THE MINIMAL FREE RESOLUTION OF THE MIGLIORE-PETERSON RINGS IN THE CASE THAT THE REFLEXIVE SHEAF HAS EVEN RANK

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Abstract. Let R be a commutative noetherian ring and $\varphi\colon F \to G$ be a homomorphism of free $R-$modules where $\operatorname{rank} F = f$ and $\operatorname{rank} G = g$. Fix an element $b_{g+1} \in \bigwedge^{g+1} F$ and a generator ω_{G^*} for $\bigwedge^g G^*$. The module action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$ produces the element $b_1 = [(\bigwedge^g \varphi^*)(\omega_{G^*})]\,(b_{g+1})$ in F. Let J denote the image of $b_1\colon F^* \to R$. Assume that $\operatorname{grade} J = f-g$, which is the largest grade possible and is attained in the generic case. The ideal J may be interpreted as the defining ideal of the degeneracy locus of a regular section of a rank $f-g$ reflexive sheaf. It may also be interpreted as the order ideal of an element in a second syzygy module of rank $f-g$. Also, J may be interpreted as the defining ideal for the symmetric algebra of a module of projective dimension two. Migliore and Peterson have studied the ideal J^{unm}, which is the unmixed part of J. Under geometric hypotheses, they have shown that R/J^{unm} is a Cohen-Macaulay ring and they have resolved this ring. Furthermore, if $f-g$ is odd, then J^{unm} is a Gorenstein ideal and is not equal to J. On the other hand, if $f-g$ is even, then $J^{\text{unm}} = J$. In the present paper, we produce the resolution of R/J by free $R-$modules in the case that $f-g$ is even and $(f-g-2)!$ is a unit in R. Our resolution is minimal whenever the data is local or homogeneous. Our resolution is built from the differential graded algebra $(\bigwedge^\bullet F^*{<}X_1,\dots,X_g{>}, d)$, where the restriction of d to $\bigwedge^\bullet F^*$ is the Koszul complex associated to $b_1\colon F^* \to R$ and the degree two divided power variables $X_1,\dots,X_g$ have been adjoined in order to kill the cycles $\varphi^*(G^*) \subseteq \bigwedge^1 F^*$. The acyclicity lemma is used to prove exactness. If $g=1$, then the ideal J is equal to the Huneke-Ulrich almost complete intersection ideal $I_1(yX)$, where y is a $1\times f$ matrix and X is an $f \times f$ alternating matrix. The resolution of this ideal is already known.

Let R be a commutative noetherian ring, and let F and G be free $R-$modules of rank f and g, respectively, with $g < f$. Consider an $R-$module homomorphism $\varphi\colon F \to G$. Let M and K represent the cokernel and kernel of φ, respectively. Assume that the $R-$ideal $I_g(\varphi)$, which is generated by the maximal minors of φ, has the largest possible grade, as permitted by the determinantal bound of Eagon and Northcott; namely, $\operatorname{grade} I_g(\varphi) = f-g+1$. In this case the entire free resolution of M is given by Eagon-Northcott complex; in particular, the next map in the resolution of M is $\eta\colon \bigwedge^{g+1} F \to F$, where every entry of the matrix representation of η is a $g \times g$ minor of φ. There are at least three ways to describe the ring R/J which is resolved in the present paper. First of all, and this is the

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approach of Migliore and Peterson (first in [13], and later, with Nagel, in [12]), one can take a regular section s of the sheaf, $\widetilde{K}$, associated to K. In this approach, the ideal J is the defining ideal of the degeneracy locus of s. Essentially, for each vector v in the column space of η, the ideal J, which is generated by the entries of v, represents a section of $\widetilde{K}$. The rank of K, as an $R-$module, is $f-g$. If the ideal J has height $f-g$, then J represents a regular section of $\widetilde{K}$. The second approach which yields the same object involves the language of order ideals. If n is an element in the $R-$module N, then the order ideal of n in N is defined to be $\{f(n) \mid f \in \operatorname{Hom}_R(N,R)\}$. It is clear that the ideal J is the order ideal of the element k in K which is represented by v. The Eisenbud-Evans Principal Ideal Conjecture (see [2, Theorem 1] or [6, Theorem 2.7]) ensures that the height of J is no more than $f-g$, provided the element k is not a minimal generator of K_P for some prime ideal P of R. Once again, we study the order ideal J, provided its height is the largest possible. The third approach to the ideal J comes from the theory of symmetric and Rees algebras. The cokernel of η^* is the $R-$module of projective dimension two which is resolved by

$$0 \to G^* \xrightarrow{\varphi^*} F^* \xrightarrow{\eta^*} \bigwedge^{g+1} F^*.$$

The ideal generated by the maximal minors of the last map in the above free resolution has grade given by the Eagon-Northcott bound for determinantal ideals. Furthermore, any module of projective dimension two whose last map attains the Eagon-Northcott bound, may be obtained in this manner. At any rate the symmetric algebra of $\operatorname{coker}\eta^*$ is equal to $R[T_1,\dots,T_n]/\mathcal{J}$, where $n = \operatorname{rank}\bigwedge^{g+1} F^*$ and $\mathcal{J}$ is generated by the entries of the product of $[T_1,\dots,T_n]$ with the matrix representation of η^*. The ideal $\mathcal{J}$ is generated by the entries of a general element of the row space of η^*; and consequently, it is equal to one of the ideals $JR[T_1,\dots,T_n]$, since these ideals are generated by the entries of a general element of the column space of η.

The quotient R/J is in general not Cohen-Macaulay; indeed, it has embedded components. Migliore and Peterson observed that the correct object to study is the unmixed part of J. (If the primary decomposition of J is $\cap\mathcal{P}_P$, where $\mathcal{P}_P$ is a $P-$primary ideal and the intersection is taken over all associated prime ideals of J, then the *unmixed part* of J is $J^{\mathrm{unm}} = \bigcap\limits_{\operatorname{ht} P = \operatorname{ht} J} \mathcal{P}_P$ where this intersection is taken over all associated prime ideals of J which have the same height as J.) In the geometric setting, J^{unm} defines the homogeneous coordinate ring of the highest dimensional component of the degeneracy locus of the regular section s. In the situation where R/J is a symmetric algebra, then the passage from J to J^{unm} kills the torsion submodule of the symmetric algebra, thereby producing the Rees algebra of the projective dimension two module $\operatorname{coker}\eta^*$. Migliore and Peterson have shown, under geometric hypotheses, that R/J^{unm} is a Cohen-Macaulay ring and they have resolved this ring. Furthermore, if $f-g$ is odd, then J^{unm} is a Gorenstein ideal and is not equal to J. On the other hand, if $f-g$ is even, then $J^{\mathrm{unm}} = J$. In the present paper, we produce the resolution $\mathbb{M}$ of R/J by free $R-$modules in the case that $f-g$ is even and $(f-g-2)!$ is a unit in R. The resolution $\mathbb{M}$ is minimal whenever the data is local or homogeneous. Our resolution arises from a different point of view than the resolution of Migliore and Peterson and it does not require the geometric hypotheses which are inherent in their approach.

We now give a brief description of $\mathbb{M}$. Let $\mathbb{K}$ be the Koszul complex $(\bigwedge^\bullet F^*, b_1)$. It is clear that $H_0(\mathbb{K}) = R/J$ and $\varphi^*(G^*) \subseteq Z_1(\mathbb{K})$. We adjoin g divided power variables of degree two in order to kill the homology represented by $\varphi^*(G^*)$. We could write $\bigwedge^\bullet F^*{<}X_1, \dots, X_g; dX_i = \gamma_1^{[i]}{>}$ for the new object, where $\gamma_1^{[1]}, \dots \gamma_1^{[g]}$ is a basis for G^*; however, instead we prefer to write the coordinate free version $(\widetilde{\mathbb{K}}, d) = (D_\bullet G^* \otimes \bigwedge^\bullet F^*, d)$, with

$$d\left(\gamma_1^{(i)} \otimes \alpha_j\right) = \gamma_1^{(i-1)} \otimes \varphi^*(\gamma_1) \wedge \alpha_j + \gamma_1^{(i)} \otimes b_1(\alpha_j).$$

For each integer n, let $\widetilde{\mathbb{K}}^{[n]}$ represent the subcomplex $\bigoplus\limits_{i+j\le n} D_i G^* \otimes \bigwedge^j F^*$ of $(\widetilde{\mathbb{K}}, d)$. Let $N = \frac{f-g}{2}$. The resolution $\mathbb{M}$ is obtained by glueing $\widetilde{\mathbb{K}}^{[N]}$ to the shifted dual $\left(\widetilde{\mathbb{K}}^{[N-1]}\right)^*[-(f-g)]$. The only complicated part of $\mathbb{M}$ is the glue which joins the two pieces.

Section 1 is a quick review of multilinear algebra. Section 2 contains the official description of $\mathbb{M}$, as well as various examples. We show that $\mathbb{M}$ is a complex in section 3. In section 4, we establish the acyclicity of $\mathbb{M}$ and record various consequences. The final section is a partial list of avenues for further study.

1. Preliminary results.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. The ideal I of R is called *perfect* if the grade of I is equal to the projective dimension of the $R-$module R/I. The inequality $\operatorname{grade} I \le \operatorname{pd}_R R/I$ always holds.

We begin with a few remarks about multilinear algebra. Let R be a commutative noetherian ring, and F be a free $R-$module of finite rank. We make much use of the exterior algebras $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$. Each element of F^* is a graded derivation on $\bigwedge^\bullet F$. In other words,

$$\alpha_1\left(a_1^{[1]} \wedge \dots \wedge a_1^{[s]}\right) = \textstyle\sum_j (-1)^{j+1} \alpha_1(a_1^{[j]}) \cdot a_1^{[1]} \wedge \dots \wedge \widehat{a_1^{[j]}} \wedge \dots \wedge a_1^{[s]} \in \bigwedge^{s-1} F,$$

for all $\alpha_1 \in F^*$ and $a_1^{[j]} \in F$. This action gives rise to the $\bigwedge^\bullet F^*-$module structure on $\bigwedge^\bullet F$. In particular,

$$(\alpha_1 \wedge \alpha_1')(a_s) = \alpha_1\left(\alpha_1'(a_s)\right),$$

for $\alpha_1, \alpha_1' \in F^*$ and $a_s \in \bigwedge^s F$. The $\bigwedge^\bullet F-$module structure on $\bigwedge^\bullet F^*$ is obtained in an analogous manner. In particular, if $a_i \in \bigwedge^i F$ and $\alpha_j \in \bigwedge^j F^*$, then

$$a_i(\alpha_j) \in \bigwedge^{j-i} F^* \qquad \text{and} \qquad \alpha_j(a_i) \in \bigwedge^{i-j} F.$$

One consequence of these two module structures is that $a_s(\alpha_s) = \alpha_s(a_s) \in R$ for all a_s in $\bigwedge^s F$ and $\alpha_s \in \bigwedge^s F^*$. The following well known formulas show more of the interaction between the two module structures. See [4, section 1], [5, Appendix], and [10, section 1].

Proposition 1.1. *Let F be a free module of rank f over a commutative noetherian ring R and let $a_r \in \bigwedge^r F$, $a_p \in \bigwedge^p F$, and $\alpha_q \in \bigwedge^q F^*$.*

(a) *If $r = 1$, then $[a_r(\alpha_q)](a_p) = a_r \wedge [\alpha_q(a_p)] + (-1)^{1+q}\alpha_q(a_r \wedge a_p)$.*
(b) *If $q = f$, then $[a_r(\alpha_q)](a_p) = (-1)^{(f-r)(f-p)}[a_p(\alpha_q)](a_r)$.*
(c) *If $p = f$, then $[a_r(\alpha_q)](a_p) = a_r \wedge \alpha_q(a_p)$.*
(d) *If $\varphi\colon F \to G$ is a homomorphism of free $R-$modules and $\gamma_{s+r} \in \bigwedge^{s+r} G^*$, then $(\bigwedge^s \varphi^*)\left[\left((\bigwedge^r \varphi)(a_r)\right)(\gamma_{s+r})\right] = a_r\left[\left(\bigwedge^{s+r}\varphi^*\right)(\gamma_{s+r})\right]$.*

Note. The exponent which is given in (b) is correct. An incorrect value has appeared elsewhere in the literature.

Remark 1.2. Let F be a free module over a commutative ring R. Let T_rF represent the $R-$module $\underbrace{F \otimes \ldots \otimes F}_{r}$. (We do not make any other use of the tensor algebra $T_\bullet F$.) The exterior algebra $\bigwedge^\bullet F$, the symmetric algebra $S_\bullet F$, and the divided power algebra $D_\bullet F$ each comes equipped with multiplication $\operatorname{mult}\colon A \otimes A \to A$ and co-multiplication $\Delta\colon A \to A \otimes A$. Co-multiplication is the algebra map which is induced by the diagonal map $F \to F \oplus F$. Often, we will use Δ to represent only one graded piece of the co-multiplication map. In other words, if $p + q = t$, then we let $\Delta\colon \bigwedge^t F \to \bigwedge^p F \otimes \bigwedge^q F$ represent the composition

$$\bigwedge^t F \xrightarrow{\text{inclusion}} \bigwedge^\bullet F \xrightarrow{\Delta} \bigwedge^\bullet F \otimes \bigwedge^\bullet F \xrightarrow{\text{projection}} \bigwedge^p F \otimes \bigwedge^q F.$$

Example 1.3. The co-multiplication map $\Delta\colon S_4F \to S_2F \otimes F \otimes F$ carries the element $a_1^{[1]} \cdot a_1^{[2]} \cdot a_1^{[3]} \cdot a_1^{[4]}$ of S_4F to

$$\begin{cases} +a_1^{[1]} \cdot a_1^{[2]} \otimes a_1^{[3]} \otimes a_1^{[4]} + a_1^{[1]} \cdot a_1^{[2]} \otimes a_1^{[4]} \otimes a_1^{[3]} + a_1^{[1]} \cdot a_1^{[3]} \otimes a_1^{[2]} \otimes a_1^{[4]} \\ +a_1^{[1]} \cdot a_1^{[3]} \otimes a_1^{[4]} \otimes a_1^{[2]} + a_1^{[1]} \cdot a_1^{[4]} \otimes a_1^{[2]} \otimes a_1^{[3]} + a_1^{[1]} \cdot a_1^{[4]} \otimes a_1^{[3]} \otimes a_1^{[2]} \\ +a_1^{[2]} \cdot a_1^{[3]} \otimes a_1^{[1]} \otimes a_1^{[4]} + a_1^{[2]} \cdot a_1^{[3]} \otimes a_1^{[4]} \otimes a_1^{[1]} + a_1^{[2]} \cdot a_1^{[4]} \otimes a_1^{[1]} \otimes a_1^{[3]} \\ +a_1^{[2]} \cdot a_1^{[4]} \otimes a_1^{[3]} \otimes a_1^{[1]} + a_1^{[3]} \cdot a_1^{[4]} \otimes a_1^{[1]} \otimes a_1^{[2]} + a_1^{[3]} \cdot a_1^{[4]} \otimes a_1^{[2]} \otimes a_1^{[1]}. \end{cases}$$

Remark 1.4. For each integer i, there are canonical perfect pairings

$$\bigwedge^i F^* \otimes \bigwedge^i F \to R \quad \text{and} \quad D_iF^* \otimes S_iF \to R.$$

For more details, see [1] or [11].

Definition 1.5. Let $\mathbf{m} = [m_1, \ldots, m_\ell]$ be a vector of non-negative integers. Define the *value* of $\mathbf{m}$ to be $|\mathbf{m}| = \sum_{h=1}^{\ell} h \cdot m_h$. We think of $\mathbf{m}$ as the partition $\underbrace{1, \ldots, 1}_{m_1}, \underbrace{2, \ldots, 2}_{m_2}, \ldots, \underbrace{\ell, \ldots, \ell}_{m_\ell}$ of $|\mathbf{m}|$. The *number of pieces* in the partition $\mathbf{m}$ is $r(\mathbf{m}) = \sum_{h=1}^{\ell} m_h$.

Definition 1.6. For each vector $\mathbf{m} = [m_1, \dots, m_\ell]$ of non-negative integers, define $\bigwedge^{\mathbf{m}} F$ to be

$$\bigwedge^{m_1\boldsymbol{\epsilon}_1} F \otimes \bigwedge^{m_2\boldsymbol{\epsilon}_2} F \otimes \dots \otimes \bigwedge^{m_\ell\boldsymbol{\epsilon}_\ell} F,$$

where $\boldsymbol{\epsilon}_h$ is the vector with 1 in position h and 0 everywhere else, and

$$\bigwedge^{m_h\boldsymbol{\epsilon}_h} F = \begin{cases} \bigwedge^{m_h}\left(\bigwedge^h F\right) & \text{if } h \text{ is odd, and} \\ S_{m_h}\left(\bigwedge^h F\right) & \text{if } h \text{ is even.} \end{cases}$$

Let $\Delta_{\mathbf{m}} = \Delta_\bullet \colon \bigwedge^{|m|} F \to \bigwedge^{\mathbf{m}} F$ represent the composition

$$\begin{array}{ccc} \bigwedge^{|m|} F & \xrightarrow{\Delta} & \bigwedge^{1m_1} F \otimes \bigwedge^{2m_2} F \otimes \dots \otimes \bigwedge^{\ell m_\ell} F \\ & & \Big\downarrow{\scriptstyle \Delta_{m_1\boldsymbol{\epsilon}_1}\otimes\Delta_{m_2\boldsymbol{\epsilon}_2}\otimes\dots\otimes\Delta_{m_\ell\boldsymbol{\epsilon}_\ell}} \\ & & \bigwedge^{m_1\boldsymbol{\epsilon}_1} F \otimes \bigwedge^{m_2\boldsymbol{\epsilon}_2} F \otimes \dots \otimes \bigwedge^{m_\ell\boldsymbol{\epsilon}_\ell} F, \end{array}$$

where Δ is the co-multiplication of Remark 1.2 and $\Delta_{m_h\boldsymbol{\epsilon}_h} \colon \bigwedge^{hm_h} F \to \bigwedge^{m_h\boldsymbol{\epsilon}_h} F$ is the natural map. If $\mathbf{m} = [m_1, \dots, m_\ell]$ is a vector of integers with $m_h < 0$, for some h, then we take $\bigwedge^{\mathbf{m}} F$ to be the zero module and $\Delta_{\mathbf{m}}$ to be the zero map.

Example 1.7. The map $\Delta_{2\boldsymbol{\epsilon}_2} \colon \bigwedge^4 F \to \bigwedge^{2\boldsymbol{\epsilon}_2} F = S_2(\bigwedge^2 F)$ carries

$$a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[3]} \wedge a_1^{[4]} \quad \text{to}$$

$$a_1^{[1]} \wedge a_1^{[2]} \otimes a_1^{[3]} \wedge a_1^{[4]} - a_1^{[1]} \wedge a_1^{[3]} \otimes a_1^{[2]} \wedge a_1^{[4]} + a_1^{[1]} \wedge a_1^{[4]} \otimes a_1^{[2]} \wedge a_1^{[3]}.$$

The map $\Delta_{2\boldsymbol{\epsilon}_3} \colon \bigwedge^6 F \to \bigwedge^{2\boldsymbol{\epsilon}_3} F = \bigwedge^2(\bigwedge^3 F)$ carries

$$a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[3]} \wedge a_1^{[4]} \wedge a_1^{[5]} \wedge a_1^{[6]} \quad \text{to}$$

$$\begin{cases} (a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[3]}) \wedge (a_1^{[4]} \wedge a_1^{[5]} \wedge a_1^{[6]}) - (a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[4]}) \wedge (a_1^{[3]} \wedge a_1^{[5]} \wedge a_1^{[6]}) \\ +(a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[5]}) \wedge (a_1^{[3]} \wedge a_1^{[4]} \wedge a_1^{[6]}) - (a_1^{[1]} \wedge a_1^{[2]} \wedge a_1^{[6]}) \wedge (a_1^{[3]} \wedge a_1^{[4]} \wedge a_1^{[5]}) \\ +(a_1^{[1]} \wedge a_1^{[3]} \wedge a_1^{[4]}) \wedge (a_1^{[2]} \wedge a_1^{[5]} \wedge a_1^{[6]}) - (a_1^{[1]} \wedge a_1^{[3]} \wedge a_1^{[5]}) \wedge (a_1^{[2]} \wedge a_1^{[4]} \wedge a_1^{[6]}) \\ +(a_1^{[1]} \wedge a_1^{[3]} \wedge a_1^{[6]}) \wedge (a_1^{[2]} \wedge a_1^{[4]} \wedge a_1^{[5]}) + (a_1^{[1]} \wedge a_1^{[4]} \wedge a_1^{[5]}) \wedge (a_1^{[2]} \wedge a_1^{[3]} \wedge a_1^{[6]}) \\ -(a_1^{[1]} \wedge a_1^{[4]} \wedge a_1^{[6]}) \wedge (a_1^{[2]} \wedge a_1^{[3]} \wedge a_1^{[5]}) + (a_1^{[1]} \wedge a_1^{[5]} \wedge a_1^{[6]}) \wedge (a_1^{[2]} \wedge a_1^{[3]} \wedge a_1^{[4]}). \end{cases}$$

Some of the interplay between the co-multiplication map Δ and the map $\Delta_{\mathbf{m}}$ of Definition 1.6 is captured in the following three results.

Observation 1.8. *If $\mathbf{m} = m_k\boldsymbol{\epsilon}_k$, then the diagram*

$$\begin{array}{ccccc} \bigwedge^{km_k} F & & \xrightarrow{\Delta_{\mathbf{m}}} & & \bigwedge^{\mathbf{m}} F \\ {\scriptstyle\Delta}\Big\downarrow & & & & \Big\downarrow{\scriptstyle m_k} \\ \bigwedge^{k(m_k-1)} F \otimes \bigwedge^k F & \xrightarrow{\Delta_{\mathbf{m}-\boldsymbol{\epsilon}_k}\otimes 1} & \bigwedge^{\mathbf{m}-\boldsymbol{\epsilon}_k} F \otimes \bigwedge^k F & \xrightarrow{\text{mult}} & \bigwedge^{\mathbf{m}} F \end{array}$$

commutes, where the map labled m_k is multiplication by the integer m_k and the map labled mult *is multiplication in the exterior algebra $\bigwedge^\bullet(\bigwedge^k F)$ (if k is odd) or multiplication in the symmetric algebra $S_\bullet(\bigwedge^k F)$ (if k is even).*

Proof. This result holds because the composition

$$A_{m_k} \xrightarrow{\Delta} A_{m_k-1} \otimes A_1 \xrightarrow{\text{mult}} A_{m_k}$$

is multiplication by m_k for $A = S_\bullet G$ or $A = \bigwedge^\bullet G$, where G is any free $R-$module. □

Observation 1.9. *If $\mathbf{m} = \sum m_h \boldsymbol{\epsilon}_h$, then the composition*

$$\bigwedge^{|\mathbf{m}|} F \xrightarrow{\Delta_{\mathbf{m}}} \bigwedge^{\mathbf{m}} F \xrightarrow{\text{mult}} \bigwedge^{|\mathbf{m}|} F$$

is multiplication by $\dfrac{|\mathbf{m}|!}{\prod\limits_{1\le h} m_h! \cdot \prod\limits_{1\le h} (h!)^{m_h}}$.

Proof. The maps

$$\bigwedge^{|\mathbf{m}|} F \xrightarrow{\Delta} \bigwedge^{1\cdot m_1} F \otimes \dots \otimes \bigwedge^{\ell\cdot m_\ell} F \xrightarrow{\text{mult}} \bigwedge^{|\mathbf{m}|} F \quad \text{and}$$
$$\bigwedge^{h\cdot m_h} F \xrightarrow{\Delta_\bullet} \bigwedge^{m_h \boldsymbol{\epsilon}_h} F \xrightarrow{\text{mult}} \bigwedge^{h\cdot m_h} F$$

are multiplication by

$$\frac{|\mathbf{m}|!}{(1\cdot m_1)!(2\cdot m_2)!\cdots(\ell\cdot m_\ell)!} \quad \text{and} \quad \frac{(h\cdot m_h)!}{(h!)^{m_h} m_h!},$$

respectively. □

Observation 1.10. *Fix $a_1 \in F$, $a_j \in \bigwedge^j F$, and a vector $\mathbf{m} = [m_1, \dots, m_\ell]$ of non-negative integers, with $|\mathbf{m}| = j+1$. For each positive integer h, let $\boldsymbol{\ell}(h)$ and $\mathbf{u}(h)$ be the lower part of $\mathbf{m}$ and the upper part of $\mathbf{m}$, respectively, with respect to h; that is,*

$$\boldsymbol{\ell}(h) = \sum_{k\le h} m_k \boldsymbol{\epsilon}_k \quad \text{and} \quad \mathbf{u}(h) = \sum_{h\le k} m_k \boldsymbol{\epsilon}_k.$$

For each h, let $\sum\limits_i a_-^{[h,i]} \otimes a^{[h,i]} \otimes a_+^{[h,i]}$ be the image of a_j under the composition

$$\begin{array}{ccc} \bigwedge^j F & \xrightarrow{\quad\Delta\quad} & \bigwedge^{|\boldsymbol{\ell}(h-1)|} F \otimes \bigwedge^{h-1} F \otimes \bigwedge^{|\mathbf{u}(h)-\boldsymbol{\epsilon}_h|} F \\ & & \Big\downarrow {\scriptstyle \Delta_\bullet \otimes 1 \otimes \Delta_\bullet} \\ & & \bigwedge^{\boldsymbol{\ell}(h-1)} F \otimes \bigwedge^{h-1} F \otimes \bigwedge^{\mathbf{u}(h)-\boldsymbol{\epsilon}_h} F, \end{array}$$

where $a_-^{[h,i]} \in \bigwedge^{\boldsymbol{\ell}(h-1)} F$, $a^{[h,i]} \in \bigwedge^{h-1} F$, and $a_+^{[h,i]} \in \bigwedge^{\mathbf{u}(h)-\boldsymbol{\epsilon}_h} F$. For each h, let

$$\mu_h \colon \bigwedge^{\boldsymbol{\ell}(h-1)} F \otimes \bigwedge^h F \otimes \bigwedge^{\mathbf{u}(h)-\boldsymbol{\epsilon}_h} F \to \bigwedge^{\mathbf{m}} F \quad \textit{be}$$

$\bigwedge^{\boldsymbol{\ell}(h-1)} F \otimes \bigwedge^h F \otimes \bigwedge^{(m_h-1)\boldsymbol{\epsilon}_h} F \otimes \bigwedge^{\mathbf{u}(h+1)} F \xrightarrow{1\otimes \text{mult} \otimes 1} \bigwedge^{\boldsymbol{\ell}(h-1)} F \otimes \bigwedge^{m_h\boldsymbol{\epsilon}_h} F \otimes \bigwedge^{\mathbf{u}(h+1)} F,$
where the multiplcation $\bigwedge^h F \otimes \bigwedge^{(m_h-1)\boldsymbol{\epsilon}_h} F \to \bigwedge^{m_h\boldsymbol{\epsilon}_h} F$ *takes place in* $S_\bullet(\bigwedge^h F)$ *or* $\bigwedge^\bullet(\bigwedge^h F)$, *depending on the parity of* h. *Then*

$$\Delta_{\mathbf{m}}(a_1 \wedge a_j) = \sum_{h,i} (-1)^{|\boldsymbol{\ell}(h-1)|} \mu_h \left(a_-^{[h,i]} \otimes a_1 \wedge a^{[h,i]} \otimes a_+^{[h,i]} \right).$$

Proof. The statement merely says that if $a^{[1]}, \dots, a^{[j+1]}$ are each in F, then every term of $\Delta_{\mathbf{m}}(a^{[1]} \wedge \dots \wedge a^{[j+1]})$ may be manipulated in order to have $a^{[1]}$ appear in the first position of $\bigwedge^{m_h\boldsymbol{\epsilon}_h} F$ for some h. $\square$

Convention 1.11. If S is a statement, then we define

$$\chi(S) = \begin{cases} 1 & \text{if } S \text{ is true,} \\ 0 & \text{if } S \text{ is false.} \end{cases}$$

For example, $\chi(i = j)$ has exactly the same meaning as the Kronecker delta $\delta_{i\,j}$.

Convention 1.12. The empty sum is zero. The empty product is one. For each positive integer r, $\mathfrak{S}_r$ is the set of permutations on $\{1, \dots, r\}$. The set $\mathfrak{S}_0$ consists of the identity permutation.

2. The definition of $\mathbb{M}$.

The following notation and assumptions are in effect everywhere.

Data 2.1. Let R be a commutative noetherian ring and $\varphi\colon F \to G$ be a homomorphism of free $R-$modules where $\operatorname{rank} F = f$ and $\operatorname{rank} G = g$. Fix generators ω_F, ω_{F^*}, ω_G, and ω_{G^*} for $\bigwedge^f F$, $\bigwedge^f F^*$, $\bigwedge^g G$ and $\bigwedge^g G^*$, respectively, with the property that $\omega_F(\omega_{F^*}) = 1$ and $\omega_G(\omega_{G^*}) = 1$. Let

$$b_{g+1} \in \bigwedge^{g+1} F, \quad \beta_g = (\bigwedge^g \varphi^*)(\omega_{G^*}) \in \bigwedge^g F^*, \quad \text{and} \quad b_1 = \beta_g(b_{g+1}) \in F.$$

Assume always that $f - g$ is even and greater than zero. Let $N = \frac{f-g}{2}$. The $R-$ideal $J = J(b_{g+1}, \varphi)$ is the image of $b_1\colon F^* \to R$.

Remark. Observe that J is the prototype for the ideals studied by Migliore and Peterson. Indeed, we saw in the introduction that such an ideal is generated by the elements of some vector v in the column space of the map η. In the notation of Data 2.1, $\eta\colon \bigwedge^{g+1} F \to F$ is the map which sends the arbitrary element b_{g+1} of $\bigwedge^{g+1} F$ to b_1 in F. Thus, b_1 is a general element in the column space of η and J is generated by the entries of b_1.

Convention 2.2. We use the following conventions:

$$a_i \in \bigwedge^i F, \quad \alpha_i \in \bigwedge^i F^*, \quad c_i \in \bigwedge^i G, \quad \gamma_i \in \bigwedge^i G^*, \quad \text{and} \quad C_i \in S_i G$$

are arbitrary elements. In particular, elements a and α are always in the exterior algebras $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$, respectively. Furthermore, the subscript of the element tells the degree of the element. When we need several elements from a particular module we identify them by using superscripts inside square brackets. For example, $a_i^{[1]}, \dots, a_i^{[k]}$ represent k elements from $\bigwedge^i F$.

Most of $\mathbb{M}$ is easy to describe. The one difficult part is the map $\Psi_{p,q,i,j}$, which may be found in Definition 2.7. We prove that $\mathbb{M}$ is a complex in Lemma 3.2; the acyclicity of $\mathbb{M}$ is established in Theorem 4.3.

Definition 2.3. Adopt Data 2.1. The modules of $\mathbb{M} = \mathbb{M}(b_{g+1}, \varphi)$ are given by

$$0 \to \mathbb{M}_{f-g} \xrightarrow{d_{f-g}} \mathbb{M}_{f-g-1} \xrightarrow{d_{f-g-1}} \dots \xrightarrow{d_2} \mathbb{M}_1 \xrightarrow{d_1} \mathbb{M}_0,$$

with

$$\mathbb{M}_t = \sum_{\substack{2i+j=t\\ i+j\le N}} D_iG^* \otimes \bigwedge^j F^* \quad \bigoplus \quad \sum_{\substack{2p+q=f-g-t\\ p+q\le N-1}} S_pG \otimes \bigwedge^q F.$$

The differential is given by

$$d\left(\gamma_1^{(i)} \otimes \alpha_j\right) = \gamma_1^{(i-1)} \otimes \varphi^*(\gamma_1) \wedge \alpha_j + \gamma_1^{(i)} \otimes b_1(\alpha_j),$$

for $\gamma_1^{(i)} \otimes \alpha_j \in D_iG^* \otimes \bigwedge^j F^*$, and

$$d\left(C_p \otimes a_q\right) = \begin{cases} C_p \otimes b_1 \wedge a_q \\ + \sum\limits_{k=1}^{q} (-1)^{k+1} \varphi(a_1^{[k]}) \cdot C_p \otimes a_1^{[1]} \wedge \dots \wedge \widehat{a_1^{[k]}} \wedge \dots \wedge a_1^{[q]} \\ + \sum\limits_{\substack{2p+2i+j+q=2N-1\\ i+j\le N}} \Psi_{p,q,i,j}(C_p \otimes a_q \otimes \_\_ \otimes \_\_) \in D_iG^* \otimes \bigwedge^j F^*, \end{cases}$$

for $C_p \otimes a_q = C_p \otimes a_1^{[1]} \wedge \dots \wedge a_1^{[q]} \in S_pG \otimes \bigwedge^q F$.

Remark. The map $\Psi_{p,q,i,j}(C_p \otimes a_q \otimes \_\_ \otimes \_\_)$ of Definition 2.7 is a homomorphism from $S_iG \otimes \bigwedge^j F$ to R; hence, it is an element of $D_iG^* \otimes \bigwedge^j F^*$.

The map $\Psi_{p,q,i,j}$ is based on two other maps; namely, Φ, which is found in Definition 2.5, and $\Delta_{\mathbf{m}}$, which is given in Definition 1.6.

Definition 2.4. Adopt Data 2.1. For each intger n, define

$$\tau_{1,n} \colon G \otimes \bigwedge^n F \to \bigwedge^{n+2} F \quad \text{and} \quad \tau_n \colon S_n(G) \to \bigwedge^{2n} F$$

by

$$\tau_{1,n}(c_1 \otimes a_n) = \left[(\bigwedge^{g-1} \varphi^*)(c_1[\omega_{G^*}])\right](b_{g+1} \wedge a_n),$$

and τ_n is the composition

$$S_nG \xrightarrow{S_n(\tau_{1,0})} S_n(\bigwedge^2 F) \xrightarrow{\text{mult}} \bigwedge^{2n} F.$$

When the domain of $\tau_{1,n}$ or τ_n is clear from context, we simply write τ.

Definition 2.5. Adopt Data 2.1. Let r, $n_1, \dots, n_r$, and p be non-negative integers which satisfy $2p + \sum\limits_{k=1}^{r} n_k = 2N - 1$. Define

$$\Phi_{p;n_1,\dots,n_r} \colon S_pG \otimes \bigwedge^{n_1} F \otimes \dots \otimes \bigwedge^{n_r} F \to R$$

to be the composition

$$S_pG \otimes \bigwedge^{n_1} F \otimes \dots \otimes \bigwedge^{n_r} F \xrightarrow{\Delta\otimes 1} S_{p+1-r}G \otimes T_{r-1}G \otimes \bigwedge^{n_1} F \otimes \dots \otimes \bigwedge^{n_r} F \to R,$$

where the last map is given by $C_{p+1-r} \otimes c_1^{[1]} \otimes \dots \otimes c_1^{[r-1]} \otimes a_{n_1} \otimes \dots \otimes a_{n_r}$ is sent to

$$\left(\tau(C_{p+1-r}) \wedge \tau(c_1^{[1]} \otimes a_{n_1}) \wedge \dots \wedge \tau(c_1^{[r-1]} \otimes a_{n_{r-1}}) \wedge a_{n_r} \wedge b_{g+1}\right)(\omega_{F^*}).$$

Remark 2.6. When the domain of $\Phi_{p;n_1,\dots,n_r}$ is clear from context, we simply write Φ. If $p \le r-2$ or $r = 0$, then $\Phi_{p;n_1,\dots,n_r}$ is the zero map. We have $\Phi_{0;n_1}(1 \otimes a_{n_1})$ is equal to $[a_{n_1} \wedge b_{g+1}](\omega_{F^*})$.

Definition 2.7. Fix p, q, i, and j with $2p+2i+j+q=2N-1$. The map

$$\Psi_{p,q,i,j}\colon S_pG\otimes \bigwedge^q F\otimes S_iG\otimes \bigwedge^j F\to R$$

is given by

$$\Psi_{p,q,i,j}(C_p\otimes a_q\otimes C_i\otimes a_j)$$

is equal to

$$\sum_{|\mathbf{m}|=q+j}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i}{N-i-j}\frac{(2N-2)!}{(2N-p-i-1)!}\Phi(C_p\cdot C_i\otimes \Delta_{\mathbf{m}}(a_q\wedge a_j)).$$

Remarks 2.8. (a) The sum $\sum\limits_{|\mathbf{m}|=q+j}$ is taken over all vectors $\mathbf{m}$, such that $\mathbf{m}$ is a partition of $q+j$, in the sense of Definition 1.5. The contribution of the partition $\mathbf{m}$ is zero, unless $r(m)\le p+i+1$.

(b) At first glance, it is not completely clear that $\Phi(C_p\cdot C_i\otimes \Delta_{\mathbf{m}}(\underline{\quad}))$ is a well defined function, because some work is required in order to show that

$$\Phi(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_r}\otimes a_{n_{r-1}})=(-1)^{n_rn_{r-1}}\Phi(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_{r-1}}\otimes a_{n_r}).$$

Nonetheless, the equation does hold; see Lemma 3.1, and the function is well defined.

(c) The coefficient in $\Psi_{p,q,i,j}$ is an integer, whenever this map appears in the complex $\mathbb{M}$ of Definition 2.3. Indeed, the only concern occurs when $p+i=0$. However, in this case, $q=N-1$, $j=N$, $\mathbf{m}=\boldsymbol{\epsilon}_{2N-1}$, and the coefficient

$$\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i}{N-i-j}\frac{(2N-2)!}{(2N-p-i-1)!}$$

is equal to $(2N-2)!(-1)^N$.

(d) Assume that the data of 2.1 is homogeneous. In particular, $R=\bigoplus_{0\le i}R_i$ is a graded algebra over the ring R_0, φ is represented by a homogeneous matrix, all of whose entries are in $R_{\boldsymbol{a}}$, and every component of b_{g+1}, with respect to any basis for F, is an element of $R_{\boldsymbol{b}}$. Then $\mathbb{M}$ is a homogeneous complex with $D_iG^*\otimes\bigwedge^j F^*$ equal to

$$R\left(-\Big[[(g+1)i+gj]\boldsymbol{a}+(i+j)\boldsymbol{b}\Big]\right)^{\binom{g+i-1}{i}\binom{f}{j}}$$

and $S_pG\otimes\bigwedge^q F$ equal to

$$R\left(-\Big[[g(f-g-1-p-q)-p]\boldsymbol{a}+(f-g-p-q)\boldsymbol{b}\Big]\right)^{\binom{g+p-1}{p}\binom{f}{q}}.$$

In particular,

$$\mathbb{M}_{f-g}=R(-N(g\boldsymbol{a}+\boldsymbol{a}+\boldsymbol{b}))^{\binom{g+N-1}{N}}\oplus R\left(-\Big[[g(f-g-1)]\boldsymbol{a}+(f-g)\boldsymbol{b}\Big]\right).$$

Example 2.9. When $f-g=2$, then the complex $\mathbb{M}$ of Definition 2.3 is

$$0 \to \begin{matrix} G^* \\ \oplus \\ R \end{matrix} \xrightarrow{d_2} F^* \xrightarrow{d_1} R,$$

where

$$d_1(\alpha_1) = b_1(\alpha_1), \quad d_2(\gamma_1) = \varphi^*(\gamma_1), \quad \text{and} \quad d_2(1) = -b_{g+1}(\omega_{F^*}).$$

The above complex is the Hilbert-Burch complex which is associated to the map d_2; consequenly, this complex is acyclic whenever $2 \le \operatorname{grade} J$. In the notation of Remark 2.8 (d), with $\boldsymbol{a}=\boldsymbol{b}=1$, $\mathbb{M}$ is

$$0 \to R(-[g+2])^{g+1} \to R(-[g+1])^{g+2} \to R.$$

Example 2.10. When $f-g=4$, then the complex $\mathbb{M}$ of Definition 2.3 is

$$0 \to \begin{matrix} D_2G^* \\ \oplus \\ R \end{matrix} \xrightarrow{d_4} \begin{matrix} G^* \otimes F^* \\ \oplus \\ F \end{matrix} \xrightarrow{d_3} \begin{matrix} \bigwedge^2 F^* \\ \oplus \\ G^* \\ \oplus \\ G \end{matrix} \xrightarrow{d_2} F^* \xrightarrow{d_1} R,$$

where

$$d_1(\alpha_1) = b_1(\alpha_1), \quad d_2 \begin{bmatrix} \alpha_2 \\ 0 \\ 0 \end{bmatrix} = b_1(\alpha_2) \quad d_2 \begin{bmatrix} 0 \\ \gamma_1 \\ 0 \end{bmatrix} = \varphi^*(\gamma_1),$$

$$d_2 \begin{bmatrix} 0 \\ 0 \\ c_1 \end{bmatrix} = P(c_1 \otimes \underline{\quad}), \quad d_3 \begin{bmatrix} \gamma_1 \otimes \alpha_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi^*(\gamma_1) \wedge \alpha_1 \\ b_1(\alpha_1) \cdot \gamma_1 \\ 0 \end{bmatrix},$$

$$d_3 \begin{bmatrix} 0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2[a_1 \wedge b_{g+1}](\omega_{F^*}) \\ -P(\underline{\quad} \otimes a_1) \\ \varphi(a_1) \end{bmatrix},$$

$$d_4 \begin{bmatrix} \gamma_1^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_1^{(1)} \otimes \varphi^*(\gamma_1) \\ 0 \end{bmatrix}, \quad \text{and} \quad d_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -P(\underline{\quad} \otimes \underline{\quad}) \\ b_1 \end{bmatrix},$$

where $P\colon G \otimes F \to R$ is given by

$$P(c_1 \otimes a_1) = \left[\left((\textstyle\bigwedge^{g-1} \varphi^*)(c_1[\omega_{G^*}])\right)(b_{g+1}) \wedge a_1 \wedge b_{g+1}\right](\omega_{F^*}).$$

In the notation of Remark 2.8 (d), with $\boldsymbol{a}=\boldsymbol{b}=1$, $\mathbb{M}$ is

$$0 \to \begin{matrix} R(-[2g+4])^{\binom{g+1}{2}} \\ \oplus \\ R(-[3g+4]) \end{matrix} \to \begin{matrix} R(-[2g+3])^{g(g+4)} \\ \oplus \\ R(-[2g+3])^{g+4} \end{matrix} \to \begin{matrix} R(-[2g+2])^{\binom{g+4}{2}} \\ \oplus \\ R(-[g+2])^{g} \\ \oplus \\ R(-[2g+2])^{g} \end{matrix} \to R(-[g+1])^{g+4} \to R.$$

Example 2.11. When $f-g=6$, then the complex $\mathbb{M}$ of Definition 2.3 is

$$0 \to \begin{matrix} D_3G^* \\ \oplus \\ R \end{matrix} \xrightarrow{d_6} \begin{matrix} D_2G^*\otimes F^* \\ \oplus \\ F \end{matrix} \xrightarrow{d_5} \begin{matrix} G^*\otimes \bigwedge^2 F^* \\ \oplus \\ D_2G^* \\ \oplus \\ \bigwedge^2 F \\ \oplus \\ G \end{matrix} \xrightarrow{d_4} \begin{matrix} \bigwedge^3 F^* \\ \oplus \\ G^*\otimes F^* \\ \oplus \\ G\otimes F \end{matrix} \xrightarrow{d_3} \begin{matrix} \bigwedge^2 F^* \\ \oplus \\ G^* \\ \oplus \\ S_2G \end{matrix} \xrightarrow{d_2} F^* \xrightarrow{d_1} R,$$

where

$$d_1(\alpha_1)=b_1(\alpha_1),\quad d_2\begin{bmatrix}\alpha_2\\0\\0\end{bmatrix}=b_1(\alpha_2),\quad d_2\begin{bmatrix}0\\\gamma_1\\0\end{bmatrix}=\varphi^*(\gamma_1),$$

$$d_2\begin{bmatrix}0\\0\\C_2\end{bmatrix}=P_1(C_2\otimes \underline{}),\quad d_3\begin{bmatrix}\alpha_3\\0\\0\end{bmatrix}=\begin{bmatrix}b_1(\alpha_3)\\0\\0\end{bmatrix},$$

$$d_3\begin{bmatrix}0\\\gamma_1\otimes\alpha_1\\0\end{bmatrix}=\begin{bmatrix}\varphi^*(\gamma_1)\wedge\alpha_1\\ b_1(\alpha_1)\cdot\gamma_1\\0\end{bmatrix},\quad d_3\begin{bmatrix}0\\0\\c_1\otimes a_1\end{bmatrix}=\begin{bmatrix}P_2(c_1\otimes a_1\wedge\underline{})\\ -P_1(c_1\cdot\underline{}\otimes a_1)\\ \varphi(a_1)\cdot c_1\end{bmatrix},$$

$$d_4\begin{bmatrix}\gamma_1\otimes\alpha_2\\0\\0\\0\end{bmatrix}=\begin{bmatrix}\varphi^*(\gamma_1)\wedge\alpha_2\\ \gamma_1\otimes b_1(\alpha_2)\\0\end{bmatrix},\quad d_4\begin{bmatrix}0\\\gamma_1^{(2)}\\0\\0\end{bmatrix}=\begin{bmatrix}0\\\gamma_1\otimes\varphi^*(\gamma_1)\\0\end{bmatrix},$$

$$d_4\begin{bmatrix}0\\0\\a_1\wedge a_1'\\0\end{bmatrix}=\begin{bmatrix}-24\Phi_{0;5}(1\otimes a_1\wedge a_1'\wedge\underline{})\\ -P_2(\underline{}\otimes a_1\wedge a_1'\wedge\underline{})\\ \varphi(a_1)\otimes a_1'-\varphi(a_1')\otimes a_1\end{bmatrix},\quad d_4\begin{bmatrix}0\\0\\0\\c_1\end{bmatrix}=\begin{bmatrix}P_2(c_1\otimes\underline{})\\ -2P_1(c_1\cdot\underline{}\otimes\underline{})\\ c_1\otimes b_1\end{bmatrix},$$

$$d_5\begin{bmatrix}\gamma_1^{(2)}\otimes\alpha_1\\0\end{bmatrix}=\begin{bmatrix}\gamma_1\otimes\varphi^*(\gamma_1)\wedge\alpha_1\\ \gamma_1^{(2)}\otimes b_1(\alpha_1)\\0\\0\end{bmatrix},\quad d_5\begin{bmatrix}0\\a_1\end{bmatrix}=\begin{bmatrix}-P_2(\underline{}\otimes a_1\wedge\underline{})\\ 2P_1(\underline{}\otimes a_1)\\ b_1\wedge a_1\\ \varphi(a_1)\end{bmatrix},$$

$$d_6\begin{bmatrix}\gamma_1^{(3)}\\0\end{bmatrix}=\begin{bmatrix}\gamma_1^{(2)}\otimes\varphi^*(\gamma_1)\\0\end{bmatrix},\quad\text{and}\quad d_6\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}P_1(\underline{}\otimes\underline{})\\ b_1\end{bmatrix},$$

where $P_1\colon S_2G\otimes\bigwedge^1 F\to R$ and $P_2\colon S_1G\otimes\bigwedge^3 F\to R$ are given by

$$P_1(C_2\otimes a_1)=-4\Phi_{2;1}(C_2\otimes a_1),\quad\text{and}$$
$$P_2(C_1\otimes a_3)=-6\Phi_{1;3}(C_1\otimes a_3)-2\Phi_{1;1,2}\left(C_1\otimes\Delta_{\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2}(a_3)\right).$$

In the notation of Remark 2.8 (d), with $\boldsymbol{a}=\boldsymbol{b}=1$, $\mathbb{M}$ is

$$0\to \begin{matrix} R(-[3g+6])^{\binom{g+2}{3}}\\ \oplus\\ R(-[5g+6])\end{matrix} \to \begin{matrix} R(-[3g+5])^{\binom{g+1}{2}(g+6)}\\ \oplus\\ R(-[4g+5])^{g+6}\end{matrix} \to \begin{matrix} R(-[3g+4])^{g\binom{g+6}{2}}\\ \oplus\\ R(-[2g+4])^{\binom{g+1}{2}}\\ \oplus\\ R(-[3g+4])^{\binom{g+6}{2}}\\ \oplus\\ R(-[4g+4])^{g}\end{matrix}$$

$$\to \begin{matrix} R(-[3g+3])^{\binom{g+6}{3}}\\ \oplus\\ R(-[2g+3])^{g(g+6)}\\ \oplus\\ R(-[3g+3])^{g(g+6)}\end{matrix} \to \begin{matrix} R(-[2g+2])^{\binom{g+6}{2}}\\ \oplus\\ R(-[g+2])^{g}\\ \oplus\\ R(-[3g+2])^{\binom{g+1}{2}}\end{matrix} \to R(-[g+1])^{g+6}\xrightarrow{d_1} R.$$

Proposition 2.12. *If $g=1$, then the complex $\mathbb{M}$ of Definition 2.3 takes the following form. Let R be a commutative notherian ring and $\varphi\colon F\to R$ be a homomorphism of $R-$modules where F is free of* $\operatorname{rank} F=f=2N+1$. *Fix a generator ω_{F^*} for $\bigwedge^f F^*$. Let $b_2\in\bigwedge^2 F$, $b_1=\varphi(b_2)\in F$, and $J=I_1(b_1)$. The complex $\mathbb{M}$ is*

$$0\to \mathbb{M}_{f-1}\xrightarrow{d_{f-1}}\mathbb{M}_{f-2}\xrightarrow{d_{f-2}}\dots\xrightarrow{d_2}\mathbb{M}_1\xrightarrow{d_1}\mathbb{M}_0,$$

with

$$\mathbb{M}_t=\sum_{\substack{2i+j=t\\ i+j\le N\\ 0\le i}} 1^{(i)}\otimes\bigwedge^j F^* \quad\bigoplus\quad \sum_{\substack{2p+q=f-1-t\\ p+q\le N-1\\ 0\le p}} 1^p\otimes\bigwedge^q F.$$

The differential is given by

$$d\left(1^{(i)}\otimes\alpha_j\right)=1^{(i-1)}\otimes\varphi\wedge\alpha_j+1^{(i)}\otimes b_1(\alpha_j),\ \text{ and}$$

$$d(1^p\otimes a_q)=\begin{cases} 1^p\otimes b_1\wedge a_q+1^{p+1}\otimes\varphi(a_q)\\ +\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{p+i}{N-1-p-q}(2N-2)!1^{(i)}\otimes\left[b_2^{(p+i+1)}\wedge a_q\right](\omega_{F^*}).\end{cases}$$

Furthermore, if $f-1\le$ grade J *and $(2N-2)!$ is a unit in R, then $\mathbb{M}$ is a resolution of R/J.*

Proof. We first show that we have correctly described the complex $\mathbb{M}$ of Definition 2.3, when $g=1$. Fix p, q, i, and j with $2p+2i+j+q=2N-1$. We must show that

$$\begin{aligned}(2.13)\qquad &\Psi_{p,q,i,j}(1^p\otimes a_q\otimes \_\_\otimes\_\_)\\ &\quad=(-1)^{N-i}\binom{p+i}{N-1-p-q}(2N-2)!1^{(i)}\otimes[b_2^{(p+i+1)}\wedge a_q](\omega_{F^*})\end{aligned}$$

in $D_iG^*\otimes\bigwedge^j F^*=1^{(i)}\otimes\bigwedge^j F^*$. Observe that

$$\Phi(1^p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})=\binom{p}{r-1}(r-1)![b_2^{p+1}\wedge\bigwedge_{k=1}^{r} a_{n_k}](\omega_{F^*}),$$

whenever $1 \le r \le p+1$ and $2p+\sum_{k=1}^{r} n_k = 2N-1$. Apply the most recent observation, together with Observation 1.9, in order to see that $\Psi_{p,q,i,j}(1^p \otimes a_q \otimes 1^i \otimes a_j)$ is equal to

$$\sum_{|\mathbf{m}|=q+j} (-1)^{N-i} \binom{p+i}{N-i-j} \frac{(2N-2)!}{(2N-p-i-1)!} \frac{|\mathbf{m}|!}{\prod\limits_{1\le h} m_h!} \binom{p+i}{r(\mathbf{m})-1} (r(\mathbf{m})-1)!$$

times $\left[b_2^{p+i+1} \wedge a_q \wedge a_j\right](\omega_{F^*})$. Recall that $b_2^n = n! b_2^{(n)}$. To complete the proof of (2.13), it suffices to show that

$$\sum_{|\mathbf{m}|=q+j} \frac{(p+i+1)!}{(2N-p-i-1)!} \frac{|\mathbf{m}|!}{\prod\limits_{1\le h} m_h!} \binom{p+i}{r(\mathbf{m})-1} (r(\mathbf{m})-1)! = 1.$$

The left side of the above line is

$$\frac{(q+j)!(p+i)!}{(2N-p-i-1)!} \sum_{\substack{|\mathbf{m}|=2N-1-2p-2i \\ r(\mathbf{m})\le p+i+1}} \frac{(p+i+1)!}{(p+i+1-r(\mathbf{m}))! \cdot \prod\limits_{1\le h} m_h!},$$

and Lemma 2.14 shows that this is equal to 1.

Now we turn to the issue of acyclicity. Assume that $f-1 \le \operatorname{grade} J$. The resolution of R/J from [9] has been recorded as $\mathbb{M}'$ in Proposition 2.15. Consider $\theta\colon \mathbb{M} \to \mathbb{M}'$ given by

$$\begin{cases} \theta(1^{(i)} \otimes \alpha_j) = h^{(i)}\alpha_j \\ \theta(1^p \otimes a_q) = (-1)^{N+1}(2N-2)!\lambda^{(N-1-p-q)}a_q. \end{cases}$$

It is easy to see that θ is a homomorphism of complexes and that θ is an isomorphism whenever $(f-3)!$ is a unit. $\square$

Lemma 2.14. *If P and M are non-negative integers, then*

$$\sum_{\mathbf{m}} \frac{(P+1)!}{(P+1-r(\mathbf{m}))! \prod\limits_{1\le h} m_h!} = \binom{M+P}{P},$$

where the sum is taken over all partitions $\mathbf{m} = \sum m_h \boldsymbol{\epsilon}_h$ with $r(\mathbf{m}) \le P+1$ and $|\mathbf{m}| = M$.

Proof. Compute the coefficient of X^M in

$$(1+X+X^2+X^3+\dots)^{P+1} = \frac{1}{(1-X)^{P+1}} = \sum_{j=0}^{\infty} \binom{P+j}{j} X^j$$

two different ways. $\square$

Proposition 2.15. *Let R be a commutative noetherian ring, N an integer, with $2\le N$, F be a free $R-$module of rank $f=2N+1$, $b_2\in\bigwedge^2 F$, and $\varphi\in F^*$. Fix an orientation element $\omega_{F^*}\in\bigwedge^f F^*$. Let b_1 be the element $\varphi(b_2)$ of F. Define*

$$(\mathbb{M}',d'):\quad 0\to\mathbb{M}'_{f-1}\xrightarrow{d'_{f-1}}\mathbb{M}'_{f-2}\xrightarrow{d'_{f-2}}\dots\xrightarrow{d'_0}\mathbb{M}'_0$$

by

$$\mathbb{M}'_t=\sum_{\substack{2i+j=t\\ i+j\le N}} h^{(i)}\bigwedge^j F^*\quad\bigoplus\quad\sum_{\substack{2p+q=t-2\\ p+q\le N-1}}\lambda^{(p)}\bigwedge^q F,$$

$$d'(h^{(i)}\alpha_j)=h^{(i-1)}\varphi\wedge\alpha_j+h^{(i)}b_1(\alpha_j),\ \textit{and}$$

$$d'(\lambda^{(p)}a_q)=\begin{cases}\lambda^{(p-1)}b_1\wedge a_q+\lambda^{(p)}\varphi(a_q)\\ +\sum\limits_{i\in\mathbb{Z}}(-1)^{i+1}\binom{N+i-q-p-1}{p}h^{(i)}\left(b_2^{(N+i-q-p)}\wedge a_q\right)(\omega_{F^*}).\end{cases}$$

Then $(\mathbb{M}',d')$ is a complex; furthermore, if $f-1\le \operatorname{grade} J$, then $(\mathbb{M}',d')$ is acyclic.

Proof. This result is a combination Propositions 2.6 and 2.16 from [9]. In order to make the notation of [9] compatible with the present notation, we have replaced ξ by ω_{F^*}, φ by b_2, Y by φ, and g by b_1. (One may think of $h^{(i)}$ as the i^{th} divided power of degree two variable or merely as a place holder. One must think of $\lambda^{(p)}$ as a place holder. At any rate, $h^{(i)}$ and $\lambda^{(p)}$ are zero unless the parenthetical exponent is a non-negative integer.) $\square$

Proposition 2.16. *If $g=0$, then the complex $\mathbb{M}$ of Definition 2.3 is isomorphic to $(\mathbb{K},\delta)$, where $\mathbb{K}_i=\bigwedge^i F^*$ for all i, and $\delta_i\colon\mathbb{K}_i\to\mathbb{K}_{i-1}$ is given by*

$$\delta_i(\alpha_i)=\begin{cases}b_1(\alpha_i) & \textit{if } i\ne N+1\\ (f-2)!b_1(\alpha_i) & \textit{if } i\ne N+1.\end{cases}$$

Furthermore, if $f\le\operatorname{grade}J$ and $(f-2)!$ is a unit in R, then $\mathbb{M}$ is a resolution of R/J.

Proof. It is easy to see that

$$\mathbb{M}_t=\begin{cases}\bigwedge^t F^* & \text{if } t\le N,\text{ and}\\ \bigwedge^{f-t}F & \text{if } N+1\le t.\end{cases}$$

The only complicated map in $\mathbb{M}$ is the map $\mathbb{M}_{N+1}\to\mathbb{M}_N$, which is given by

$$a_{N-1}\mapsto\Psi_{0,N-1,0,N}(1\otimes a_{N-1}\otimes 1\otimes \underline{}).$$

Remark 2.6 shows that the contribution to $\Psi_{0,N-1,0,N}$ of the partition $|m|$ of $2N-1$ is zero for all $\mathbf{m}$ except $\mathbf{m}=\boldsymbol{\epsilon}_{2N-1}$. It follows that

$$\Psi_{0,N-1,0,N}(1\otimes a_{N-1}\otimes 1\otimes\underline{})=\pm(2N-2)!\,b_1(a_{N-1}[\omega_{F^*}]).$$

The map $\bigwedge^{f-t}F\to\bigwedge^t F^*$, which is given by $a_{f-t}\mapsto a_{f-t}(\omega_{F^*})$, provides an isomorphism from $\mathbb{M}$ to $\mathbb{K}$. If $(f-2)!$ is a unit in R, then $\mathbb{K}$ is isomorphic to the Koszul complex. $\square$

3. $\mathbb{M}$ is a complex.

Data 2.1 is in effect throughout the entire section. In this section we prove the following two results.

Lemma 3.1. *Fix $C_p \in S_pG$ and $a_i \in \bigwedge^i F$. The functions*

$$\Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r}) \quad \textit{and} \quad \Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r} \wedge \beta_g(a_i))$$

are both graded-symmetric in the terms $a_{n_1}, \ldots, a_{n_r}$.

Lemma 3.2. *The maps and modules of Definition 2.3 form a complex.*

We derive Lemmas 3.1 and 3.2 from Lemmas 3.3 and 3.4; and then we spend the rest of the section establishing Lemmas 3.3 and 3.4.

Lemma 3.3. *Let r, $n_1, \ldots, n_r$, and p be non-negative integers and let J equal $2p + 1 + \sum\limits_{k=1}^{r} n_k$. Consider the two maps*

$$Q_1 \text{ and } Q_2 \colon S_pG \otimes \textstyle\bigwedge^{n_1} F \otimes \ldots \otimes \bigwedge^{n_r} F \to \bigwedge^J F,$$

which are given by

$$S_pG \otimes \textstyle\bigwedge^{n_1} F \otimes \ldots \otimes \bigwedge^{n_r} F \xrightarrow{\Delta\otimes 1} S_{p-r}G \otimes T_rG \otimes \bigwedge^{n_1} F \otimes \ldots \otimes \bigwedge^{n_r} F \xrightarrow{q_1} \bigwedge^J F$$

and

$$S_pG \otimes \textstyle\bigwedge^{n_1} F \otimes \ldots \otimes \bigwedge^{n_r} F \xrightarrow{\Delta\otimes 1} S_{p+1-r}G \otimes T_{r-1}G \otimes \bigwedge^{n_1} F \otimes \ldots \otimes \bigwedge^{n_r} F \xrightarrow{q_2} \bigwedge^J F,$$

respectively, where $q_1 \left(C_{p-r} \otimes c_1^{[1]} \otimes \ldots \otimes c_1^{[r]} \otimes a_{n_1} \otimes \ldots \otimes a_{n_r}\right)$ is equal to

$$b_1 \wedge \tau(C_{p-r}) \wedge \tau(c_1^{[1]} \otimes a_{n_1}) \wedge \ldots \wedge \tau(c_1^{[r]} \otimes a_{n_r})$$

and $q_2 \left(C_{p+1-r} \otimes c_1^{[1]} \otimes \ldots \otimes c_1^{[r-1]} \otimes a_{n_1} \otimes \ldots \otimes a_{n_r}\right)$ is equal to

$$\sum_{\ell=1}^{r} \tau(C_{p+1-r}) \wedge \bigwedge_{k=1}^{\ell-1} (-1)^{n_k} \tau(c_1^{[k]} \otimes a_{n_k}) \wedge \beta_g(b_{g+1} \wedge a_{n_\ell}) \wedge \bigwedge_{k=\ell}^{r-1} \tau(c_1^{[k]} \otimes a_{n_{k+1}}).$$

If $J = 2N$, then $\beta_g\left[\left(\Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r})\right)(\omega_F)\right]$ is equal to

$$(-1)^{g+1}[Q_1(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r}) + Q_2(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r})].$$

Lemma 3.4. *If $2p+2i+q+j=2N-2$, $a_q=a_1^{[1]}\wedge\ldots\wedge a_1^{[q]}$, and a_j is equal to $a_1^{[q+1]}\wedge\ldots\wedge a_1^{[q+j]}$, then*

$$0=\begin{cases}\Psi_{p,q+1,i,j}(C_p\otimes b_1\wedge a_q\otimes C_i\otimes a_j)\\ +\Psi_{p,q,i,j+1}(C_p\otimes a_q\otimes C_i\otimes a_j\wedge b_1)\\ +\sum\limits_{k=1}^{q}(-1)^{k+1}\Psi_{p+1,q-1,i,j}(\varphi(a_1^{[k]})\cdot C_p\otimes a_1^{[1]}\wedge\ldots\wedge\widehat{a_1^{[k]}}\wedge\ldots\wedge a_1^{[q]}\otimes C_i\otimes a_j)\\ +\sum\limits_{k=q+1}^{q+j}(-1)^{k}\Psi_{p,q,i+1,j-1}(C_p\otimes a_q\otimes C_i\cdot\varphi(a_1^{[k]})\otimes a_1^{[q+1]}\wedge\ldots\wedge\widehat{a_1^{[k]}}\wedge\ldots\wedge a_1^{[q+j]}).\end{cases}$$

Proof of Lemma 3.1. Definition 2.5 shows that both maps are graded-symmetric in the terms $a_{n_1},\ldots,a_{n_{r-1}}$. Let α_0 be

$$\Phi(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_r}\otimes a_{n_{r-1}})-(-1)^{n_rn_{r-1}}\Phi(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_{r-1}}\otimes a_{n_r}).$$

It suffices to show that $\alpha_0=0$ and that

$$\begin{aligned}&\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_r}\otimes a_{n_{r-1}}\wedge\beta_g(a_i)\right)\\ &=(-1)^{n_{r-1}n_r}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-2}}\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge\beta_g(a_i)\right).\end{aligned}$$

The second equation follows from the first equation together with the fact that

$$\tau(c_1\otimes a_n\wedge\beta_g(a_i))=\tau(c_1\otimes a_n)\wedge\beta_g(a_i). \tag{3.5}$$

If we prove that $\alpha_0=0$ whenever R is a polynomial ring of the form $\mathbb{Z}[X_1,\ldots,X_m]$, then we may apply a ring homomorphism in order to conclude that $\alpha_0=0$ over an arbitrary commutative ring. Consequently, we assume that R is a domain and β_g is a non-zero element of $\bigwedge^g F^*$. Proposition 1.1 (b) or (c) shows that $[\beta_g(\alpha_0(\omega_F))](\omega_{F^*})=\alpha_0\cdot\beta_g$; therefore, it suffices to show that $\beta_g(\alpha_0(\omega_F))=0$. This statement is established by Lemma 3.3. $\square$

Proof of Lemma 3.2. The only interesting part of this calculation is the composition

$$S_pG\otimes\bigwedge^q F\xrightarrow{\text{incl}}\mathbb{M}_t\xrightarrow{d}\mathbb{M}_{t-1}\xrightarrow{d}\mathbb{M}_{t-2}\xrightarrow{\text{proj}}D_iG^*\otimes\bigwedge^j F^*,$$

where $p+q\le N-1$, $i+j\le N$, and $2i+j-2=2N-2p-q$. Let $C_p\otimes a_q\in S_pG\otimes\bigwedge^q F$ and $C_i\otimes a'_j\in S_iG\otimes\bigwedge^j F$. Write $a_q=a_1^{[1]}\wedge\ldots\wedge a_1^{[q]}$ and $a'_j=a_1'^{[1]}\wedge\ldots\wedge a_1'^{[j]}$. Observe that when the element

$$(\text{proj}\circ d\circ d\circ\text{incl})(C_p\otimes a_q)$$

of $D_iG^*\otimes\bigwedge^j F^*$ is applied to $C_i\otimes a'_j$, the result is

$$\begin{cases}\Psi_{p,q+1,i,j}(C_p\otimes b_1\wedge a_q\otimes C_i\otimes a'_j)\\ +\sum\limits_{k=1}^{q}(-1)^{k+1}\Psi_{p+1,q-1,i,j}(\varphi(a_1^{[k]})\cdot C_p\otimes a_1^{[1]}\wedge\ldots\wedge\widehat{a_1^{[k]}}\wedge\ldots\wedge a_1^{[q]}\otimes C_i\otimes a_j)\\ +\sum\limits_{k=1}^{j}(-1)^{k+j}\Psi_{p,q,i+1,j-1}(C_p\otimes a_q\otimes C_i\cdot\varphi(a_1'^{[k]})\otimes a_1'^{[1]}\wedge\ldots\wedge\widehat{a_1'^{[k]}}\wedge\ldots\wedge a_1'^{[j]})\\ +\Psi_{p,q,i,j+1}(C_p\otimes a_q\otimes C_i\otimes a'_j\wedge b_1).\end{cases}$$

It is clear that the top two lines are correct. The third line is the image of $\Psi_{p,q,i+1,j-1}(C_p \otimes a_q \otimes \_\_ \otimes \_\_) \in D_{i+1}G^* \otimes \bigwedge^{j-1} F^*$ under the composition

$$\gamma_1^{(i+1)} \otimes \alpha_{j-1} \mapsto \gamma_1^{(i)} \otimes \varphi^*(\gamma_1) \wedge \alpha_{j-1} \mapsto \gamma_1^{(i)}(C_i) \cdot [\varphi^*(\gamma_1) \wedge \alpha_{j-1}](a'_j) \in R.$$

The fourth line is the image of $\Psi_{p,q,i,j+1}(C_p \otimes a_q \otimes \_\_ \otimes \_\_) \in D_iG^* \otimes \bigwedge^{j+1} F^*$ under the composition

$$\gamma_1^{(i)} \otimes \alpha_{j+1} \mapsto \gamma_1^{(i)} \otimes b_1(\alpha_{j+1}) \mapsto \gamma_1^{(i)}(C_i) \cdot [b_1(\alpha_{j+1})](a'_j) \in R.$$

Lemma 3.4 completes the proof. $\square$

The next four results are used in the proof of Lemma 3.3.

Lemma 3.6. *If α_{g-1} and α'_{g-1} are in the image of $\bigwedge^{g-1} \varphi^*$, then*

(a) $\alpha_{g-1}(\alpha_{g-1}(a_i) \wedge a_j) = (-1)^{(g+1)(i+1)}\alpha_{g-1}(a_i) \wedge \alpha_{g-1}(a_j)$, *and*

(b) $\alpha_{g-1}(\alpha'_{g-1}(a_i) \wedge a_j) + \alpha'_{g-1}(\alpha_{g-1}(a_i) \wedge a_j)$
$= (-1)^{(g+1)(i+1)}[\alpha_{g-1}(a_i) \wedge \alpha'_{g-1}(a_j) + \alpha'_{g-1}(a_i) \wedge \alpha_{g-1}(a_j)]$,

for all $a_i \in \bigwedge^i F$ and $a_j \in \bigwedge^j F$.

Proof. Assertion (b) is obtained by applying (a) to $\alpha_{g-1} + \alpha'_{g-1}$. To prove (a), we let $\gamma_1^{[1]}, \dots, \gamma_1^{[g]}$ be a basis of G^*, and $\alpha_1^{[i]} = \varphi^*(\gamma_1^{[i]})$. The general case quickly boils down to the case $\alpha_{g-1} = \alpha_{g-2} \wedge \alpha_1^{[g-1]} + \alpha_{g-2} \wedge \alpha_1^{[g]}$, where $\alpha_{g-2} = \alpha_1^{[1]} \wedge \dots \wedge \alpha_1^{[g-2]}$. In this case the assertion is obvious. $\square$

Corollary 3.7. *If α_{g-1} is in the image of $\bigwedge^{g-1} \varphi^*$, then*

$$\beta_g(\alpha_{g-1}(a_i) \wedge a_j) = (-1)^{(g+1)(i+1)}\beta_g(a_i) \wedge \alpha_{g-1}(a_j) + (-1)^{gi}\alpha_{g-1}(a_i) \wedge \beta_g(a_j),$$

for all $a_i \in \bigwedge^i F$ and $a_j \in \bigwedge^j F$.

Proof. Let $\alpha_{g-1} = (\bigwedge^{g-1} \varphi^*)(c_1[\omega_{G^*}])$ for some $c_1 \in G$. The proposed equation is linear in c_1; consequently, it suffices to establish the equation when c_1 is part of a basis for G. In this case, there exists α_1 such that $\alpha_1 \wedge \alpha_{g-1} = \beta_g$. Apply Lemma 3.6 to complete the proof. $\square$

Corollary 3.8. *If $\alpha_{g-1}^{[1]}, \dots, \alpha_{g-1}^{[s]}$ are in the image of $\bigwedge^{g-1} \varphi^*$, then*

$$\beta_g\left(\alpha_{g-1}^{[1]}(b_{g+1}) \wedge \dots \wedge \alpha_{g-1}^{[s]}(b_{g+1}) \wedge a_j\right)$$

is equal to

$$\begin{cases} \sum_{\ell=1}^{s} b_1 \wedge \alpha_{g-1}^{[1]}(b_{g+1}) \wedge \dots \wedge \widehat{\alpha_{g-1}^{[\ell]}(b_{g+1})} \wedge \dots \wedge \alpha_{g-1}^{[s]}(b_{g+1}) \wedge \alpha_{g-1}^{[\ell]}(a_j) \\ +\alpha_{g-1}^{[1]}(b_{g+1}) \wedge \dots \wedge \alpha_{g-1}^{[s]}(b_{g+1}) \wedge \beta_g(a_j), \end{cases}$$

for all $a_j \in \bigwedge^j F$.

Proof. The result follows from repeated application of Corollary 3.7 together with the observation that

$$b_1 \wedge \alpha_{g-1}(\alpha'_{g-1}(b_{g+1}) \wedge a_j) = b_1 \wedge \alpha'_{g-1}(b_{g+1}) \wedge \alpha_{g-1}(a_j)$$

for α_{g-1} and α'_{g-1} in the image of $\bigwedge^{g-1} \varphi^*$. This last observation holds because if α_1 is in the image of φ^*, then $\alpha_1(\alpha'_{g-1}(b_{g+1})) = r\, b_1$ for some $r \in R$. $\square$

Definition 3.9. Let $t\colon G\otimes\bigwedge^i F\to\bigwedge^{i-g+1}F$ be given by

$$t(c_1\otimes a_i)=\left[(\textstyle\bigwedge^{g-1}\varphi^*)(c_1[\omega_{G^*}])\right](a_i).$$

Remark. Notice that $\tau(c_1\otimes a_i)=t(c_1\otimes b_{g+1}\wedge a_i)$.

Lemma 3.10. *Fix a positive integer r. If $c_1^{[k]}\in G$, $a_{n_k}\in\bigwedge^{n_k}F$, and $m_\ell=\sum\limits_{k=1}^{\ell-1}n_k$, then*

(a) $\sum\limits_{\sigma\in\mathfrak{S}_r} t\left(c_1^{[\sigma(1)]}\otimes\bigwedge\limits_{k=2}^{r}t(c_1^{[\sigma(k)]}\otimes a_{n_{k-1}})\wedge a_{n_r}\right)$ *is equal to*

$$(-1)^{(g+1)(m_r+r+1)}\sum_{\sigma\in\mathfrak{S}_r}\bigwedge_{k=1}^{r}t(c_1^{[\sigma(k)]}\otimes a_{n_k}),\quad \textit{and}$$

(b) $\sum\limits_{\sigma\in\mathfrak{S}_{r-1}}\beta_g\left(\bigwedge\limits_{k=1}^{r-1}t(c_1^{[\sigma(k)]}\otimes a_{n_k})\wedge a_{n_r}\right)$ *is equal to*

$$\sum_{\sigma\in\mathfrak{S}_{r-1}}\sum_{\ell=1}^{r}(-1)^{(g+1)(m_r+r+\ell)+m_\ell}\bigwedge_{k=1}^{\ell-1}t(c_1^{[\sigma(k)]}\otimes a_{n_k})\wedge\beta_g(a_{n_\ell})\wedge\bigwedge_{k=\ell}^{r-1}t(c_1^{[\sigma(k)]}\otimes a_{n_{k+1}}).$$

Proof. Both results are obvious for $r=1$. (See Convention 1.12.) Lemma 3.6 establishes (a) when $r=2$. The proof of (a) is completed by induction on r. Corollary 3.7 shows that the left side of (b) is equal to $L'+L''$, where

$$L'=(-1)^{(g+1)(n_1+1)}\sum_{\sigma\in\mathfrak{S}_{r-1}}\beta_g(a_{n_1})\wedge t\left(c_1^{[\sigma(1)]}\otimes\bigwedge_{k=2}^{r-1}t(c_1^{[\sigma(k)]}\otimes a_{n_k})\wedge a_{n_r}\right)$$

and

$$L''=(-1)^{gn_1}\sum_{\sigma\in\mathfrak{S}_{r-1}}t(c_1^{[\sigma(1)]}\otimes a_{n_1})\wedge\beta_g\left(\bigwedge_{k=2}^{r-1}t(c_1^{[\sigma(k)]}\otimes a_{n_k})\wedge a_{n_r}\right).$$

Complete the proof by applying (a) to L' and induction to L''. □

Proof of Lemma 3.3. Fix $C_p,a_{n_1},\dots,a_{n_r}$, with $J=2N$. Let

$$Q_0=\beta_g\left[(\Phi(C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r}))(\omega_F)\right]\quad\text{and}\quad Q_i=Q_i(C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r}),$$

for $i=1,2$. If

$$\sum_i C_{p+1-r}^{[i]}\otimes c_1^{[i,1]}\otimes\dots\otimes c_1^{[i,r-1]}$$

is the image of C_p under $\Delta\colon S_pG\to S_{p+1-r}G\otimes T_{r-1}G$, then Definition 2.5 shows that Q_0 is equal to

$$\sum_i\beta_g\left(\tau(C_{p+1-r}^{[i]})\wedge\tau(c_1^{[i,1]}\otimes a_{n_1})\wedge\dots\wedge\tau(c_1^{[i,r-1]}\otimes a_{n_{r-1}})\wedge a_{n_r}\wedge b_{g+1}\right).$$

Let $\sum_h C^{[i;h]}_{p-r} \otimes c^{[i;h]}_1$ be the image of $C^{[i]}_{p+1-r}$ under $\Delta\colon S_{p+1-r}G \to S_{p-r}G \otimes S_1G$. Apply Corollary 3.8 to see that $Q_0 = Q_0' + Q_0''$, where Q_0' is equal to

$$\sum_{i,h} b_1 \wedge \tau(C^{[i;h]}_{p-r}) \wedge t\left(c^{[i;h]}_1 \otimes \tau(c^{[i,1]}_1 \otimes a_{n_1}) \wedge \ldots \wedge \tau(c^{[i,r-1]}_1 \otimes a_{n_{r-1}}) \wedge a_{n_r} \wedge b_{g+1}\right)$$

and Q_0'' is equal to

$$\sum_i \tau(C^{[i]}_{p+1-r}) \wedge \beta_g \left(\tau(c^{[i,1]}_1 \otimes a_{n_1}) \wedge \ldots \wedge \tau(c^{[i,r-1]}_1 \otimes a_{n_{r-1}}) \wedge a_{n_r} \wedge b_{g+1}\right).$$

Co-multiplication is co-associative; thus,

$$\sum_{i,h} C^{[i;h]}_{p-r} \otimes c^{[i;h]}_1 \otimes c^{[i,1]}_1 \otimes \ldots \otimes c^{[i,r-1]}_1 = \sum_j C^{[j]}_{p-r} \otimes c^{[j,1]}_1 \otimes \ldots \otimes c^{[j,r]}_1,$$

where the right side is the image of C_p under $\Delta\colon S_pG \to S_{p-r}G \otimes T_rG$. It follows that Q_0' is equal to $(-1)^{(g+1)n_r}$ times

$$\sum_j b_1 \wedge \tau(C^{[j]}_{p-r}) \wedge t\left(c^{[j,1]}_1 \otimes \tau(c^{[j,2]}_1 \otimes a_{n_1}) \wedge \ldots \wedge \tau(c^{[j,r]}_1 \otimes a_{n_{r-1}}) \wedge b_{g+1} \wedge a_{n_r}\right).$$

Recall that $\sum\limits_{k=1}^{r} n_k$ is odd. Apply Lemma 3.10 to see that $Q_0' = (-1)^{g+1}Q_1$ and $Q_0'' = (-1)^{g+1}Q_2$. $\square$

We now gather the definitions and lemmas which are used in the proof of Lemma 3.4.

Lemma 3.11. *If $2p + \sum\limits_{k=1}^{r} n_k = 2N-1$ and $a_{n_i} = 1$ for some i, then*

$$\Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r}) = (p+2-r)\Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes \widehat{a_{n_i}} \otimes \ldots \otimes a_{n_r}).$$

Proof. The hypothesis forces $2 \le r$. In light of Lemma 3.1, we may assume that $i = 1 < r$. The co-associativity of Δ shows that the composition

$$S_pG \xrightarrow{\Delta} S_{p+2-r}G \otimes T_{r-2}G \xrightarrow{\Delta\otimes 1} S_{p+1-r}G \otimes G \otimes T_{r-2}G \tag{3.12}$$

is equal to

$$S_pG \xrightarrow{\Delta} S_{p+1-r}G \otimes T_{r-1}G. \tag{3.13}$$

The first map in (3.12) is the first step in the calculation of $\Phi(C_p \otimes a_{n_2} \otimes \ldots \otimes a_{n_r})$. The map of (3.13) is the first step in the calculation of $\Phi(C_p \otimes a_{n_1} \otimes \ldots \otimes a_{n_r})$. We complete the proof by observing that if $C_{p+2-r} = c^{[1]}_1 \cdots c^{[p+2-r]}_1$ in $S_{p+2-r}G$, then $\Delta(C_{p+2-r})$ is equal to $\sum_j c^{[1]}_1 \cdots \widehat{c^{[j]}_1} \cdots c^{[p+2-r]}_1 \otimes c^{[j]}_1$ in $S_{p+1-r}G \otimes G$ and

$$\sum_j \tau(c^{[1]}_1 \cdots \widehat{c^{[j]}_1} \cdots c^{[p+2-r]}_1) \wedge \tau(c^{[j]}_1 \otimes a_{n_1}) = (p+2-r)\tau(C_{p+2-r}). \quad \square$$

Definition 3.14. Let r, i, $n_1,\dots,n_r$, and p be non-negative integers with $2\le r$ and $2p+i-g+\sum\limits_{h=1}^{r} n_h=2N-1$. Define the maps

$$\Xi_k,\ \Xi_{\boldsymbol{\epsilon}_h},\ \text{and } \Xi_{\text{all}}\colon \bigwedge^i F\otimes S_pG\otimes \bigwedge^{n_1}F\otimes\dots\otimes\bigwedge^{n_r}F\to R$$

by $\Xi_k(a_i\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})$ is equal to

$$\begin{cases}\Phi(C_p\otimes \bigotimes\limits_{h=1}^{k-1}a_{n_h}\otimes \beta_g(a_i\wedge a_{n_k})\wedge a_{n_{k+1}}\otimes \bigotimes\limits_{h=k+2}^{r}a_{n_h}) & \text{if } 1\le k\le r-1\\ \Phi(C_p\otimes \bigotimes\limits_{h=1}^{r-2}a_{n_h}\otimes a_{r-1}\wedge\beta_g(a_i\wedge a_{n_r})) & \text{if } k=r\\ 0 & \text{for any other } k,\end{cases}$$

$\Xi_{\boldsymbol{\epsilon}_h}(a_i\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})$ is equal to

$$\sum_{k=1}^{r}\chi(n_k=h)(-1)^{(i-g)(n_k+\dots+n_r)}\Xi_k(a_i\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r}),$$

and $\Xi_{\text{all}}=\sum\limits_h \Xi_{\boldsymbol{\epsilon}_h}$.

Remark. The observation of (3.5) shows that $\Xi_k(a_i\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})$ is equal to

$$(-1)^{(n_k+i-g)(n_{k+1}+\dots+n_r)}\Phi(C_p\otimes a_{n_1}\otimes\dots\otimes\widehat{a_{n_k}}\otimes\dots\otimes a_{n_r}\wedge\beta_g(a_i\wedge a_{n_k})),$$

for $1\le k\le r-1$.

Lemma 3.15. *If r, $n_1,\dots,n_r$, and p are non-negative integers with $2\le r$ and $2p+\sum\limits_{h=1}^{r}n_h=2N-2$, then*

$$\Phi(C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge b_1)+\Xi_{\text{all}}(b_{g+1}\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})=0.$$

Proof. The argument below (3.5) shows that it suffices to prove that $T_1+T_2=0$, where

$$T_1=(-1)^{g+1}\beta_g([\Phi(C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge b_1)](\omega_F))\quad\text{and}$$
$$T_2=(-1)^{g+1}\beta_g([\Xi_{\text{all}}(b_{g+1}\otimes C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})](\omega_F)).$$

Apply Lemma 3.3 to see that $T_1=Q_2(C_p\otimes a_{n_1}\otimes\dots\otimes a_{n_r})\wedge b_1$ and $T_2=T_2^{[1]}+T_2^{[2]}$, where

$$T_2^{[i]}=\sum_{k=1}^{r}(-1)^{n_k(1+n_{k+1}+\dots+n_r)}Q_i(C_p\otimes a_{n_1}\otimes\dots\otimes\widehat{a_{n_k}}\otimes\dots\otimes a_{n_r})\wedge\beta_g(b_{g+1}\wedge a_{n_k}).$$

A straightforward calculation, using the definition of the maps Q_i, shows that $T_1+T_2^{[1]}=0$ and $T_2^{[2]}=0$. $\square$

Remark 3.16. When $r=1$, Lemma 3.15 becomes: if $2p+n_1=2N-2$, then

$$\Phi(C_p\otimes\beta_g(b_{g+1}\wedge a_{n_1}))=(-1)^{n_1+1}(p+1)\Phi(C_p\otimes a_{n_1}\wedge b_1).$$

(The proof is obtained by combining Corollary 3.8 and the trick below (3.5).) In particular, if $p=N-1$, then $\Phi(C_p\otimes b_1)=0$.

Lemma 3.17. *If* $2p+\sum_{k=1}^{r} n_k = 2N-3$*, then* $\Phi\left(C_p\cdot\varphi(a_1)\otimes a_{n_1}\otimes\ldots\otimes a_{n_r}\right)$ *is equal to*

$$\begin{cases}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_r}\wedge\beta_g(b_{g+1}\wedge a_1)\right)\\ +\chi(2\le r)\cdot\Xi_{\text{all}}(b_{g+1}\wedge a_1\otimes C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_r})\\ +\chi(r=1)\cdot\Phi\left(C_p\otimes a_{n_1}\wedge a_1\wedge b_1\right).\end{cases}$$

Proof. Definition 2.5, together with Proposition 1.1 (d), shows that the left side is

$$\begin{cases}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_r}\wedge[a_1(\beta_g)](b_{g+1})\right)\\ +\sum_{k=1}^{r-1}(-1)^{n_k(n_{k+1}+\cdots+n_r)}\Phi\Big(C_p\otimes a_{n_1}\otimes\ldots\otimes\widehat{a_{n_k}}\otimes\ldots\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge[a_1(\beta_g)](b_{g+1}\wedge a_{n_k})\Big)\end{cases}$$

Apply Proposition 1.1 (a) to write

$$[a_1(\beta_g)](b_{g+1}\wedge a_n)=a_1\wedge\beta_g(b_{g+1}\wedge a_n)+\beta_g(b_{g+1}\wedge a_1\wedge a_n).$$

The proof is complete when $r=1$. Henceforth, we assume that $2\le r$. Lemma 3.15 shows that

$$\begin{cases}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_r}\wedge a_1\wedge b_1\right)\\ +\sum_{k=1}^{r-1}(-1)^{n_k(n_{k+1}+\cdots+n_r)}\Phi\Big(C_p\otimes a_{n_1}\otimes\ldots\otimes\widehat{a_{n_k}}\otimes\ldots\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge a_1\wedge\beta_g(b_{g+1}\wedge a_{n_k})\Big)\end{cases}$$

is equal to

$$(-1)^{n_r}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_{r-1}}\wedge\beta_g(b_{g+1}\wedge a_{n_r}\wedge a_1)\right).$$

When the most recent expression is added to

$$\sum_{k=1}^{r-1}(-1)^{n_k(n_{k+1}+\cdots+n_r)}\Phi\left(C_p\otimes a_{n_1}\otimes\ldots\otimes\widehat{a_{n_k}}\otimes\ldots\otimes a_{n_{r-1}}\otimes a_{n_r}\wedge\beta_g(b_{g+1}\wedge a_1\wedge a_{n_k})\right),$$

the result is $\Xi_{\text{all}}(b_{g+1}\wedge a_1\otimes C_p\otimes a_{n_1}\otimes\ldots\otimes a_{n_r})$. □

Lemma 3.18. *If* $1\le j$*,* $2p+j=2N-2$*, and* $\mathbf{m}=[m_1,\ldots,m_\ell]$ *is a vector of non-negative integers with* $|\mathbf{m}|=j+1$*, in the sense of Definition 1.5, then* $\Phi(C_p\otimes\Delta_{\mathbf{m}}(a_j\wedge b_1))$ *is equal to*

$$\begin{cases}(p+2-r(\mathbf{m}))\Phi\left(C_p\otimes\Delta_{\mathbf{m}-\boldsymbol{\epsilon}_1}(a_j)\wedge b_1\right)\\ +\sum_{1\le k}(m_k+1)\Phi\left(C_p\otimes\Delta_{\mathbf{m}+\boldsymbol{\epsilon}_k-\boldsymbol{\epsilon}_{k+1}}(a_j)\wedge b_1\right).\end{cases}$$

Remark. Recall, from Definition 1.6, that $\Delta_{\mathbf{m}-\boldsymbol{\epsilon}_1}$ is the zero map unless $1\le m_1$. If $X=x_{n_1}\otimes\ldots\otimes x_{n_r}$, then $X\wedge b_1$ means $x_{n_1}\otimes\ldots\otimes x_{n_r}\wedge b_1$. Lemma 3.1 guarantees that the expression $\Phi\left(C_p\otimes\Delta_{\mathbf{m}-\boldsymbol{\epsilon}_1}(a_j)\wedge b_1\right)$ is well defined.

Proof. In the notation of Observation 1.10, we have $\Phi(C_p\otimes\Delta_{\mathbf{m}}(a_j\wedge b_1))$ is equal to

$$(-1)^j\sum_{h,i}(-1)^{|\boldsymbol{\ell}(h-1)|}\Phi(C_p\otimes a_-^{[h,i]}\otimes b_1\wedge a^{[h,i]}\otimes a_+^{[h,i]}),$$

which, by (3.5), is equal to

$$\sum_{h,i} \Phi(C_p \otimes a_-^{[h,i]} \otimes a^{[h,i]} \otimes a_+^{[h,i]} \wedge b_1) = T_1 + T_2,$$

where T_1 is the summand with $h = 1$ and T_2 is the summand with $2 \le h$. Apply Lemma 3.11 and Observation 1.8 to see that

$$\begin{aligned} T_1 &= (p + 2 - r(\mathbf{m}))\Phi\left(C_p \otimes \Delta_{\mathbf{m}-\boldsymbol{\epsilon}_1}(a_j) \wedge b_1\right) \quad \text{and} \\ T_2 &= \sum_{2 \le h} (m_{h-1} + 1)\Phi\left(C_p \otimes \Delta_{\mathbf{m}+\boldsymbol{\epsilon}_{h-1}-\boldsymbol{\epsilon}_h}(a_j) \wedge b_1\right). \quad \square \end{aligned}$$

Lemma 3.19. *Fix h, i, j, p, and $\mathbf{m}$ with $|\mathbf{m}| = j$ and $2p + j + i - g = 2N - 1$. Let $Q\colon \bigwedge^i F \otimes S_pG \otimes \bigwedge^j F \to R$ be the composition*

$$\bigwedge^i F \otimes S_pG \otimes \bigwedge^j F \xrightarrow{1\otimes 1\otimes \Delta} \bigwedge^i F \otimes S_pG \otimes \bigwedge^{j-h} F \otimes \bigwedge^h F \xrightarrow{q} R,$$

where $q(a_i \otimes C_p \otimes a_{j-h} \otimes a_h) = \Phi(C_p \otimes \Delta_{\mathbf{m}-\boldsymbol{\epsilon}_h}(a_{j-h}) \wedge \beta_g(a_i \wedge a_h))$. Then

$$\Xi_{\boldsymbol{\epsilon}_h}(a_i \otimes C_p \otimes \Delta_{\mathbf{m}}(a_j)) = (-1)^{(i-g)h} Q(a_i \otimes C_p \otimes a_j).$$

Proof. The result follows quickly from Definition 3.14. $\square$

Corollary 3.20. *If $|\mathbf{m}| = j - 1$, $2p + j = 2N - 2$, and $a_j = a_1^{[1]} \wedge \dots \wedge a_1^{[j]}$, then*

$$\sum_{k=1}^{j} (-1)^{k+1} \Phi(C_p \otimes \Delta_{\mathbf{m}}(a_1^{[1]} \wedge \dots \wedge \widehat{a_1^{[k]}} \wedge \dots \wedge a_1^{[j]}) \wedge \beta_g(b_{g+1} \wedge a_1^{[k]}))$$

is equal to $(-1)^j \Xi_{\boldsymbol{\epsilon}_1}(b_{g+1} \otimes C_p \otimes \Delta_{\mathbf{m}+\boldsymbol{\epsilon}_1}(a_j))$.

Proof. Apply Lemma 3.19. $\square$

Lemma 3.21. *If $|\mathbf{m}| = j - 1$, $2p + j = 2N - 2$, $1 \le h$, and $a_j = a_1^{[1]} \wedge \dots \wedge a_1^{[j]}$, then*

$$\sum_{k=1}^{j} (-1)^{k+1} \Xi_{\boldsymbol{\epsilon}_h}(b_{g+1} \wedge a_1^{[k]} \otimes C_p \otimes \Delta_{\mathbf{m}}(a_1^{[1]} \wedge \dots \wedge \widehat{a_1^{[k]}} \wedge \dots \wedge a_1^{[j]}))$$

is equal to $(-1)^j (h + 1) \Xi_{\boldsymbol{\epsilon}_{h+1}}(b_{g+1} \otimes C_p \otimes \Delta_{\mathbf{m}-\boldsymbol{\epsilon}_h+\boldsymbol{\epsilon}_{h+1}}(a_j))$.

Proof. Use Lemma 3.19 to convert the result into the following commutative diagram:

$$\begin{array}{ccc} \bigwedge^j F & \xrightarrow{\Delta} & \bigwedge^{j-1-h} F \otimes \bigwedge^1 F \otimes \bigwedge^h F \\ \Delta \Big\downarrow & & \Big\downarrow 1\otimes\text{mult} \\ \bigwedge^{j-1-h} F \otimes \bigwedge^{h+1} F & \xrightarrow{h+1} & \bigwedge^{j-1} \otimes \bigwedge^{h+1} F. \end{array} \quad \square$$

Proof of Lemma 3.4. Observe that $q+j$ is even. Let S be the sum of the top two lines of the right side of the proposed identity and T be the sum of the bottom two lines. Use Definition 2.7, together with $\binom{p+i}{N-i-j-1}+\binom{p+i}{N-i-j}=\binom{p+i+1}{N-i-j}$, in order to see that S is equal to

$$\sum_{|\mathbf{m}|=q+j+1}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-1)!}\Phi(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j\wedge b_1)).$$

Lemma 3.18 shows S that is equal to

$$\begin{cases}\sum\limits_{|\mathbf{m}|=q+j}\prod\limits_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-1)!}(p+i+1-r(\mathbf{m}))\Phi\big(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)\wedge b_1\big)\\ +\sum\limits_{|\mathbf{m}|=q+j}\sum\limits_{1\le k}\prod\limits_{1\le h}(h!)^{m_h}(k+1)(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-1)!}m_k\Phi\big(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)\wedge b_1\big)\end{cases}$$

Recall that $\sum(k+1)m_k=|\mathbf{m}|+r(\mathbf{m})=q+j+r(\mathbf{m})$ and that $p+i+1+q+j$ is equal to $2N-p-i-1$. It follows that S is equal to

$$\sum_{|\mathbf{m}|=q+j}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Phi\left(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)\wedge b_1\right).$$

Write $S=S_1+S_2$, where S_1 has $\mathbf{m}=\boldsymbol{\epsilon}_{q+j}$ and S_2 involves all other $\mathbf{m}$. In other words, S_1 is

$$(q+j)!\,(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Phi\left(C_p\cdot C_i\otimes a_q\wedge a_j\wedge b_1\right)$$

and S_2 is

$$\sum_{\substack{|\mathbf{m}|=q+j\\ 2\le r(\mathbf{m})}}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Phi\left(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)\wedge b_1\right).$$

If $q+j=0$, then $T=0$ and $S=S_1$; furthermore, Remark 3.16 shows that $S_1=0$. Henceforth, we assume that $2\le q+j$.

Definition 2.7 yields that T is equal to

$$\sum_{k=1}^{j+q}(-1)^{k+1+N-i}\sum_{|\mathbf{m}|=q+j-1}\prod_{1\le h}(h!)^{m_h}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Phi(\varphi(a_1^{[k]})\cdot C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_1^{[1]}\wedge\dots\wedge\widehat{a_1^{[k]}}\wedge\dots\wedge a_1^{[q+j]})).$$

Apply Lemma 3.17 to write $T=T_1+T_2+T_3$, where T_1 is

$$\sum_{k=1}^{j+q}(-1)^{k+1+N-i}(q+j-1)!\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Phi(C_p\cdot C_i\otimes a_1^{[1]}\wedge\dots\wedge\widehat{a_1^{[k]}}\wedge\dots\wedge a_1^{[q+j]}\wedge a_1^{[k]}\wedge b_1),$$

$$T_2=\begin{cases}\sum\limits_{k=1}^{j+q}(-1)^{k+1}\sum\limits_{|\mathbf{m}|=q+j-1}\prod\limits_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\\ \qquad\Phi(C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_1^{[1]}\wedge\dots\wedge\widehat{a_1^{[k]}}\wedge\dots\wedge a_1^{[q+j]})\wedge\beta_g(b_{g+1}\wedge a_1^{[k]})),\end{cases}$$

and

$$T_3=\begin{cases}\sum\limits_{k=1}^{j+q}(-1)^{k+1}\sum\limits_{\substack{|\mathbf{m}|=q+j-1\\ 2\le r(\mathbf{m})}}\prod\limits_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\\ \qquad\Xi_{\mathrm{all}}(b_{g+1}\wedge a_1^{[k]}\otimes C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_1^{[1]}\wedge\dots\wedge\widehat{a_1^{[k]}}\wedge\dots\wedge a_1^{[q+j]})).\end{cases}$$

Observe that $S_1+T_1=0$. Lemma 3.21 yields that T_3 is

$$\sum_{\substack{|\mathbf{m}|=q+j\\ 2\le r(\mathbf{m})}}\sum_{1\le k}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\tbinom{p+i+1}{N-i-j}\tfrac{(2N-2)!}{(2N-p-i-2)!}\Xi_{\boldsymbol{\epsilon}_{k+1}}(b_{g+1}\otimes C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)).$$

In a similar manner, Corollary 3.20 yields that T_2 is

$$\sum_{\substack{|\mathbf{m}|=q+j\\ 2\le r(\mathbf{m})}}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Xi_{\boldsymbol{\epsilon}_1}(b_{g+1}\otimes C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)).$$

We see that T_2+T_3 is equal to

$$\sum_{\substack{|\mathbf{m}|=q+j\\ 2\le r(\mathbf{m})}}\prod_{1\le h}(h!)^{m_h}(-1)^{N-i}\binom{p+i+1}{N-i-j}\frac{(2N-2)!}{(2N-p-i-2)!}\Xi_{\mathrm{all}}(b_{g+1}\otimes C_p\cdot C_i\otimes\Delta_{\mathbf{m}}(a_q\wedge a_j)).$$

Apply Lemma 3.15 to conclude that $S_2+T_2+T_3=0$. $\square$

4. The complex $\mathbb{M}$ is acyclic.

In this section we prove Theorem 4.3, which establishes the acyclicity of $\mathbb{M}$. At the end of the section, we record a few consequences of the fact that J is a perfect ideal. In the proof of Theorem 4.3 we use "generic data", which is described below. Such data forces one to deal with bases. In fact, bases play only a very minor role in the present section; however, they play a significant role in section 5. We have recorded all of our conventions about bases at this time. These conventions are different than the ambient conventions of 2.2.

Convention 4.1. Adopt Data 2.1. Fix dual bases $\boldsymbol{f}_1,\dots,\boldsymbol{f}_f$ and $\boldsymbol{f}_1^*,\dots,\boldsymbol{f}_f^*$ for F and F^*, respectively, and $\boldsymbol{g}_1,\dots,\boldsymbol{g}_g$ and $\boldsymbol{g}_1^*,\dots,\boldsymbol{g}_g^*$ for G and G^*, respectively. For each index set $I=\{i_1,\dots,i_\ell\}$, with $i_1<\dots<i_\ell$, we write $|I|=\ell$,

$$\boldsymbol{f}_I=\boldsymbol{f}_{i_1}\wedge\ldots\wedge\boldsymbol{f}_{i_\ell},\quad\text{and}\quad \boldsymbol{f}_I^*=\boldsymbol{f}_{i_\ell}^*\wedge\ldots\wedge\boldsymbol{f}_{i_1}^*.$$

Let $[\ell]$ represent the index set $\{1,\dots,\ell\}$.

Generic Data 4.2. Adopt Convention 4.1. Assume that R_0 is a commutative noetherian ring, φ is represented by a $g\times f$ matrix of indeterminates (x_{ij}), b_{g+1} is the element $\sum v_I\boldsymbol{f}_I$, where I varies over all index sets with $|I|=g+1$ and $\{v_I\}$ is a set of indeterminates, and R is the polynomial ring $R_0[\{x_{ij}\},\{v_I\}]$.

Theorem 4.3. *Adopt Data 2.1. If J is a proper ideal of R with $f-g\le\operatorname{grade}J$, and $(f-g-2)!$ is a unit in R, then J is a perfect ideal of grade $f-g$, and $\mathbb{M}$ is a resolution of R/J. Furthermore, if the data of 2.1 is local (i.e., $(R,\mathfrak{m})$ is a local ring, $\operatorname{im}\varphi\in\mathfrak{m}G$, and $b_{g+1}\in\mathfrak{m}\bigwedge^{g+1}F$) or homogeneous in the sense of Remark 2.8 (d) with $1\le\boldsymbol{a},\boldsymbol{b}$, then the resolution $\mathbb{M}$ is minimal.*

Proof. The assertion about minimality is obvious. We first establish acyclicity for generic data, as in 4.2, with $R_0=\mathbb{Z}[1/(f-g-2)!]$. Once the result is established in this case, then the principal of the transfer of perfection; see, for example [3, Theorem 3.5], yields the result in general.

Observe that the hypothesis $f-g \le \operatorname{grade} J$ holds in the generic situation. We prove this by showing that there exists a regular sequence of linear forms with $f-g \le \operatorname{grade} \bar{J}$, where $\bar{}$ represents reduction modulo the regular sequence. Indeed, the specialization

$$\bar{\varphi} = \begin{bmatrix} 0 & x_1 & x_2 & \dots & x_{f-g} & 0 & 0 & \dots & 0 \\ 0 & 0 & x_1 & x_2 & \dots & x_{f-g} & 0 & \dots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \dots & 0 & 0 & x_1 & x_2 & \dots & x_{f-g} \end{bmatrix} \text{ and } \bar{b}_{g+1} = \sum_{i=1}^{f-g} x_i \boldsymbol{f}_i \wedge \boldsymbol{f}_{i+1} \wedge \dots \wedge \boldsymbol{f}_{i+g}$$

yields $\bar{b}_1 = \sum_{i=1}^{f-g} (\pm x_i^{g+1} + y_i)\boldsymbol{f}_i + a_1$, for some elements $y_i \in (x_1, \dots, x_{i-1})$ and a_1 in $R\boldsymbol{f}_{f-g+1} \oplus \dots \oplus R\boldsymbol{f}_f$; and therefore, the radical of $\bar{J}$ is equal to the $(x_1, \dots, x_{f-g})$.

Fix a prime ideal P in R with $\operatorname{grade} P \le f-g-1$. Since

$$\operatorname{grade} P < f-g+1 = \operatorname{grade} I_{f-g}(\varphi),$$

we know that some $g \times g$ minor of φ is a unit in R_P. For the time being, we work over the ring R_P. There exists an isomorphism $\theta\colon F \to F$, of determinant 1, such that $\varphi \circ \theta$ is equal to $\varphi' = [\, I \quad 0\,]$. Lemma 4.4 shows that $\mathbb{M}_P$ is isomorphic to a complex $\mathbb{M}'$, formed using φ'. Lemma 4.5 shows that the homology of $\mathbb{M}'$ is isomorphic to the homology of a complex $\mathbb{M}''$, which is formed with $g=0$. Proposition 2.16 shows that $\mathbb{M}''$ is the Koszul complex which is associated to the ideal $J=R$. It follows that $\mathbb{M}''$ is split exact; and therefore, $\mathbb{M}_P$ is also split exact. Now we return to the ring R. The acyclicity lemma [4, Corollary 4.2] yields that $\mathbb{M}$ is a resolution. Also, we know that

$$\operatorname{grade} J \le \operatorname{pd}_R R/J \le f-g \le \operatorname{grade} J,$$

and the proof is complete. $\square$

Lemma 4.4. *Adopt Data 2.1. Let $\theta\colon F \to F$ be an isomorphism of determinant one. If $\varphi' = \varphi \circ \theta$ and $b'_{g+1} = (\bigwedge^{g+1} \theta^{-1})(b_{g+1})$, then the complexes $\mathbb{M} = \mathbb{M}(b_{g+1}, \varphi)$ and $\mathbb{M}' = \mathbb{M}(b'_{g+1}, \varphi')$ are isomorphic.*

Proof. Define $\Theta\colon \mathbb{M}' \to \mathbb{M}$ by

$$\begin{cases} \Theta(\gamma_1^{(i)} \otimes \alpha_j) = \gamma_1^{(i)} \otimes (\bigwedge^j \theta^{-1*})(\alpha_j) \\ \Theta(C_p \otimes a_q) = C_p \otimes (\bigwedge^q \theta)(a_q). \end{cases}$$

It is clear that Θ is an isomorphism of modules. It is not difficult to show that it is a map of complexes. The most interesting step involves showing that

$$\Phi'[C_p \otimes (\textstyle\bigwedge^{n_1} \theta^{-1})(a_{n_1}) \otimes \dots \otimes (\bigwedge^{n_r} \theta^{-1})(a_{n_r})] = \Phi(C_p \otimes a_{n_1} \otimes \dots \otimes a_{n_r}).$$

This holds because

$$\tau'[c_1 \otimes (\textstyle\bigwedge^n \theta^{-1})(a_n)] = (\bigwedge^{n+2} \theta^{-1})[\tau(c_1 \otimes a_n)],$$

and θ has determinant 1. $\square$

Lemma 4.5. *Adopt Data 2.1. Suppose that φ decomposes as*

$$\varphi = \left[\begin{smallmatrix} \varphi' & 0 \\ 0 & 1 \end{smallmatrix}\right] : F = F' \oplus R\boldsymbol{a}_1 \to G = G' \oplus G\boldsymbol{c}_1,$$

where F' and G' are free $R-$modules of rank $f-1$ and $g-1$, respectively. Let

$$b_{g+1} = b'_g \wedge \boldsymbol{a}_1 + b'_{g+1},$$

with $b'_g \in \bigwedge^g F'$ and $b'_{g+1} \in \bigwedge^{g+1} F'$, be the corresponding decomposition of b_{g+1}. Then the complexes $\mathbb{M} = \mathbb{M}(b_{g+1}, \varphi)$ and $\mathbb{M}' = \mathbb{M}(b'_g, \varphi')$ have the same homology.

Proof. Let $F'^* \oplus R\boldsymbol{\alpha}_1$ and $G'^* \oplus R\boldsymbol{\gamma}_1$ be the corresponding decompositions of F^* and G^*, respectively. It follows that

$$\varphi^* = \left[\begin{smallmatrix} \varphi'^* & 0 \\ 0 & 1 \end{smallmatrix}\right] : G^* = G'^* \oplus R\boldsymbol{\gamma}_1 \to F^* = F'^* \oplus R\boldsymbol{\alpha}_1.$$

Write $\omega_F = \omega_{F'} \wedge \boldsymbol{a}_1$, $\omega_G = \omega_{G'} \wedge \boldsymbol{c}_1$, $\omega_{F^*} = \boldsymbol{\alpha}_1 \wedge \omega_{F'^*}$, and $\omega_{G^*} = \boldsymbol{\gamma}_1 \wedge \omega_{G'^*}$. Let $\beta'_{g-1} = (\bigwedge^{g-1} \varphi'^*)(\omega_{G'^*})$ and $b'_1 = \beta'_{g-1}(b'_g)$. Observe that

$$\beta_g = \boldsymbol{\alpha}_1 \wedge \beta'_{g-1} \quad \text{and} \quad b_1 = -b'_1.$$

Let $\mathbb{A}$ be the sum of all modules from $\mathbb{M}$ of the form

$$D_i G'^* \boldsymbol{\gamma}_1^{(k)} \otimes \bigwedge^j F'^* \wedge \boldsymbol{\alpha}_1^{\ell}, \quad \text{with } 1 \le k \text{ or } 1 = \ell.$$

Observe that $d(\mathbb{A}) \subseteq \mathbb{A}$. It follows that $\mathbb{A}$ is a complex with differential given by the restriction of d to $\mathbb{A}$. Consider the map $s\colon \mathbb{A} \to \mathbb{A}$, given by

$$\begin{cases} s(\gamma_1'^{(i)} \boldsymbol{\gamma}_1^{(k)} \otimes \alpha'_j) = 0 \\ s(\gamma_1'^{(i)} \boldsymbol{\gamma}_1^{(k)} \otimes \boldsymbol{\alpha}_1 \wedge \alpha'_j) = \gamma_1'^{(i)} \boldsymbol{\gamma}_1^{(k+1)} \otimes \alpha'_j. \end{cases}$$

It is easy to see that $ds + sd = \mathrm{id}_{\mathbb{A}}$; and therefore, $\mathbb{A}$ is a split exact complex. At this point we know that $\mathbb{M}/\mathbb{A}$ is a complex with $H_i(\mathbb{M}/\mathbb{A})$ isomorphic to $H_i(\mathbb{M})$ for all i.

We next show that the map $\rho\colon \mathbb{M}' \to \mathbb{M}/\mathbb{A}$, which is given by

$$\begin{cases} \rho(\gamma_1'^{(i)} \otimes \alpha'_j) = (-1)^{i+j} \gamma_1'^{(i)} \otimes \alpha'_j \\ \rho(C'_p \otimes a'_q) = (-1)^{p+q+1} C'_p \otimes a'_q, \end{cases}$$

is a map of complexes. The only complicated step in this calculation involves showing that

$$(-1)^{p+q+1} \Psi_{p,q,i,j}(C'_p \otimes a'_q \otimes \underline{} \otimes \underline{}) = \rho \Psi'_{p,q,i,j}(C'_p \otimes a'_q \otimes \underline{} \otimes \underline{})$$

in $D_i G'^* \otimes \bigwedge^j F'^*$, when $C'_p \otimes a'_q \in S_p G' \otimes \bigwedge^q F'$ and

$$2p + 2i + j + q = 2N - 1, \quad p + q \le N - 1, \quad \text{and} \quad i + j \le N.$$

On the other hand, the above equation is equivalent to

$$\Psi_{p,q,i,j}(C'_p \otimes a'_q \otimes C'_i \otimes a'_j) = (-1)^{i+p}\Psi'_{p,q,i,j}(C'_p \otimes a'_q \otimes C'_i \otimes a'_j)$$

for $C'_i\otimes a'_j \in S_iG'\otimes\bigwedge^j F'$, and this equation holds because $\tau(c'_1\otimes a'_n) = -\tau'(c'_1\otimes a'_n)$.

Define $\mathbb{B}$ to be the cokernel of ρ. Notice that $\mathbb{B}$ consists of the modules

$$S_pG' \cdot \boldsymbol{c}_1^r \otimes \bigwedge^q F' \wedge \boldsymbol{a}_1^\ell, \quad \text{with } 1 \le r \text{ or } 1 = \ell.$$

The differential on $\mathbb{B}$ is given by $d(C'_p \cdot \boldsymbol{c}_1^r \otimes a'_q \wedge \boldsymbol{a}_1^\ell)$ is equal to

$$\begin{cases} \quad C'_p \cdot \boldsymbol{c}_1^r \otimes b_1 \wedge a'_q \wedge \boldsymbol{a}_1^\ell \\ + \sum\limits_{k=1}^{q} (-1)^{k+1}\varphi'(a_1'^{[k]}) \cdot C'_p \cdot \boldsymbol{c}_1^r \otimes a_1'^{[1]} \wedge \ldots \wedge \widehat{a_1'^{[k]}} \wedge \ldots \wedge a_1'^{[q]} \wedge \boldsymbol{a}_1^\ell \\ +\delta_{\ell 1}(-1)^q C'_p \cdot \boldsymbol{c}_1^{r+1} \otimes a'_q, \end{cases}$$

where $a'_q = a_1'^{[1]} \wedge \ldots \wedge a_1'^{[q]}$. Consider the map $s\colon \mathbb{B} \to \mathbb{B}$, which is given by

$$\begin{cases} s\left(C'_p \cdot \boldsymbol{c}_1^r \otimes a'_q \wedge \boldsymbol{a}_1\right) = 0 \\ s\left(C'_p \cdot \boldsymbol{c}_1^r \otimes a'_q\right) = C'_p \cdot \boldsymbol{c}_1^{r-1} \otimes \boldsymbol{a}_1 \wedge a'_q. \end{cases}$$

There is no difficulty in showing that $s \circ d + d \circ s = \mathrm{id}_{\mathbb{B}}$. Thus, $\mathbb{B}$ is split exact and the short exact sequence of complexes

$$0 \to \mathbb{M}' \xrightarrow{\rho} \mathbb{M}/\mathbb{A} \to \mathbb{B} \to 0,$$

shows that the proof is complete. $\square$

We conclude the section by recording a few consequences of the fact that J is a perfect ideal. First, we estimate the size of the singular locus of R/J in the generic situation.

Proposition 4.6. *Adopt Data 4.2, with $(f-g-2)!$ a unit in R_0. If P is a prime ideal of R with* $\operatorname{depth} R_P \le f-g+3$*, then $(R/J)_P$ is a localization of a polynomial ring over R_0.*

Proof. The hypothesis on $\operatorname{depth} R_P$ ensures that some $g-1 \times g-1$ minor of φ is not an element of P. For notational convenience, we assume that $\delta \notin P$, where δ is the determinant of the lower left hand $g-1 \times g-1$ submatrix of φ. Let $\theta_1\colon F \to F$ and $\theta_2\colon G \to G$ be isomorphisms so that

$$\theta_2 \circ \varphi \circ \theta_1 = \begin{bmatrix} 0 & \varphi' \\ I & 0 \end{bmatrix}.$$

It is easy to see that the ideal $J((\bigwedge^{g+1} \theta_1^{-1})(b_{g+1}), \theta_2 \circ \varphi \circ \theta_1)$ is equal to J. Let b'_2 be the element $\boldsymbol{f}_{[g-1]}[(\bigwedge^{g+1} \theta_1^{-1})(b_{g+1})]$ of $\bigwedge^2(\bigoplus_{g\le k} R\boldsymbol{f}_k)$. Lemma 4.5 shows that J is equal $J(b'_2, \varphi')$. Let R_1 be the polynomial ring which is obtained by adjoining complement of

$$\{x_{1k} \mid g \le k \le f\} \cup \{v_{[g-1],p,q} \mid g \le p < q \le f\},$$

in $\{x_{ij}\} \cup \{v_I\}$, to R_0. A careful analysis of the matrix θ_1 shows that θ_1 is an invertible map over $(R_1)_\delta$, the coefficients of φ' and b_2' are algebraically independent over $(R_1)_\delta$, and R_δ is equal to the polynomial ring obtained by adjoining the coefficients of φ' and b_2' to $(R_1)_\delta$. The ideal $I_1(b_2'^{(N)})$ in R_δ is generated by the maximal order pfaffians of a generic alternating matrix of odd size. It follows that $I_1(\varphi') + I_1(b_2'^{(N)})$ has grade $f-g+4$ in R_δ; therefore, the hypothesis on depth R_P ensures that $I_1(\varphi') + I_1(b_2'^{(N)})$ is not contained in PR_δ. We have successfully reduced to the case $g = 1$, where the result is both well known and easy to prove. $\square$

Corollary 4.7. *Adopt Data 4.2, with $(f-g-2)!$ a unit in R_0. Let $\overline{R} = R/J$.*

(a) *If R_0 is a domain, then so is $\overline{R}$.*
(b) *Let k be an integer with $k \le 3$.*
   (*i*) *If R_0 satisfies the Serre condition (S_{k+1}), then so does $\overline{R}$.*
   (*ii*) *If R_0 satisfies the Serre conditions (R_k) and (S_{k+1}), then so does $\overline{R}$.*

In particular, if the ring R_0 is reduced, then so is $\overline{R}$; if the ring R_0 is normal, then so is $\overline{R}$.

Proof. This result follows from Proposition 4.6 by way of a standard argument. See [3, Theorem 2.10] for (a), and [11, Theorem 9.4] or [10, Corollary 5.4] for (b). $\square$

Finally, we describe the canonical module of R/J.

Corollary 4.8. *Adopt Data 2.1 with J a proper ideal of R, $4 \le f-g \le \operatorname{grade} J$, and $(f-g-2)!$ a unit in R. Fix an element a_g in $\bigwedge^g F$. Let L be the ideal*

$$L = \frac{(\{[\tau(C_N) \wedge a_g](\omega_{F^*}) \mid C_N \in S_N G\}, \beta_g(a_g), J)}{J}$$

of R/J. Then there exists an R/J-module surjection $\lambda\colon \operatorname{Ext}_R^{f-g}(R/J, R) \twoheadrightarrow L$. Furthermore, if $1 \le \operatorname{grade} L$, then λ is an isomorphism.

Remarks. (a) If R is a Gorenstein ring, then $\operatorname{Ext}_R^{f-g}(R/J,R)$ is the canonical module of R/J.

(b) In the generic situation of 4.2, the hypothesis $1 \le \operatorname{grade} L$ does hold because the ideal $I_g(\varphi)$ has positive grade in R/J and $\beta_g(a_g)$ can be taken to be a $g \times g$ minor of φ.

(c) If $2 = f-g$, then the canonical module of R/J is already well understood.

Proof. Let K denote $\operatorname{Ext}_R^{f-g}(R/J,R)$. Theorem 4.3 ensures that $\mathbb{M}^*$ is a resolution of K. Use the hypotheses about $(f-g-2)!$ and $f-g$ to see that K is presented by

$$(S_{N-1}G \otimes F) \oplus F^* \xrightarrow{d} S_N G \oplus R \to K \to 0,$$

where

$$d \begin{bmatrix} C_{N-1} \otimes a_1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} C_{N-1} \cdot \varphi(a_1) \\ -[\tau(C_{N-1}) \wedge b_{g+1} \wedge a_1](\omega_{F^*}) + b_1(\alpha_1) \end{bmatrix}.$$

Consider the map $\boldsymbol{\lambda}\colon S_NG\oplus R\to R$, which is given by

$$\boldsymbol{\lambda}(C_N)=[\tau(C_N)\wedge a_g](\omega_{F^*})\quad\text{and}\quad \boldsymbol{\lambda}(1)=\beta_g(a_g).$$

The calculations of section 3 show that $\tau(\varphi(a_1))=a_1\wedge b_1+\beta_g(b_{g+1}\wedge a_1)$ and

$$0=\begin{cases}\tau(C_{N-1})\wedge\beta_g(a_{g+2})\wedge a_g\\ -\tau(C_{N-1})\wedge a_{g+2}\wedge\beta_g(a_g)\\ +\sum\limits_{\ell} b_1\wedge\tau(c_1^{[1]})\wedge\dots\wedge\widehat{\tau(c_1^{[\ell]})}\wedge\dots\wedge\tau(c_1^{[N-1]})\wedge t(c_1^{[\ell]}\otimes a_{g+2})\wedge a_g,\end{cases}$$

for $C_{N-1}=c_1^{[1]}\cdots c_1^{[N-1]}$. (Use Corollaries 3.8 and 3.7 and the idea below (3.5).) It follows that the image of $\boldsymbol{\lambda}\circ d$ is in J, and therefore, $\boldsymbol{\lambda}$ induces the desired $R/J-$module surjection $\lambda\colon K\twoheadrightarrow L$. The last assertion is obvious because K is a torsion-free $R/J-$module of rank one. $\square$

5. Further questions.

There are at least three promising directions for further study.

1. We hope that the resolution $\mathbb{M}$ is able to shed light on the case when $f-g$ is odd. In particular, we want to know the extra generators of $J^{\text{unm}}\supsetneq J$. The case $g=1$ provides encouragement for this hope. We have seen that when $g=1$ and f is odd, then J is a Huneke-Ulrich almost complete intersection ideal. On the other hand, when $g=1$ and f is even, then J^{unm} is generated by J together with the pfaffian $b_2^{(2)}$. In other words, J^{unm} is a Huneke-Ulrich deviation two Gorenstein ideal. Furthermore, the resolution of a Huneke-Ulrich almost complete intersection [9] is very similar to the resolution of a Huneke-Ulrich Gorenstein ideal [8, 14, 7]. (The similarity is best exhibited in [8].)

2. A great deal is already known about the Rees algebra of an ideal. (See, for example [15], especially its extensive bibliography.) In the present paper, we resolve the Rees algebra of certain projective dimension two modules. We hope to use the insights we have gained to learn more about the Rees algebra of modules.

3. We are hopeful that the complex $\mathbb{M}$ may be modified in such a way that the hypothesis "$(f-g-2)!$ is a unit in R" may be removed from Theorem 4.3. No modification is needed when $f-g=2$; see Example 2.9. The next result shows the flavor of the desired result.

Theorem 5.1. *Adopt Data 2.1 with $f=g+4$. Let $(\mathbb{M},d)$ be the complex of Example 2.10 and let $(\mathbb{M}',d')$ be the sequence of maps and modules with $\mathbb{M}'_i=\mathbb{M}_i$ and d'_i is equal to d_i, except*

$$d'_2\begin{bmatrix}0\\0\\c_1\end{bmatrix}=p(c_1\otimes \underline{}),\quad d'_3\begin{bmatrix}0\\a_1\end{bmatrix}=\begin{bmatrix}[a_1\wedge b_{g+1}](\omega_{F^*})\\ -p(\underline{}\otimes a_1)\\ \varphi(a_1)\end{bmatrix},\quad\text{and}\quad d'_4\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-p(\underline{}\otimes\underline{})\\ b_1\end{bmatrix}.$$

Then there exists a map $p\colon G\otimes F\to R$ such that

(1) *$\mathbb{M}'$ is a complex,*
(2) *$\mathbb{M}'$ is exact whenever $4\le\operatorname{grade}J$, (the commutative noetherian ring R is arbitrary), and*
(3) *if 2 is a unit in R, then $\mathbb{M}'$ and $\mathbb{M}$ are isomorphic.*

Proof. The key step in this argument is that we prove that there exists a map $p\colon G \otimes F \to R$ and an element $\xi \in D_2 G^*$ such that

$$P(c_1 \otimes a_1) = 2p(c_1 \otimes a_1) + \xi(c_1 \cdot \varphi(a_1)), \tag{5.2}$$

for all $c_1 \otimes a_1 \in G \otimes F$. For the time being, let us assume that (5.2) holds. Let $\tau\colon \mathbb{M} \to \mathbb{M}'$ be given by $\tau_0 = \mathrm{id}$, $\tau_1 = \mathrm{id}$,

$$\tau_2 \begin{bmatrix} \alpha_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ 0 \\ 0 \end{bmatrix}, \quad \tau_2 \begin{bmatrix} 0 \\ \gamma_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_1 \\ 0 \end{bmatrix}, \quad \tau_2 \begin{bmatrix} 0 \\ 0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1(\xi) \\ 2c_1 \end{bmatrix},$$

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \tau_4 = \begin{bmatrix} 1 & -\xi \\ 0 & 2 \end{bmatrix}.$$

It is easy to verify that τ is a homomorphism of complexes, and it is clear that τ is an isomorphism whenever $1/2 \in R$. The rest of this paragraph is devoted to establishing assertion (2). Exactly as in the proof of Theorem 4.3, it suffices to prove that $\mathbb{M}'$ is acyclic when the data is generic, in the sense of 4.2, with $R_0 = \mathbb{Z}$. We are not interested in establishing analogues of Lemmas 4.4 and 4.5 for $\mathbb{M}'$. Instead, we use the long exact sequence of homology which is associated to to the short exact sequence of complexes:

$$0 \to \mathbb{M} \xrightarrow{\tau} \mathbb{M}' \to \mathbb{N} \to 0,$$

where $\mathbb{N}$, which is defined to be $\operatorname{coker}\tau$, is

$$0 \to \bar{R} \xrightarrow{n_4} \bar{F} \xrightarrow{n_3} \bar{G} \xrightarrow{n_2} 0 \xrightarrow{n_1} 0,$$

with $n_4(\bar{1}) = \bar{b}_1$ and $n_3(\bar{a}_1) = \overline{\varphi(a_1)}$. (We take $\overline{}$ to mean modulo 2.) By the acyclicity lemma, it suffices to show that $\mathbb{M}'_\delta$ is acyclic, whenever δ is a $g \times g$ minor of φ. Fix such a δ, localize at δ, and change notation. (That is, write R in place of R_δ, $\mathbb{M}$ in place of $\mathbb{M}_\delta$, $\mathbb{N}$ in place of $\mathbb{N}_\delta$, etc.) Pick a basis for F so that $\varphi = [\,I \quad 0\,]$. It follows that $b_1 = \sum_{i=1}^4 r_i \boldsymbol{f}_{g+i}$ for some $r_i \in R$. It also follows that r_1, r_2, r_3, r_4 is a regular sequence which generates J. Let $F' = \bigoplus_{i=1}^4 R\boldsymbol{f}_{g+i}$. We know, from Proposition 2.16, that

$$H_i(\mathbb{M}) \equiv \begin{cases} 0 & \text{if } i = 1,\ 3, \text{ or } 4 \\ \frac{b_1(\bigwedge^3 F')}{2b_1(\bigwedge^3 F')} & \text{if } i = 2. \end{cases}$$

Furthermore, the isomorphism $\sigma\colon H_2(\mathbb{M}) \to \frac{b_1(\bigwedge^3 F')}{2b_1(\bigwedge^3 F')}$ is induced by the map which sends the cycle

$$\begin{bmatrix} \alpha_2 \\ \gamma_1 \\ c_1 \end{bmatrix} \in \mathbb{M}_2$$

to $\overline{(\bigwedge^2 \pi^*)(\alpha_2)}$, where $\pi\colon F \to F'$ is the projection which annihilates $\bigoplus_{i=1}^g R\boldsymbol{f}_i$. On the other hand, the map n_3 is a surjection, so we see that

$$H_i(\mathbb{N}) = \begin{cases} 0 & \text{if } i = 1,\ 2, \text{ or } 4 \\ \frac{F'}{2F'+Rb_1} & \text{if } i = 3. \end{cases}$$

The connecting homomorphism $\partial\colon H_3(\mathbb{N}) \to H_2(\mathbb{M})$ carries $\bar{a}_1 \in \mathbb{N}_3$, with $a_1 \in F'$, to the class of

$$\begin{bmatrix} [a_1 \wedge b_{g+1}](\omega_{F^*}) \\ \gamma_1 \\ c_1 \end{bmatrix}$$

in $\mathrm{H}_2(\mathbb{M})$, for some $\gamma_1 \in G^*$ and some $c_1 \in G$. Thus,

$$\sigma \circ \partial(\bar{a}_1) = \pm[a_1 \wedge b_1](\omega_{F'^*}) = \pm b_1(a_1[\omega_{F'^*}]).$$

It follows that $\partial\colon H_3(\mathbb{N}) \to H_2(\mathbb{M})$ is an isomorphism, and $\mathbb{M}'$ is exact.

Return to the original notation. The rest of the proof is devoted to establishing (5.2). We use the basis convention of 4.1. Let $\xi \in D_2G^*$ be the element

$$-\sum_{i<j}\sum_{\ell=1}^{g-1}\left[\boldsymbol{f}_\ell \wedge \left(\boldsymbol{f}_\ell^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge \boldsymbol{f}_{[\ell-1]}\left((\textstyle\bigwedge^{g-2}\varphi^*)\,[(\boldsymbol{g}_i \wedge \boldsymbol{g}_j)(\omega_{G^*})]\right)\right)(b_{g+1}) \wedge b_{g+1}\right](\omega_{F^*})\cdot \boldsymbol{g}_i^*\boldsymbol{g}_j^*,$$

and $p\colon G \otimes F \to R$ be the map $p(c_1 \otimes a_1) = \sum_{i=1}^{3} p_i(c_1 \otimes a_1)$, where $p_1(c_1 \otimes a_1)$, $p_2(c_1 \otimes a_1)$, and $p_3(c_1 \otimes a_1)$ are equal to

$$\sum_{|I|=g-1}\left[\boldsymbol{f}_I\left((\textstyle\bigwedge^{g-1}\varphi^*)\,[c_1(\omega_{G^*})]\right)\cdot a_1 \wedge \boldsymbol{f}_I \wedge [\boldsymbol{f}_I^*(b_{g+1})]^{(2)}\right](\omega_{F^*}),$$

$$\begin{cases} -\displaystyle\sum_{\ell=1}^{g-1}\sum_k\sum_{|I|=g-\ell-1}(\boldsymbol{f}_k \wedge \boldsymbol{f}_I \wedge \boldsymbol{f}_{[\ell-1]})\left((\textstyle\bigwedge^{g-1}\varphi^*)\,[c_1(\omega_{G^*})]\right)\cdot \boldsymbol{f}_k^*(a_1)\cdot \\ \qquad \left[\boldsymbol{f}_\ell \wedge \boldsymbol{f}_k \wedge \boldsymbol{f}_I \wedge \boldsymbol{f}_{[\ell-1]} \wedge (\boldsymbol{f}_\ell^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge \boldsymbol{f}_I^*)(b_{g+1}) \wedge (\boldsymbol{f}_k^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge \boldsymbol{f}_I^*)(b_{g+1})\right](\omega_{F^*}), \end{cases}$$

and

$$\begin{cases} \displaystyle\sum_{j<i}\sum_{\ell=1}^{g-1}\left[\boldsymbol{f}_\ell \wedge \left(\boldsymbol{f}_\ell^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge \boldsymbol{f}_{[\ell-1]}\left((\textstyle\bigwedge^{g-2}\varphi^*)\,[(\boldsymbol{g}_i \wedge \boldsymbol{g}_j)(\omega_{G^*})]\right)\right)(b_{g+1}) \wedge b_{g+1}\right](\omega_{F^*}) \\ \qquad \cdot \boldsymbol{g}_i^*(c_1)\cdot \boldsymbol{g}_j^*(\varphi(a_1)), \end{cases}$$

respectively. (In the sum for p_1, I varies over all subsets of $\{1,\dots,f\}$ with $g-1$ elements.) The interplay between the symmetric object $(\boldsymbol{g}_i^*\boldsymbol{g}_j^*)(c_1c_1')$ and the alternating object $\boldsymbol{g}_i \wedge \boldsymbol{g}_j$ yields

$$\sum_{i<j}(\boldsymbol{g}_i^*\boldsymbol{g}_j^*)(c_1c_1')\cdot \boldsymbol{g}_i \wedge \boldsymbol{g}_j = c_1 \wedge c_1' - 2\sum_{j<i}\boldsymbol{g}_i^*(c_1)\cdot \boldsymbol{g}_j^*(c_1')\cdot \boldsymbol{g}_i \wedge \boldsymbol{g}_j;$$

and therefore, we see that $\xi(c_1 \cdot \varphi(a_1)) = T_1 + 2p_3(c_1 \otimes a_1)$, where T_1 is equal to

$$-\sum_{\ell=1}^{g-1}\left[\boldsymbol{f}_\ell \wedge \left(\boldsymbol{f}_\ell^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge \boldsymbol{f}_{[\ell-1]}\left((\textstyle\bigwedge^{g-2}\varphi^*)\,[(c_1 \wedge \varphi(a_1))(\omega_{G^*})]\right)\right)(b_{g+1}) \wedge b_{g+1}\right](\omega_{F^*}).$$

Apply Proposition 1.1 (d) to see that

$$T_1 = \sum_{\ell=1}^{g-1}\left[\boldsymbol{f}_\ell \wedge \left(\boldsymbol{f}_\ell^* \wedge \boldsymbol{f}_{[\ell-1]}^* \wedge (\boldsymbol{f}_{[\ell-1]} \wedge a_1)\left((\textstyle\bigwedge^{g-1}\varphi^*)\,[c_1(\omega_{G^*})]\right)\right)(b_{g+1}) \wedge b_{g+1}\right](\omega_{F^*}).$$

Lemmas 5.3 and 5.4 now yield $P(c_1 \otimes a_1) - T_1 = 2[p_1(c_1 \otimes a_1) + p_2(c_1 \otimes a_1)]$. $\square$

Lemma 5.3. *Adopt Data 2.1 with $f=g+4$. Then $\alpha_{g-1}(b_{g+1})\wedge a_1\wedge b_{g+1}$ is equal to*

$$\begin{cases} 2\sum\limits_{|I|=g-1} \boldsymbol{f}_I(\alpha_{g-1})\cdot a_1\wedge \boldsymbol{f}_I\wedge[\boldsymbol{f}_I^*(b_{g+1})]^{(2)} \\ +\sum\limits_k \sum\limits_{|I|=g-2} (\boldsymbol{f}_I\wedge \boldsymbol{f}_k)(\alpha_{g-1})\cdot \boldsymbol{f}_k^*(a_1)\cdot \boldsymbol{f}_k\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_I^*)(b_{g+1})\wedge b_{g+1}. \end{cases}$$

Proof. First we fix k and I, with $|I|=g-1$. Consider

$$T_1=\boldsymbol{f}_k\wedge \boldsymbol{f}_I^*(b_{g+1})\wedge b_{g+1},\quad T_2=2\boldsymbol{f}_k\wedge \boldsymbol{f}_I\wedge[\boldsymbol{f}_I^*(b_{g+1})]^{(2)},\ \text{and}$$
$$T_3=\boldsymbol{f}_k\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_k(\boldsymbol{f}_I^*))(b_{g+1})\wedge b_{g+1}$$

Observe that $T_1=T_2+T_3$. Indeed, if $k\in I$, then $T_2=0$ and T_3 is obviously equal to T_1. If $k\notin I$, then $T_3=0$ and, since T_1 and T_2 are both in $\bigwedge^f F$, it suffices to notice that the equality holds after applying $\boldsymbol{f}_I^*$. Now we allow k and I to vary. Apply $\sum\limits_k\sum\limits_{|I|=g-1}\boldsymbol{f}_k^*(a_1)\cdot \boldsymbol{f}_I(\alpha_{g-1})$ to both sides of $T_1=T_2+T_3$. The proof is complete because

$$\sum_{|I|=g-1}\boldsymbol{f}_I\otimes \boldsymbol{f}_k^*\wedge \boldsymbol{f}_k(\boldsymbol{f}_I^*)=\sum_{|I|=g-2}\boldsymbol{f}_I\wedge \boldsymbol{f}_k\otimes \boldsymbol{f}_k^*\wedge \boldsymbol{f}_I^*.\quad\square$$

Lemma 5.4. *Adopt Data 2.1 with $f=g+4$. Then the sum*

$$\begin{cases} \sum\limits_k\sum\limits_{|I|=g-2}(\boldsymbol{f}_I\wedge \boldsymbol{f}_k)(\alpha_{g-1})\cdot \boldsymbol{f}_k^*(a_1)\cdot \boldsymbol{f}_k\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_I^*)(b_{g+1})\wedge b_{g+1} \\ -\sum\limits_{\ell=1}^{g-1}\boldsymbol{f}_\ell\wedge\Big(\boldsymbol{f}_\ell^*\wedge \boldsymbol{f}_{[\ell-1]}^*\wedge(\boldsymbol{f}_{[\ell-1]}\wedge a_1)(\alpha_{g-1})\Big)(b_{g+1})\wedge b_{g+1} \\ +2\sum\limits_{\ell=1}^{g-1}\sum\limits_k\sum\limits_{|I|=g-\ell-1}(\boldsymbol{f}_k\wedge \boldsymbol{f}_I\wedge \boldsymbol{f}_{[\ell-1]})(\alpha_{g-1})\cdot \boldsymbol{f}_k^*(a_1)\cdot \\ \qquad \boldsymbol{f}_\ell\wedge \boldsymbol{f}_k\wedge \boldsymbol{f}_I\wedge \boldsymbol{f}_{[\ell-1]}\wedge(\boldsymbol{f}_\ell^*\wedge \boldsymbol{f}_{[\ell-1]}^*\wedge \boldsymbol{f}_I^*)(b_{g+1})\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_{[\ell-1]}^*\wedge \boldsymbol{f}_I^*)(b_{g+1}). \end{cases}$$

is equal to zero.

Proof. First we fix k, ℓ, and J with $k,\ell\notin J$ and $|J|=g-2$. Observe that

$$0=\begin{cases} \boldsymbol{f}_k\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_J^*)(b_{g+1})\wedge b_{g+1} \\ -\boldsymbol{f}_\ell\wedge(\boldsymbol{f}_\ell^*\wedge \boldsymbol{f}_J^*)(b_{g+1})\wedge b_{g+1} \\ +2(-1)^g\boldsymbol{f}_\ell\wedge \boldsymbol{f}_k\wedge \boldsymbol{f}_J\wedge(\boldsymbol{f}_\ell^*\wedge \boldsymbol{f}_J^*)(b_{g+1})\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_J^*)(b_{g+1}). \end{cases}\tag{5.5}$$

Indeed, the assertion is obvious if $k=\ell$. If $k\neq\ell$, then it suffices to prove equality after applying $\boldsymbol{f}_J^*\wedge \boldsymbol{f}_k^*\wedge \boldsymbol{f}_\ell^*$. This calculation is long, but straightforward.

Now we turn to the identity from the statement of the lemma. Each index set I from the first sum has the form $I=[\ell-1]\cup I'$, where $|I'|=g-\ell-1$ and $\ell\notin I'$, for some ℓ. It follows that the first sum is equal to

$$\sum_{\ell=1}^{g-1}\sum_k\sum_{\substack{|I|=g-\ell-1\\ \ell\notin I}}(\boldsymbol{f}_I\wedge \boldsymbol{f}_{[\ell-1]}\wedge \boldsymbol{f}_k)(\alpha_{g-1})\cdot \boldsymbol{f}_k^*(a_1)\cdot \boldsymbol{f}_k\wedge(\boldsymbol{f}_k^*\wedge \boldsymbol{f}_{[\ell-1]}^*\wedge \boldsymbol{f}_I^*)(b_{g+1})\wedge b_{g+1}.$$

Expand $(\boldsymbol{f}_{[\ell-1]} \wedge a_1)(\alpha_{g-1})$ and a_1 in terms of the bases to see that the second sum is equal to

$$-\sum_{\ell=1}^{g-1}\sum_{k}\sum_{|I|=g-\ell-1}(\boldsymbol{f}_I\wedge\boldsymbol{f}_{[\ell-1]}\wedge\boldsymbol{f}_k)(\alpha_{g-1})\cdot\boldsymbol{f}_k^*(a_1)\cdot\boldsymbol{f}_\ell\wedge\left(\boldsymbol{f}_\ell^*\wedge\boldsymbol{f}_{[\ell-1]}^*\wedge\boldsymbol{f}_I^*\right)(b_{g+1})\wedge b_{g+1}.$$

Apply (5.5) to complete the proof. □

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