{n},\boldsymbol{r}})_4$.

	$\mathfrak{P}(-(1,1,\underline{0}))^1$	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_1))^1$	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_2))^1$	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_3))^1$	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_4))^1$
$\mathfrak{P}$	$\mathfrak{T}_{1,1,1}^{1,1,1}$	$y_1\mathfrak{T}_{1,1,2}^{\boldsymbol{d},1,1}$	$y_2\mathfrak{T}_{1,1,3}^{\boldsymbol{d},1,1}$	$y_3\mathfrak{T}_{1,1,4}^{\boldsymbol{d},1,1}$	$y_4\mathfrak{T}_{1,1,5}^{\boldsymbol{d},1,1}$

TABLE 13. The matrix for $(\mathfrak{l}_{\boldsymbol{d}})_1$.

		1 $\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_1))$	2 $\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_2))$	3 $\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_3))$	4 $\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_4))$
1	$\mathfrak{P}(-(1,1,\underline{0}))^1$	$y_1\mathfrak{T}_{2,1,1}^{\boldsymbol{d},1,1}$	$y_2\mathfrak{T}_{2,1,2}^{\boldsymbol{d},1,1}$	$y_3\mathfrak{T}_{2,1,3}^{\boldsymbol{d},1,1}$	$y_4\mathfrak{T}_{2,1,4}^{\boldsymbol{d},1,1}$
2	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_1))^1$	$\mathfrak{T}_{2,2,1}^{1,1,1}$	$w_1y_2\mathfrak{T}_{2,2,2}^{0,1,1}$	$w_1y_3\mathfrak{T}_{2,2,3}^{0,1,1}$	$w_1y_4\mathfrak{T}_{2,2,4}^{0,1,1}$
3	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_2))^1$	$y_1w_2\mathfrak{T}_{2,3,1}^{0,1,1}$	$\mathfrak{T}_{2,3,2}^{1,1,1}$	$w_2y_3\mathfrak{T}_{2,3,3}^{0,1,1}$	$w_2y_4\mathfrak{T}_{2,3,4}^{0,1,1}$
4	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_3))^1$	$y_1w_3\mathfrak{T}_{2,4,1}^{0,1,1}$	$y_2w_3\mathfrak{T}_{2,4,2}^{0,1,1}$	$\mathfrak{T}_{2,4,3}^{1,1,1}$	$w_3y_4\mathfrak{T}_{2,4,4}^{0,1,1}$
5	$\mathfrak{P}(-(\boldsymbol{d}+1,\boldsymbol{d},z_4))^1$	$y_1w_4\mathfrak{T}_{2,5,1}^{0,1,1}$	$y_2w_4\mathfrak{T}_{2,5,2}^{0,1,1}$	$y_3w_4\mathfrak{T}_{2,5,3}^{0,1,1}$	$\mathfrak{T}_{2,5,4}^{1,1,1}$

TABLE 14. The left side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_2$.

	5 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}-1,\underline{1}))^{\boldsymbol{d}}$	6 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_2))^{\boldsymbol{d}+1}$	7 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_3))^{\boldsymbol{d}+1}$	8 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_4))^{\boldsymbol{d}+1}$
1	$y_1y_2y_3y_4\mathfrak{T}_{2,1,5}^{2\boldsymbol{d}-2,1,\boldsymbol{d}}$	$y_1y_2\mathfrak{T}_{2,1,6}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1y_3\mathfrak{T}_{2,1,7}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1y_4\mathfrak{T}_{2,1,8}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$
2	$y_2y_3y_4\mathfrak{T}_{2,2,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$y_2\mathfrak{T}_{2,2,6}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_3\mathfrak{T}_{2,2,7}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_4\mathfrak{T}_{2,2,8}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
3	$y_1y_3y_4\mathfrak{T}_{2,3,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$y_1\mathfrak{T}_{2,3,6}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_1w_2y_3\mathfrak{T}_{2,3,7}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1w_2y_4\mathfrak{T}_{2,3,8}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$
4	$y_1y_2y_4\mathfrak{T}_{2,4,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$y_1y_2w_3\mathfrak{T}_{2,4,6}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1\mathfrak{T}_{2,4,7}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_1w_3y_4\mathfrak{T}_{2,4,8}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$
5	$y_1y_2y_3\mathfrak{T}_{2,5,5}^{\boldsymbol{d}-1,1,\boldsymbol{d}}$	$y_1y_2w_4\mathfrak{T}_{2,5,6}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1y_3w_4\mathfrak{T}_{2,5,7}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_1\mathfrak{T}_{2,5,8}^{\boldsymbol{d},1,\boldsymbol{d}+1}$

TABLE 15. The middle of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_2$.

	9 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_2+z_3))^{\boldsymbol{d}+1}$	10 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_2+z_4))^{\boldsymbol{d}+1}$	11 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_3+z_4))^{\boldsymbol{d}+1}$	12 $\mathfrak{P}(-(2\boldsymbol{d}+1,2\boldsymbol{d}+1,\underline{0}))^{\boldsymbol{d}+1}$
1	$y_2y_3\mathfrak{T}_{2,1,9}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_2y_4\mathfrak{T}_{2,1,10}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_3y_4\mathfrak{T}_{2,1,11}^{2\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$\mathfrak{T}_{2,1,12}^{2\boldsymbol{d},1,\boldsymbol{d}+1}$
2	$w_1y_2y_3\mathfrak{T}_{2,2,9}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$w_1y_2y_4\mathfrak{T}_{2,2,10}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$w_1y_3y_4\mathfrak{T}_{2,2,11}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$w_1\mathfrak{T}_{2,2,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
3	$y_3\mathfrak{T}_{2,3,9}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_4\mathfrak{T}_{2,3,10}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$w_2y_3y_4\mathfrak{T}_{2,3,11}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$w_2\mathfrak{T}_{2,3,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
4	$y_2\mathfrak{T}_{2,4,9}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_2w_3y_4\mathfrak{T}_{2,4,10}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_4\mathfrak{T}_{2,4,11}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$w_3\mathfrak{T}_{2,4,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$
5	$y_2y_3w_4\mathfrak{T}_{2,5,9}^{\boldsymbol{d}-1,1,\boldsymbol{d}+1}$	$y_2\mathfrak{T}_{2,5,10}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$y_3\mathfrak{T}_{2,5,11}^{\boldsymbol{d},1,\boldsymbol{d}+1}$	$w_4\mathfrak{T}_{2,5,12}^{\boldsymbol{d},1,\boldsymbol{d}+1}$

TABLE 16. The right side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_2$.

		1 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_1))^{2\boldsymbol{d}+1}$	2 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_2))^{2\boldsymbol{d}+1}$	3 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_3))^{2\boldsymbol{d}+1}$	4 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_4))^{2\boldsymbol{d}+1}$
1	$\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_1))$	$\mathfrak{T}_{3,1,1}^{\boldsymbol{d},1,2\boldsymbol{d}+1}$	$w_1y_2\mathfrak{T}_{3,1,2}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$w_1y_3\mathfrak{T}_{3,1,3}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$w_1y_4\mathfrak{T}_{3,1,4}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
2	$\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_2))$	$y_1w_2\mathfrak{T}_{3,2,1}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$\mathfrak{T}_{3,2,2}^{\boldsymbol{d},1,2\boldsymbol{d}+1}$	$w_2y_3\mathfrak{T}_{3,2,3}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$w_2y_4\mathfrak{T}_{3,2,4}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
3	$\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_3))$	$y_1w_3\mathfrak{T}_{3,3,1}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$y_2w_3\mathfrak{T}_{3,3,2}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$\mathfrak{T}_{3,3,3}^{\boldsymbol{d},1,2\boldsymbol{d}+1}$	$w_3y_4\mathfrak{T}_{3,3,4}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$
4	$\mathfrak{P}(-(\boldsymbol{d}+2,\boldsymbol{d}+1,z_4))$	$y_1w_4\mathfrak{T}_{3,4,1}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$y_2w_4\mathfrak{T}_{3,4,2}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$y_3w_4\mathfrak{T}_{3,4,3}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}$	$\mathfrak{T}_{3,4,4}^{\boldsymbol{d},1,2\boldsymbol{d}+1}$
5	$\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}-1,\underline{1}))^{\boldsymbol{d}}$	0	0	0	0
6	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_2))^{\boldsymbol{d}+1}$	$w_2\mathfrak{T}_{3,6,1}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$w_1\mathfrak{T}_{3,6,2}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	0
7	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_3))^{\boldsymbol{d}+1}$	$w_3\mathfrak{T}_{3,7,1}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	$w_1\mathfrak{T}_{3,7,3}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0
8	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_1+z_4))^{\boldsymbol{d}+1}$	$w_4\mathfrak{T}_{3,8,1}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	0	$w_1\mathfrak{T}_{3,8,4}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
9	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_2+z_3))^{\boldsymbol{d}+1}$	0	$w_3\mathfrak{T}_{3,9,2}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$w_2\mathfrak{T}_{3,9,3}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0
10	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_2+z_4))^{\boldsymbol{d}+1}$	0	$w_4\mathfrak{T}_{3,10,2}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	0	$w_2\mathfrak{T}_{3,10,4}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
11	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d},z_3+z_4))^{\boldsymbol{d}+1}$	0	0	$w_4\mathfrak{T}_{3,11,3}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$w_3\mathfrak{T}_{3,11,4}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$
12	$\mathfrak{P}(-(2\boldsymbol{d}+1,2\boldsymbol{d}+1,\underline{0}))^{\boldsymbol{d}+1}$	$y_1\mathfrak{T}_{3,12,1}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$y_2\mathfrak{T}_{3,12,2}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$y_3\mathfrak{T}_{3,12,3}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$	$y_4\mathfrak{T}_{3,12,4}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}$

TABLE 17. The left side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_3$.

	5 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},0,1,1,1))^{2\boldsymbol{d}+1}$	6 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,0,1,1))^{2\boldsymbol{d}+1}$	7 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,0,1))^{2\boldsymbol{d}+1}$	8 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,1,0))^{2\boldsymbol{d}+1}$
1	$w_1y_2y_3y_4\mathfrak{T}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}_{3,1,5}$	$y_3y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,6}$	$y_2y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,7}$	$y_2y_3\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,1,8}$
2	$y_3y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,5}$	$y_1w_2y_3y_4\mathfrak{T}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}_{3,2,6}$	$y_1y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,7}$	$y_1y_3\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,2,8}$
3	$y_2y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,5}$	$y_1y_4\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,6}$	$y_1y_2w_3y_4\mathfrak{T}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}_{3,3,7}$	$y_1y_2\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,3,8}$
4	$y_2y_3\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,5}$	$y_1y_3\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,6}$	$y_1y_2\mathfrak{T}^{\boldsymbol{d}-1,1,2\boldsymbol{d}+1}_{3,4,7}$	$y_1y_2y_3w_4\mathfrak{T}^{\boldsymbol{d}-2,1,2\boldsymbol{d}+1}_{3,4,8}$
5	$w_1\mathfrak{T}^{0,\boldsymbol{d},2\boldsymbol{d}+1}_{3,5,5}$	$w_2\mathfrak{T}^{0,\boldsymbol{d},2\boldsymbol{d}+1}_{3,5,6}$	$w_3\mathfrak{T}^{0,\boldsymbol{d},2\boldsymbol{d}+1}_{3,5,7}$	$w_4\mathfrak{T}^{0,\boldsymbol{d},2\boldsymbol{d}+1}_{3,5,8}$
6	0	0	$y_4\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,6,7}$	$y_3\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,6,8}$
7	0	$y_4\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,7,6}$	0	$y_2\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,7,8}$
8	0	$y_3\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,8,6}$	$y_2\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,8,7}$	0
9	$y_4\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,9,5}$	0	0	$y_1\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,9,8}$
10	$y_3\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,10,5}$	0	$y_1\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,10,7}$	0
11	$y_2\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,11,5}$	$y_1\mathfrak{T}^{0,\boldsymbol{d}+1,2\boldsymbol{d}+1}_{3,11,6}$	0	0
12	0	0	0	0

TABLE 18. The right side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_3$.

		1 $\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+2,\underline{0}))^{\boldsymbol{d}}$	2 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_1+z_2))^{\boldsymbol{d}}$	3 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_1+z_3))^{\boldsymbol{d}}$	4 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_1+z_4))^{\boldsymbol{d}}$
1	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_1))^{2\boldsymbol{d}+1}$	$w_1\mathfrak{T}_{4,1,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_2\mathfrak{T}_{4,1,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_3\mathfrak{T}_{4,1,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_4\mathfrak{T}_{4,1,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
2	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_2))^{2\boldsymbol{d}+1}$	$w_2\mathfrak{T}_{4,2,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_1\mathfrak{T}_{4,2,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0
3	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_3))^{2\boldsymbol{d}+1}$	$w_3\mathfrak{T}_{4,3,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$y_1\mathfrak{T}_{4,3,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
4	$\mathfrak{P}(-(2\boldsymbol{d}+2,2\boldsymbol{d}+1,z_4))^{2\boldsymbol{d}+1}$	$w_4\mathfrak{T}_{4,4,1}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0	$y_1\mathfrak{T}_{4,4,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
5	$\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},0,1,1,1))^{2\boldsymbol{d}+1}$	0	0	0	0
6	$\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,0,1,1))^{2\boldsymbol{d}+1}$	0	0	$w_4\mathfrak{T}_{4,6,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$w_3\mathfrak{T}_{4,6,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
7	$\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,0,1))^{2\boldsymbol{d}+1}$	0	$w_4\mathfrak{T}_{4,7,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$w_2\mathfrak{T}_{4,7,4}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$
8	$\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d},1,1,1,0))^{2\boldsymbol{d}+1}$	0	$w_3\mathfrak{T}_{4,8,2}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$w_2\mathfrak{T}_{4,8,3}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0

TABLE 19. The left side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_4$.

	5 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_2+z_3))^{\boldsymbol{d}}$	6 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_2+z_4))^{\boldsymbol{d}}$	7 $\mathfrak{P}(-(2\boldsymbol{d}+3,2\boldsymbol{d}+1,z_3+z_4))^{\boldsymbol{d}}$	8 $\mathfrak{P}(-(2\boldsymbol{d}+4,2\boldsymbol{d},\underline{1}))^{\boldsymbol{d}+1}$
1	0	0	0	0
2	$y_3\mathfrak{T}_{4,2,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_4\mathfrak{T}_{4,2,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0
3	$y_2\mathfrak{T}_{4,3,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$y_4\mathfrak{T}_{4,3,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
4	0	$y_2\mathfrak{T}_{4,4,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_3\mathfrak{T}_{4,4,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0
5	$w_4\mathfrak{T}_{4,5,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$w_3\mathfrak{T}_{4,5,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$w_2\mathfrak{T}_{4,5,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_1\mathfrak{T}_{4,5,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
6	0	0	$w_1\mathfrak{T}_{4,6,7}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	$y_2\mathfrak{T}_{4,6,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
7	0	$w_1\mathfrak{T}_{4,7,6}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	$y_3\mathfrak{T}_{4,7,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$
8	$w_1\mathfrak{T}_{4,8,5}^{0,2\boldsymbol{d}+1,\boldsymbol{d}}$	0	0	$y_4\mathfrak{T}_{4,8,8}^{0,2\boldsymbol{d}+1,\boldsymbol{d}+1}$

TABLE 20. The right side of the matrix for $(\mathfrak{l}_{\boldsymbol{d}})_4$.

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