THE COHOMOLOGY OF THE KOSZUL COMPLEXES ASSOCIATED TO THE TENSOR PRODUCT OF TWO FREE MODULES

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Abstract. Let E and G be free modules of rank e and g, respectively, over a commutative noetherian ring R. The identity map on $E^* \otimes G$ induces the Koszul complex

$$\dots \to S_m E^* \otimes S_n G \otimes \bigwedge^p (E^* \otimes G) \to S_{m+1} E^* \otimes S_{n+1} G \otimes \bigwedge^{p-1} (E^* \otimes G) \to \dots$$

and its dual

$$\dots \to D_{m+1} E \otimes D_{n+1} G^* \otimes \bigwedge^{p-1} (E \otimes G^*) \to D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*) \to \dots .$$

Let $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ be the homology of the above complexes at

$$S_m E^* \otimes S_n G \otimes \bigwedge^p (E^* \otimes G) \quad \text{and} \quad D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*),$$

respectively. In this paper, we investigate the modules $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ when $-e \le m-n \le g$. We record the fact, already implicitly calculated by Bruns and Guerrieri, that $\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p')$, provided $m+m' = g-1$, $n+n' = e-1$, $p+p' = (e-1)(g-1)$, and $1-e \le m-n \le g-1$. If $m-n$ is equal to either g or $-e$, then we prove that the only non-zero modules of the form $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ appear in one of the split exact sequences

$$0 \to \mathrm{H}_{\mathcal{M}}(g,0,p') \to \bigwedge^{g+p'} (E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(0,e,p) \to 0, \text{ or}$$
$$0 \to \mathrm{H}_{\mathcal{M}}(0,e,p') \to \bigwedge^{e+p'} (E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(g,0,p) \to 0,$$

where $p+p' = (e-1)(g-1)-1$. The modules that we study are not always free modules. Indeed, if $m = n$, then the module $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is equal to a homogeneous summand of the graded module $\mathrm{Tor}^P_{\bullet,\bullet}(T,R)$, where P is a polynomial ring in eg variables over R and T is the determinantal ring defined by the 2×2 minors of the corresponding $e \times g$ matrix of indeterminates. Hashimoto's work shows that if e and g are both at least five, then $\mathrm{H}_{\mathcal{N}}(2,2,3)$ is not a free module when R is $\mathbb{Z}$, and when R is a field, the rank of this module depends on the characteristic of R. When the modules $\mathrm{H}_{\mathcal{M}}(m,n,p)$ are free, they are summands of the resolution of the universal ring for finite length modules of projective dimension two.

2000 *Mathematics Subject Classification.* 13D25, 13C20, 13C40.

Key words and phrases. Buchsbaum-Rim complex, Class group, Determinantal ring, Duality, Eagon-Northcott complex, Finite free resolution, Koszul complex, Koszul homology, Perfect module, Universal resolution.

Throughout the paper, R is a commutative noetherian ring with one, E and G are free R-modules of rank e and g, respectively, and α is the integer $(e-1)(g-1)$. The identity map on $E^* \otimes G$ induces the Koszul complex

$$(0.1)\quad \cdots \to \mathcal{N}(m-1,n-1,p+1) \to \mathcal{N}(m,n,p) \to \mathcal{N}(m+1,n+1,p-1) \to \cdots$$

and its dual

$$(0.2)\quad \cdots \to \mathcal{M}(m+1,n+1,p-1) \to \mathcal{M}(m,n,p) \to \mathcal{M}(m-1,n-1,p+1) \to \cdots .$$

where $\mathcal{N}(m,n,p) = S_m E^* \otimes S_n G \otimes \bigwedge^p (E^* \otimes G)$ and

$$\mathcal{M}(m,n,p) = D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*).$$

Let $\mathrm{H}_{\mathcal{N}}(m,n,p)$ be the homology of (0.1) at $\mathcal{N}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ be the cohomology of (0.2) at $\mathcal{M}(m,n,p)$. We investigate the modules $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ when $-e \le m-n \le g$. Theorem 1.1 records the fact, already implicitly calculated by Bruns and Guerrieri [2], that $\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p')$, provided $m+m' = g-1$, $n+n' = e-1$, $p+p' = \alpha$, and $1-e \le m-n \le g-1$. If $m-n$ is equal to either g or $-e$, then we prove in Theorem 2.1 that the only non-zero modules of the form $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ appear in one of the split exact sequences

$$0 \to \mathrm{H}_{\mathcal{M}}(g,0,p') \to \textstyle\bigwedge^{g+p'}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(0,e,p) \to 0, \text{ or}$$
$$0 \to \mathrm{H}_{\mathcal{M}}(0,e,p') \to \textstyle\bigwedge^{e+p'}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(g,0,p) \to 0,$$

where $p+p' = \alpha - 1$.

The cohomology modules $\mathrm{H}_{\mathcal{M}}(m,n,p)$ have forced themselves into consideration. Hochster [7, Thm. 7.2] established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution

$$(0.3)\qquad \mathbb{U}:\quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g \to 0,$$

such that for any commutative noetherian ring S and any resolution

$$\mathbb{V}:\quad 0 \to S^e \to S^f \to S^g \to 0,$$

there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$. In [10], we found a free resolution $\mathbb{F}$ of the universal ring $\mathcal{R}$ over an integral polynomial ring, $\mathcal{P}$, in the case that the module resolved by $\mathbb{U}$ has finite length. The resolution $\mathbb{F}$ is comprised of maps from four Koszul complexes, one of which is (0.2). The resolution $\mathbb{F}$ is coordinate free and straightforward, but much too large. It is imperative to understand the homology of (0.2) in order to split $\mathbb{F}$ down to a manageable size. In particular, the ring $\mathcal{R}$ is Gorenstein of projective dimension $eg+1$. The module in position $eg+1$ in the resolution of $\mathcal{R}$ is $\mathrm{H}_{\mathcal{M}}(0,e,eg-e)$, which, according to Theorem 2.1, is isomorphic to $\bigwedge^{eg}(E \otimes G^*) \cong R$. Section 5 of the present paper contains many results that are used in [10] to split summands off $\mathbb{F}$. One of the more general results along these lines is the vanishing statement in Theorem 5.4. In section 4 we use the Eagon-Northcott and Buchsbaum-Rim complexes to calculate the homology of (0.1) when $e = 2$.

The *grade* of a proper ideal I in a commutative noetherian ring $\mathcal{P}$ is the length of the longest regular sequence on $\mathcal{P}$ in I. An $\mathcal{P}$-module M is called *perfect* if the grade of the annihilator of M is equal to the projective dimension of M. If M is a perfect $\mathcal{P}$-module of projective dimension c, then $\operatorname{Ext}^c_{\mathcal{P}}(M,\mathcal{P})$ is also a perfect $\mathcal{P}$-module of projective dimension c; furthermore, if $\mathbb{F}$ is a length c projective resolution of M, then $\mathbb{F}^* = \operatorname{Hom}_{\mathcal{P}}(\mathbb{F},\mathcal{P})$ is a resolution of $\operatorname{Ext}^c_{\mathcal{P}}(M,\mathcal{P})$. We record below one well-known and very useful property of perfect modules. An excellent reference on perfect modules is [4, Sect. 16C].

Observation 0.4. *If M and N are perfect $\mathcal{P}$-modules of projective dimension c, then the $\mathcal{P}$ modules*

$$\operatorname{Ext}^j_{\mathcal{P}}(M,\operatorname{Ext}^c_{\mathcal{P}}(N,\mathcal{P})) \cong \operatorname{Ext}^j_{\mathcal{P}}(N,\operatorname{Ext}^c_{\mathcal{P}}(M,\mathcal{P}))$$

for all j.

Proof. Let $\mathbb{F}$ be a length c resolution of M, $\mathbb{G}$ be a length c resolution of N, and $\mathbb{T}$ be the total complex of the double complex $\mathbb{X} = \operatorname{Hom}_{\mathcal{P}}(\mathbb{F}\otimes\mathbb{G},\mathcal{P})$. When one views $\mathbb{X}$ as $\operatorname{Hom}(\mathbb{F},\mathbb{G}^*)$, one sees that

$$\operatorname{Ext}^j_{\mathcal{P}}(M,\operatorname{Ext}^c_{\mathcal{P}}(N,\mathcal{P})) = \mathrm{H}^{2c-j}(\mathbb{T}).$$

On the other hand, when one views $\mathbb{X}$ as $\operatorname{Hom}(\mathbb{G},\mathbb{F}^*)$, then one obtains

$$\operatorname{Ext}^j_{\mathcal{P}}(N,\operatorname{Ext}^c_{\mathcal{P}}(M,\mathcal{P})) = \mathrm{H}^{2c-j}(\mathbb{T}). \quad \square$$

If $\mathcal{P} = \bigoplus_i \mathcal{P}_i$ is a graded ring, and $M = \bigoplus_i M_i$ and $N = \bigoplus_i N_i$ are graded $\mathcal{P}$-modules, then the module $\operatorname{Tor}^{\mathcal{P}}(M,N)$ is a bi-graded $\mathcal{P}$-module. Indeed, if

$$\mathbb{X}\colon \ \cdots \to X_1 \to X_0 \to M$$

is a $\mathcal{P}$-free resolution of M, homogeneous of degree zero, then

$$\operatorname{Tor}^{\mathcal{P}}_{p,q}(M,N) = \frac{\ker[(X_p\otimes N)_q \to (X_{p-1}\otimes N)_q]}{\operatorname{im}[(X_{p+1}\otimes N)_q \to (X_p\otimes N)_q]}.$$

In particular, if $N = \mathcal{P}/\mathcal{P}_+$ is a field, $\mathbb{X}$ is a minimal resolution, and X_p is equal to $\bigoplus_i \mathcal{P}(-i)^{\beta_{p,i}}$, then $\operatorname{Tor}^{\mathcal{P}}_{p,q}(M,N)$ is equal to $N^{\beta_{p,q}}$. Recall, also, (see, for example [3, Thm. 1.5.9]) that ${}^*\operatorname{Hom}_{\mathcal{P}}(M,N)$ is the graded $\mathcal{P}$-module

$$\bigoplus_j {}^*\operatorname{Hom}_{\mathcal{P}}(M,N)_j,$$

where ${}^*\operatorname{Hom}_{\mathcal{P}}(M,N)_j$ consists of all $\mathcal{P}$-module homomorphisms $\varphi\colon M \to N$ with $\varphi(M_i) \subseteq N_{i+j}$ for all i. In particular, if N is concentrated in degree zero, then

$$\text{(0.5)} \qquad {}^*\operatorname{Hom}_{\mathcal{P}}(M,N)_j = \operatorname{Hom}_{\mathcal{P}_0}(M_{-j},N).$$

The statement and proof of the next result illustrate the importance of the modules $\mathrm{H}_{\mathcal{N}}(m,n,p)$ as well as the fact that these modules are significantly more delicate than they might appear at first glance.

Observation 0.6. *The R-modules* $\mathrm{H}_{\mathcal{N}}(m,n,p)$ *are not always free.*

Proof. (This argument was suggested by a referee of an earlier version of this paper. The author uses it with much thanks.) Let S be the R-algebra $S_\bullet E^* \otimes S_\bullet G$. If we fix bases $v_1,\dots,v_e$ for E^*, and $x_1,\dots,x_g$ for G, then one may think of S as the polynomial ring $S = R[v_1,\dots,v_e,x_1,\dots,x_g]$. Let T be the subring

$$T = \sum_m S_m E^* \otimes S_m G$$

of S. One may think of T as the subring $R[v_i x_j]$ of S. Let $\mathcal{P}$ be the R-algebra $S_\bullet(E^* \otimes G)$. One may think of $\mathcal{P}$ as a polynomial ring over R in the eg indeterminates $\{v_i \otimes x_j\}$. It is convenient to let z_{ij} represent the element $v_i \otimes x_j$ of $\mathcal{P}$. The identity map on $E^* \otimes G$ induces a surjective map $\varphi\colon \mathcal{P} \to T$. Let Z be the $e \times g$ matrix whose entry in row i column j is the indeterminate z_{ij}. It is clear that the grade α perfect ideal $I_2(Z)$ is contained in the kernel of φ. Dimension considerations show that if $R = \mathbb{Z}$, then $\ker\varphi = I_2(Z)$; and therefore, it follows that $\ker\varphi$ is equal to $I_2(Z)$ for all choices of R. The R-algebras S, T, and $\mathcal{P}$ all are graded using "the degree in G"; that is, $S_m E^* \otimes S_n G$ has grade n in S; $S_n E^* \otimes S_n G$ has grade n in T, and $S_n(E^* \otimes G^*)$ has grade n in $\mathcal{P}$. The map φ is a homogeneous map of degree zero of graded rings, and S is a graded module over both T and $\mathcal{P}$. Significant information about the modules $\mathrm{H}_{\mathcal{N}}(n,n,p)$ is contained in the graded module $\mathrm{Tor}^{\mathcal{P}}_{\bullet,\bullet}(T,R)$, where R is the graded $\mathcal{P}$-module $\mathcal{P}/\mathcal{P}_+$ concentrated in degree zero. Hashimoto [6] is interested in the $\mathcal{P}$-resolution of the determinantal ring T. He has shown that if R is equal to the ring of integers, and e and g are both at least five, then $\mathrm{Tor}^{\mathcal{P}}_{3,5}(T,R)$ is not a free R-module. On the other hand, the Koszul complex $\mathcal{P} \otimes_R \bigwedge^\bullet(E^* \otimes G)$ is a homogeneous resolution of the $\mathcal{P}$-module R. It follows that $\mathrm{Tor}^{\mathcal{P}}_{\bullet,\bullet}(T,R)$ may be computed as the graded homology of

$$T \otimes_R {\textstyle\bigwedge}^\bullet(E^* \otimes G) = \bigoplus\nolimits_{n,p} \mathcal{N}(n,n,p).$$

The summand $\mathcal{N}(n,n,p)$ has homological position p and is part of the graded summand of grade $n+p$. In other words,

$$\mathrm{Tor}^{\mathcal{P}}_{p,n+p}(T,R) = \mathrm{H}_{\mathcal{N}}(n,n,p). \quad \square \tag{0.7}$$

Corollary 0.8. *Assume that* e *and* g *are both at least* 5. *If* R *is equal to* $\mathbb{Z}$, *then* $\mathrm{H}_{\mathcal{N}}(2,2,3)$ *and* $\mathrm{H}_{\mathcal{M}}(1,1,4)$ *are not free R-modules. If R is a field, then the dimension of* $\mathrm{H}_{\mathcal{N}}(2,2,3)$, $\mathrm{H}_{\mathcal{N}}(1,1,4)$, $\mathrm{H}_{\mathcal{M}}(2,2,3)$, *and* $\mathrm{H}_{\mathcal{M}}(1,1,4)$ *all depend on the characteristic of R.*

Proof. Hashimoto proved that if $R = \mathbb{Z}$, then $\mathrm{Tor}^{\mathcal{P}}_{3,5}(T,R)$, which is equal to $\mathrm{H}_{\mathcal{N}}(2,2,3)$, has a summand of $(\mathbb{Z}/(3))^N$, for some positive integer N. The complexes

$$\dots \to \mathcal{N}(1,1,4) \to \mathcal{N}(2,2,3) \to \dots \quad \text{and} \quad \dots \to \mathcal{M}(2,2,3) \to \mathcal{M}(1,1,4) \to \dots$$

are dual to one another. The only places in these complexes with non-zero homology are the places labeled $(2,2,3)$ or $(1,1,4)$. The Euler characteristic of the homology of these complexes does not change when one changes the characteristic of the base field. □

See [12, Chapt. 4] for more insight into the connection between the non-freeness of $\operatorname{Tor}^{\mathcal{P}}_{\bullet\bullet}(T, R)$, when R is $\mathbb{Z}$, and the dependence of $\operatorname{Tor}^{\mathcal{P}}_{\bullet\bullet}(T, R)$ on the characteristic of R, when R is a field. One consequence of Corollary 0.8 is that the universal ring $\mathcal{R}$ of (0.3) does not posses a generic minimal resolution over $\mathbb{Z}$ when e and g are both at least 5; see [10, Thm. 6.3].

1. Duality inside the Cohen-Macaulay range.

We begin by summarizing and extending the ideas in the proof of Observation 0.6. Let E^* and G be free modules of rank e and g, respectively, over the commutative noetherian ring R. Let S be the ring $S_\bullet E^* \otimes S_\bullet G$, T be the subring

$$T = \sum_m S_m E^* \otimes S_m G$$

of S, and for each integer ℓ, let M_ℓ be the T-submodule

$$M_\ell = \sum_{m-n=\ell} S_m E^* \otimes S_n G$$

of S. Give S a grading by saying that $S_m E^* \otimes S_n G$ has grade n, for all m and n. We see that T is a graded ring, and $\bigoplus M_\ell$ is a direct sum decomposition of S into graded T-submodules. Let $\mathcal{P}$ be the polynomial ring $S_\bullet(E^* \otimes G)$. The ring $\mathcal{P}$ is graded; each element of $S_n(E^* \otimes G)$ is homogeneous of grade n. The identity map on $E^* \otimes G$ induces a graded ring homomorphism φ from $\mathcal{P}$ onto T. Each graded T-module is automatically a graded $\mathcal{P}$-module.

Theorem 1.1. [Bruns and Guerrieri] *Let R be an arbitrary commutative noetherian ring and fix ℓ with $1 - e \le \ell \le g - 1$. Adopt the above notation. The following statements hold.*

(a) *The $\mathcal{P}$-module M_ℓ is perfect of projective dimension α.*
(b) *The $\mathcal{P}$-module M_{g-e} is isomorphic to $\operatorname{Ext}^{\alpha}_{\mathcal{P}}(T, \mathcal{P})$.*
(c) *The $\mathcal{P}$-modules $M_{g-e-\ell}$ and $\operatorname{Ext}^{\alpha}_{\mathcal{P}}(M_\ell, \mathcal{P})$ are isomorphic.*
(d) *If R is a field, then the Hilbert series of M_ℓ is*

$$H_{M_\ell}(t) = \frac{\sum_u \binom{e-1+\ell}{u+\ell}\binom{g-1-\ell}{u} t^u}{(1-t)^{e+g-1}}.$$

(e) *If $\mathbb{F}$ is a graded $\mathcal{P}$-free resolution of M_ℓ of length α, then $\mathbb{F}^*[-\alpha, g-eg]$ is graded $\mathcal{P}$-free resolution of $M_{g-e-\ell}$. The shift $-\alpha$ is the homological shift. The other shift is the shift as a graded module.*
(f) *If $m + m' = g - 1$, $n + n' = e - 1$, $p + p' = \alpha$, and $1 - e \le m - n \le g - 1$, then*

$$\mathrm{H}_{\mathcal{N}}(m, n, p) \cong \mathrm{H}_{\mathcal{M}}(m', n', p').$$

Remarks.

(1) The present approach for proving (f) was suggested to the author by a referee of an earlier version of this paper. Much appreciation is offered to the referee.
(2) If R is a field, then $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ are isomorphic for all triples (m,n,p); consequently, in this situation, the conclusion of (f) could read

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{N}}(m',n',p').$$

Notice, however, that the most recent display is false over $\mathbb{Z}$. The correct way to express the result is given in (f).

Proof. Fix ℓ with $1-e \le \ell \le g-1$. Assume first that R is a field. Bruns and Guerrieri [2, Cor. 3] proved that M_ℓ is a Cohen-Macaulay T-module. It is clear that M_ℓ is faithful over T and graded over $\mathcal{P}$. It follows (see, for example, [4, Prop. 16.19]) that M_ℓ is a perfect $\mathcal{P}$-module of projective dimension equal to

$$\operatorname{proj.dim.}_{\mathcal{P}} T = \operatorname{grade} I_2(Z) = \alpha.$$

Now take $R=\mathbb{Z}$. The module M_ℓ is free over $\mathbb{Z}$ and $M_\ell \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is a perfect $\mathcal{P} \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ module for all prime integers p. It follows (see, for example, [4, Thm. 3.3]) that M_ℓ is a perfect $\mathcal{P}$-module. A very readable discussion of the divisor class group of a normal domain may be found in [1]. The class group of T is known (see, for example, [4, Cor. 8.4]) to be $\mathbb{Z}$ and [2] shows why $n \mapsto [M_n]$ is an isomorphism from $\mathbb{Z} \to \mathrm{C}\ell\,(T)$. The canonical class of T is $[M_{g-e}]$, (see, for example, [4, Thm. 8.8]). A given divisorial class contains exactly one rank one reflexive T-module, up to isomorphism; consequently, $\operatorname{Ext}^{\alpha}_{\mathcal{P}}(\mathcal{P},T)$ and M_{g-e} are isomorphic as T-modules and as $\mathcal{P}$-modules. The arithmetic in the class group $\mathrm{C}\ell\,(T)$ is given by

$$[\operatorname{Hom}_T(M_n, M_m)] = [M_{m-n}].$$

When ℓ is in the Cohen-Macaulay range, $1-e \le \ell \le g-1$, then $\operatorname{Hom}_T(M_\ell, M_{g-e})$ and $M_{g-e-\ell}$ are both automatically reflexive; so,

$$M_{g-e-\ell} \cong \operatorname{Hom}_T(M_\ell, M_{g-e}) = \operatorname{Hom}_T(M_\ell, \operatorname{Ext}^{\alpha}_{\mathcal{P}}(T,\mathcal{P})) \cong \operatorname{Ext}^{\alpha}_{\mathcal{P}}(M_\ell,\mathcal{P}).$$

The right most isomorphism is Observation 0.4 with N and j taken to be T and 0, respectively. Keep in mind that there is no difference between a T-module homomorphism and a $\mathcal{P}$-module homomorphism of T-modules.

Let $\mathbb{F}$ be a length α resolution of M_ℓ by free $\mathcal{P}$-modules. In this case, $\mathbb{F}^*$ is a resolution of $M_{g-e-\ell}$. The $\mathcal{P}$-modules M_ℓ and $M_{g-e-\ell}$ are generically perfect. Use the transfer of perfection [4, Thm. 3.5] to see that $\mathbb{F} \otimes_{\mathbb{Z}} R$ and $\mathbb{F}^* \otimes_{\mathbb{Z}} R$ are free $\mathcal{P} \otimes_{\mathbb{Z}} R$ resolutions of $M_\ell \otimes R$ and $M_{g-e-\ell} \otimes R$ for all commutative rings R. Assertions (a), (b), and (c) are established in full generality.

The formula for $H_{M_\ell}(t)$, from (d), when R is a field, is given in [2]. We have shifted the indices to make the grading in M_ℓ be given by "the degree in G". In particular, the generators for M_ℓ live in degree zero when $0 \le \ell$, but live in degree $-\ell$, when $\ell \le 0$.

We now prove (e). It is clear that $\mathbb{F}^*[-\alpha, N]$ is a homogeneous resolution of $M_{g-e-\ell}$, for some integer N. We must determine the value of N. The formula for

$H_{M_\ell}(t)$ in (d) continues to be meaningful for all choices of commutative noetherian ring R. Recall that the graded module M_ℓ is equal to $\sum_n (M_\ell)_n$, where the homogeneous summand of grade n is the free R-module $(M_\ell)_n = S_{\ell+n}E^* \otimes S_nG$. It follows that the given formula for $H_{M_\ell}(t)$ is always equal to the formal power series

$$\sum_n \operatorname{rank}_R(M_\ell)_n t^n.$$

It is easy to calculate that

$$H_{M_{g-e-\ell}}(t) = (-1)^{e+g-1}t^{-g}H_{M_\ell}(1/t). \tag{1.2}$$

One may read the rational function $H_{M_\ell}(t)$ from the resolution $\mathbb{F}$ of M_ℓ. One may also read $H_{M_{g-e-\ell}}(t)$ from the homogeneous resolution $\mathbb{F}^*[-\alpha, N]$ of $M_{g-e-\ell}$. Apply (1.2) to see that $N = g - eg$.

We prove (f). Fix $\ell = m - n$. We know that M_ℓ and $M_{g-e-\ell}$ are perfect $\mathcal{P}$-modules of projective dimension α. Let $\mathbb{F}$ be a resolution of M_ℓ of length α and let $\mathbb{G}$ be the resolution $\mathbb{F}^*[-\alpha, g-eg]$ of $M_{g-e-\ell}$. In other words, $G_{\alpha-p} = F_p^*[g-eg]$. The inclusion map $R \hookrightarrow \mathcal{P}$ makes every graded $\mathcal{P}$-module become a graded R-module. Consider the double complex

$$\mathbb{D}^{\bullet,\bullet} = {}^\star\operatorname{Hom}_R(\mathbb{F} \otimes_R \textstyle\bigwedge^\bullet(E^* \otimes G), R),$$

with $\mathbb{D}^{i,j} = {}^\star\operatorname{Hom}_R(F_i \otimes \bigwedge^j(E^* \otimes G), R)$. Let $\mathbb{T}$ be the total complex of $\mathbb{D}$. The proof consists of computing the cohomology of $\mathbb{T}$ in two different ways.

First, we focus on the column $\mathbb{D}^{\bullet,j}$, for some fixed j, with $0 \le j \le eg$. We know that $0 \to \mathbb{F} \to M_\ell \to 0$ is an exact complex of $\mathcal{P}$-modules. Every module in this complex is a free R-module; so, the complex is a split exact complex of R-modules. It follows that

$$0 \to \mathbb{F} \otimes_R \textstyle\bigwedge^j(E^* \otimes G) \to M_\ell \otimes_R \bigwedge^j(E^* \otimes G) \to 0$$

and

$$0 \to {}^\star\operatorname{Hom}_R(M_\ell \otimes_R \textstyle\bigwedge^j(E^* \otimes G), R) \to \mathbb{D}^{0,j} \to \cdots \to \mathbb{D}^{\alpha,j} \to 0$$

are split exact sequences of free R-modules. So, the only non-zero cohomology in the column $\mathbb{D}^{\bullet,j}$ occurs at position zero and is equal to ${}^\star\operatorname{Hom}_R(M_\ell \otimes_R \bigwedge^j(E^* \otimes G), R)$. It follows that the cohomology of $\mathbb{T}$ is equal to the cohomology of the complex

$$\begin{aligned} 0 &\to {}^\star\operatorname{Hom}_R(M_\ell \otimes_R \textstyle\bigwedge^0(E^* \otimes G), R) \to {}^\star\operatorname{Hom}_R(M_\ell \otimes_R \bigwedge^1(E^* \otimes G), R) \to \cdots \\ \cdots &\to {}^\star\operatorname{Hom}_R(M_\ell \otimes_R \textstyle\bigwedge^{eg}(E^* \otimes G), R) \to 0. \end{aligned}$$

Apply (0.5) to see that $\mathrm{H}^{p,q}(\mathbb{T})$ is equal to the cohomology of

$$\begin{aligned} \operatorname{Hom}_R((M_\ell \otimes \textstyle\bigwedge^{p-1}(E^* \otimes G))_{-q}, R) &\to \operatorname{Hom}_R(((M_\ell \otimes \textstyle\bigwedge^{p}(E^* \otimes G))_{-q}, R) \\ &\to \operatorname{Hom}_R((M_\ell \otimes \textstyle\bigwedge^{p+1}(E^* \otimes G))_{-q}, R). \end{aligned}$$

Recall that the summand $\mathcal{N}(s+\ell,s,t)$ of the graded module $M_\ell\otimes\bigwedge^\bullet(E^*\otimes G)$ has grade $s+t$. Recall also, that $\mathcal{M}(m,n,p)$ is the R-dual of $\mathcal{N}(m,n,p)$. We conclude that

$$\mathrm{H}^{p,q}(\mathbb{T})=\mathrm{H}_{\mathcal{M}}(\ell-q-p,-q-p,p). \tag{1.3}$$

Now we look at the i^{th} row $\mathbb{D}^{i,\bullet}$ of $\mathbb{D}$. The Koszul complex

$$\mathcal{P}\otimes_R\bigwedge\nolimits^\bullet(E^*\otimes G):\quad 0\to\mathcal{P}\otimes_R\bigwedge\nolimits^{eg}(E^*\otimes G)\to\cdots\to\mathcal{P}\otimes_R\bigwedge\nolimits^0(E^*\otimes G)\to R\to 0$$

is an exact sequence of $\mathcal{P}$-modules, which is also a split exact sequence of R-modules. It follows that

$$0\to F_i\otimes_R\bigwedge\nolimits^\bullet(E^*\otimes G)\to F_i\otimes_R R\to 0$$

and

$$0\to{}^\star\mathrm{Hom}_R(F_i\otimes R,R)\to\mathbb{D}^{i,0}\to\cdots\to\mathbb{D}^{i,eg}\to 0$$

are split exact sequence of R-modules. The only non-zero cohomology in the row $\mathbb{D}^{i,\bullet}$ of $\mathbb{D}$ occurs at position zero and is equal to ${}^\star\mathrm{Hom}_R(F_i\otimes R,R)$. Thus, the cohomology of $\mathbb{T}$ is equal to the cohomology of the complex

$$0\to{}^\star\mathrm{Hom}_R(F_0\otimes R,R)\to{}^\star\mathrm{Hom}_R(F_1\otimes R,R)\to\cdots\to{}^\star\mathrm{Hom}_R(F_\alpha\otimes R,R)\to 0.$$

Notice that

$${}^\star\mathrm{Hom}_R(F_i\otimes_{\mathcal{P}}R,R)=\mathrm{Hom}_R(F_i\otimes_{\mathcal{P}}R,R)$$

because the graded R-modules $F_i\otimes_{\mathcal{P}}R$ and R are concentrated in grade zero. Furthermore, the natural $\mathcal{P}$-module homomorphism

$$\mathrm{Hom}_{\mathcal{P}}(F_i,\mathcal{P})\to\mathrm{Hom}_{\mathcal{P}}(F_i\otimes_{\mathcal{P}}R,R),$$

which is given by $\varphi\mapsto\varphi\otimes 1$, induces an isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(F_i,\mathcal{P})\otimes_{\mathcal{P}}R\cong\mathrm{Hom}_R(F_i\otimes_{\mathcal{P}}R,R).$$

Thus, the cohomology $\mathrm{H}^{p,q}(\mathbb{T})$ is the cohomology of

$$(F^*_{p-1}\otimes R)_q\to(F^*_p\otimes R)_q\to(F^*_{p+1}\otimes R)_q.$$

We have seen that $F^*_p=G_{\alpha-p}[eg-g]$, where $\mathbb{G}$ is a homogeneous $\mathcal{P}$-resolution of $M_{g-e-\ell}$. It follows that

$$\mathrm{H}^{p,q}(\mathbb{T})=\mathrm{Tor}_{\alpha-p,eg-g+q}(M_{g-e-\ell},R).$$

The same thinking that led to (0.7) tells us that

$$\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell,R)=\mathrm{H}_{\mathcal{N}}(\ell+q-p,q-p,p). \tag{1.4}$$

We conclude that $\mathrm{H}^{p,q}(\mathbb{T})=\mathrm{H}_{\mathcal{N}}(g-1-\ell+q+p,e-1+q+p,\alpha-p)$, and the proof is completed by comparing this result with (1.3). □

2. Duality at the boundary of the Cohen-Macaulay range.

In Theorem 1.1 (f) we learned about the homology $\mathrm{H}_{\mathcal{N}}(m,n,p)$, provided the parameters satisfy $1-e \le m-n \le g-1$. The main result of the present section is the following theorem.

Theorem 2.1. *Assume that $m-n$ is equal to either g or $-e$. The only non-zero modules of the form $\mathrm{H}_{\mathcal{N}}(m,n,p)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ appear in one of the split exact sequences*

$$0 \to \mathrm{H}_{\mathcal{M}}(g,0,p') \to \textstyle\bigwedge^{g+p'}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(0,e,p) \to 0, \text{ or}$$
$$0 \to \mathrm{H}_{\mathcal{M}}(0,e,p') \to \textstyle\bigwedge^{e+p'}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(g,0,p) \to 0,$$

where $p+p' = \alpha - 1$. All of the modules in the above exact sequences are free modules.

We may use (1.4) to translate the above result into a statement about the homogeneous resolutions of the modules M_g and M_{-e} of Theorem 1.1. Recall that $\mathcal{P}/I_2(Z) = T$ is the determinantal ring defined by the 2×2 minors of a generic $e\times g$ matrix Z over the commutative noetherian ring R. The set $\{M_\ell \mid \ell \in \mathbb{Z}\}$ is a family of rank one reflexive T-modules, with M_ℓ a perfect $\mathcal{P}$-module of projective dimension α, for $1-e \le \ell \le g-1$.

Corollary 2.2. *Adopt the language of Theorem 1.1.*

(a) *The projective dimension of the $\mathcal{P}$-module M_g is $eg-g$, and $\operatorname{pd}_{\mathcal{P}} M_{-e} = eg-e$.*

(b) *The homogeneous resolution of M_ℓ by free $\mathcal{P}$-modules is linear for ℓ equal to g or $-e$.*

(c) *The betti numbers in the resolutions of* (b) *satisfy*

$$\beta_{p,p+e}(M_{-e}) + \beta_{p',p'}(M_g) = \binom{eg}{e+p},$$

provided $p+p' = eg-e-g$. In particular,

$$\beta_{p,p+e}(M_{-e}) = \binom{eg}{e+p} \quad \text{for } \alpha \le p \le eg-e, \text{ and}$$
$$\beta_{p,p}(M_g) = \binom{eg}{g+p} \quad \text{for } \alpha \le p \le eg-g.$$

Remark. The argument in [2] shows that the $eg-g$ is the projective dimension of M_ℓ for all sufficiently large ℓ. It is interesting to notice that the maximal possible projective dimension is already attained at the least possible ℓ.

Examples 2.3. **(a)** If $e = g = 3$, then the betti numbers

$$\begin{array}{rr} & \beta_{6,6}(M_3)=1\\ & \beta_{5,5}(M_3)=9\\ & \beta_{4,4}(M_3)=36\\ \beta_{0,3}(M_{-3})=10 & \beta_{3,3}(M_3)=74\\ \beta_{1,4}(M_{-3})=45 & \beta_{2,2}(M_3)=81\\ \beta_{2,5}(M_{-3})=81 & \beta_{1,1}(M_3)=45\\ \beta_{3,6}(M_{-3})=74 & \beta_{0,0}(M_3)=10\\ \beta_{4,7}(M_{-3})=36 & \\ \beta_{5,8}(M_{-3})=9 & \\ \beta_{6,9}(M_{-3})=1 & \end{array}$$

are exhibited in [11, Ex. 5.5]. Notice that $10 + 74 = \binom{9}{3}$ and $45 + 81 = \binom{9}{4}$, as expected.

(b) If $e = 3$ and $g = 2$, then the computer program Macaulay will quickly calculate the betti numbers

$$\begin{array}{rr} & \beta_{4,4}(M_2)=1\\ & \beta_{3,3}(M_2)=6\\ & \beta_{2,2}(M_2)=15\\ \beta_{0,3}(M_{-3})=4 & \beta_{1,1}(M_2)=16\\ \beta_{1,4}(M_{-3})=9 & \beta_{0,0}(M_2)=6\\ \beta_{2,5}(M_{-3})=6 & \\ \beta_{3,6}(M_{-3})=1 & \end{array}$$

Once again, we see that $4 + 16 = \binom{6}{3}$ and $9 + 6 = \binom{6}{4}$, as expected.

The proof of Theorem 2.1 appears at the end of the section. We begin by giving names to the complexes which first appeared as (0.1) and (0.2).

Definition 2.4. Fix integers P and Q. Let $\mathbb{N}(P,Q)$ be the complex

$$0 \to \mathcal{N}(P-eg, Q-eg, eg) \to \dots \to \mathcal{N}(P-1, Q-1, 1) \to \mathcal{N}(P, Q, 0) \to 0,$$

and $\mathbb{M}(P,Q)$ be

$$0 \to \mathcal{M}(P, Q, 0) \to \mathcal{M}(P-1, Q-1, 1) \to \dots \to \mathcal{M}(P-eg, Q-eg, eg) \to 0.$$

The module $\mathcal{N}(P,Q,0)$ is in position zero in $\mathbb{N}(P,Q)$; $\mathcal{M}(P,Q,0)$ is in position $P+Q+1$ in $\mathbb{M}(P,Q)$. The differential $\mathcal{N}(m,n,p) \to \mathcal{N}(m+1, n+1, p-1)$ sends

$$\begin{aligned} &U \otimes Y \otimes (u_1 \otimes y_1) \wedge \dots \wedge (u_p \otimes y_p) \\ &\mapsto \sum_{k=1}^{p} (-1)^k u_k U \otimes y_k Y \otimes (u_1 \otimes y_1) \wedge \dots \wedge \widehat{(u_k \otimes y_k)} \wedge \dots \wedge (u_p \otimes y_p). \end{aligned}$$

Remarks 2.5.

(a) The module $\mathcal{N}(m,n,p)$ is equal to $[\mathbb{N}(P,Q)]_i$ if $i = p$, $Q = n+p$, and $P = m+p$.
(b) The module $\mathcal{M}(m,n,p)$ is equal to $[\mathbb{M}(P,Q)]_i$ if $i = m+n+p+1$, $Q = n+p$, and $P = m+p$.
(c) The graded complexes $\mathbb{N}(P,Q)^*[-(P+Q+1)]$ and $\mathbb{M}(P,Q)$ are isomorphic, where $\_^*$ is the functor $\operatorname{Hom}_R(\_, R)$.

Lemma 2.6. *If $Q = P + e$, then the homology of $\mathbb{N}(P,Q)$ and the cohomology of $\mathbb{M}(P,Q)$ are free and concentrated in position $(0,e,P)$. If $P = g + Q$, then the homology of $\mathbb{N}(P,Q)$ and the cohomology of $\mathbb{M}(P,Q)$ are free and concentrated in position $(g,0,Q)$.*

Proof. We prove that $Q = P + e$ implies that the homology of $\mathbb{N}(P,Q)$ is concentrated in $\mathrm{H}_{\mathcal{N}}(0,e,P)$. The proof is by induction on the rank of G. If G has rank one, then $\mathbb{N}(P,Q)$ is the Koszul complex

$$0 \to S_0E^* \otimes \bigwedge^P E^* \to \cdots \to S_PE^* \otimes \bigwedge^0 E^* \to 0,$$

and the assertion is clear. For G of large rank, decompose G as $\hat{G} \oplus Rx$. Form the modules $\hat{\mathcal{N}}(m,n,p)$ and the complexes $\hat{\mathbb{N}}(P,Q)$ using the rank $g-1$ free module $\hat{G}$ in place of G. Multiplication $x\colon S_{n-1}G \to S_nG$ gives a short exact sequence

$$0 \to \mathcal{N}(m,n-1,p) \to \mathcal{N}(m,n,p) \to \sum_\ell \hat{\mathcal{N}}(m,n,p-\ell) \otimes \bigwedge^\ell E^* \to 0$$

of modules, which induces a short exact sequence of complexes

$$0 \to \mathbb{N}(P,Q-1) \to \mathbb{N}(P,Q) \to \sum_\ell \hat{\mathbb{N}}(P-\ell,Q-\ell) \otimes \bigwedge^\ell E^* \to 0.$$

Part (f) of Theorem 1.1 shows that the homology of $\mathbb{N}(P,Q-1)$ is concentrated in $\mathrm{H}_{\mathcal{N}}(0,e-1,P)$. Induction on g shows that the homology of $\hat{\mathbb{N}}(P-\ell,Q-\ell) \otimes \bigwedge^\ell E^*$ is concentrated at $\mathrm{H}_{\mathcal{N}}(0,e,P-\ell) \otimes \bigwedge^\ell E^*$. The long exact sequence of homology completes the proof.

The homology of the complex $\mathbb{N}(P,P+e)$, which is equal to $\mathrm{H}_{\mathcal{N}}(0,e,P)$, is a submodule of the free module $\mathcal{N}(0,e,P)$. The complex $\mathbb{N}(P,P+e)$, over the arbitrary ring R, is obtained from the complex $\mathbb{N}(P,P+e)$, over the ring $\mathbb{Z}$, by way of a base change. It follows that, for any commutative noetherian ring R, $\mathrm{H}_{\mathcal{N}}(0,e,P)$ is a free module R-module and $\mathbb{N}(P,P+e)$ is the direct sum of a split exact sequence plus $\mathrm{H}_{\mathcal{N}}(0,e,P)$. The analogous result about $\mathbb{N}(g+Q,Q)$ may be obtained using similar methods. The assertions about the complexes $\mathbb{M}(P,Q)$ then follow using Remark 2.5 (c). □

Lemma 2.6 proves most of Theorem 2.1. To finish the proof, we consider complexes $\mathfrak{C}^{0,p}$ which concatenate $\mathbb{N}(p,p+e)$ and $\mathbb{M}(p'+g,p')$. The complexes are introduced in Definition 2.9. The relevant maps are in Definition 2.8. Every free R-module that we consider is oriented; in the sense that, if F is a free module of rank f, then ω_F is the name of our preferred generator for $\bigwedge^f F$. The orientations of F and F^* are always compatible in the sense that $\omega_F(\omega_{F^*}) = 1$ and $\omega_{F^*}(\omega_F) = 1$.

Notation 2.7. Let m be an integer. Each pair of elements (U,Y), with $U \in D_mE$ and $Y \in \bigwedge^m G^*$, gives rise to an element of $\bigwedge^m(E \otimes G^*)$, which we denote by $U \bowtie Y$. We now give the definition of $U \bowtie Y$. Consider the composition

$$D_mE \otimes T_mG^* \xrightarrow{\Delta\otimes 1} T_mE \otimes T_mG^* \xrightarrow{\psi} \bigwedge^m(E \otimes G^*),$$

where $\psi\Big((U_1\otimes\ldots\otimes U_m)\otimes(Y_1\otimes\ldots\otimes Y_m)\Big) = (U_1\otimes Y_1)\wedge\ldots\wedge(U_m\otimes Y_m)$, for $U_i\in E$ and $Y_i\in G^*$. It is easy to see that the above composition factors through $D_mE\otimes\bigwedge^m G^*$. Let $U\otimes Y\mapsto U\bowtie Y$ be the resulting map from $D_mE\otimes\bigwedge^m G^*$ to $\bigwedge^m(E\otimes G^*)$. Notice, for example, that if $u\in E$ and $Y_i\in G^*$, then

$$u^{(m)}\bowtie(Y_1\wedge\ldots\wedge Y_m) = (u\otimes Y_1)\wedge\ldots\wedge(u\otimes Y_m).$$

The map

$$\textstyle\bigwedge^m E\otimes D_mG^*\to\bigwedge^m(E\otimes G^*),$$

which sends $U\otimes Y$ to $U\bowtie Y$, for $U\in\bigwedge^m E$ and $Y\in D_mG^*$, is defined in a completely analogous manner.

Definition 2.8. Fix an integer p. Define p' by $p+p'=\alpha-1$. Define homomorphisms

$$\begin{aligned}\gamma\colon \mathcal{M}(0,e,p)&\to \textstyle\bigwedge^{e+p}(E\otimes G^*),\\ \gamma\colon \mathcal{M}(g,0,p)&\to \textstyle\bigwedge^{e+p}(E\otimes G^*),\\ \Gamma\colon \textstyle\bigwedge^{p}(E\otimes G^*)&\to \mathcal{N}(g,0,p'+e),\ \text{and}\\ \Gamma\colon \textstyle\bigwedge^{p}(E\otimes G^*)&\to \mathcal{N}(0,e,p'+g).\end{aligned}$$

If $1\otimes Y\otimes Z$ is in $\mathcal{M}(0,e,p)$ and $U\otimes 1\otimes Z$ is in $\mathcal{M}(g,0,p)$, then

$$\gamma(1\otimes Y\otimes Z) = (\omega_E\bowtie Y)\wedge Z\quad\text{and}\quad \gamma(U\otimes 1\otimes Z) = (U\bowtie\omega_{G^*})\wedge Z.$$

If Z is in $\bigwedge^p(E\otimes G^*)$, then $\Gamma(Z)$ is the element of $\mathcal{N}(g,0,p'+e)$ (or $\mathcal{N}(0,e,p'+g)$, respectively) with

$$[\Gamma(Z)](T) = [\gamma(T)\wedge Z](\omega_{E^*\otimes G})$$

for all T in $\mathcal{M}(g,0,p'+e)$ (or $\mathcal{M}(0,e,p'+g)$, respectively).

Definition 2.9. Fix an integer p. Define p' by $p+p'=\alpha-1$. Define $\mathfrak{C}^{0,p}$ to be the complex

$$0\to\mathcal{M}(p'+g,p',0)\to\cdots\to\mathcal{M}(g,0,p')\xrightarrow{\gamma}\textstyle\bigwedge^{p'+g}(E\otimes G^*)$$

$$\xrightarrow{\Gamma}\mathcal{N}(0,e,p)\to\cdots\to\mathcal{N}(p,p+e,0)\to 0,$$

with $\mathcal{N}(p,p+e,0)$ in position zero. Define $\mathfrak{C}^{e+g,p+g}$ to be the complex

$$0\to\mathcal{M}(p',p'+e,0)\to\cdots\to\mathcal{M}(0,e,p')\xrightarrow{\gamma}\textstyle\bigwedge^{p'+e}(E\otimes G^*)$$

$$\xrightarrow{\Gamma}\mathcal{N}(g,0,p)\to\cdots\to\mathcal{N}(p+g,p,0)\to 0,$$

with $\mathcal{N}(p+g,p,0)$ is position zero.

Remark 2.10. The right side of $\mathfrak{C}^{0,p}$ is equal to $\mathbb{N}(p,p+e)$. The left side of $\mathfrak{C}^{0,p}$ is equal to a shift of $\mathbb{M}(p'+g,p')$. The complexes $\mathfrak{C}^{0,p}$ and $\mathfrak{C}^{e+g,p'+g^*}$ are isomorphic to one another (after an appropriate shift). Observation 2.11 shows that $\mathfrak{C}^{r,s}$ is a complex for all integers s when $r=0$ or $e+g$.

Observation 2.11. *If $Y \in D_e G^*$ and $U \in D_g E$, then*

$$(\omega_E \bowtie Y) \wedge (U \bowtie \omega_{G^*})$$

is equal to zero in $\bigwedge^{e+g}(E \otimes G^*)$.

Proof. It suffices to prove the result when the base ring is the ring of integers. In light of Lemma 2.12, we may assume that $U = u^{(g)}$ and $Y = y^{(e)}$ for some non-zero $u \in E$ and $y \in G^*$. There exists $v \in E^*$, with $v(u)$ not equal to zero. Use the identity $u \wedge v(\omega_E) = [u(v)](\omega_E)$ to see that $v(u)$ times the indicated expression is

$$\begin{aligned}&(u \wedge v(\omega_E) \bowtie y^{(e)}) \wedge (u^{(g)} \bowtie \omega_{G^*})\\ &= (-1)^{e-1}(v(\omega_E) \bowtie y^{(e-1)}) \wedge (u^{(g+1)} \bowtie [y \wedge \omega_{G^*}]) = 0. \quad \square\end{aligned}$$

The next result is well-known. One proof of it appears in [10].

Lemma 2.12. *Suppose R is a polynomial ring over the ring of integers, E and G are free $R-$modules, and $\varphi\colon D_m E \to G$ is an $R-$module homomorphism. If $\varphi(u^{(m)}) = 0$ for all $u \in E$, then φ is identically zero.*

In Lemma 2.17 we prove that the complex $\mathfrak{C}^{e+g,p+g}$ is exact at $\mathcal{N}(g,0,p)$ by taking advantage of the enormous homogeneity of the maps of the Koszul complex $\mathbb{N}(p+g,p)$. Fix a basis $x_1,\dots,x_g$ for G. In Observation 2.16 we show that the complex $\mathbb{N}(p+g,p)$ decomposes into a direct sum of subcomplexes, one for each monomial of degree p in the variables $x_1,\dots,x_g$. We set up the notation in 2.13 — 2.15.

Definition 2.13. Let $\boldsymbol{q} = (q_1,\dots,q_g)$ be a g-tuple of integers.

(a) Let $|\boldsymbol{q}|$ represent $\sum_{i=1}^{g} q_i$.
(b) Let $\bigwedge^{\boldsymbol{q}} E^*$ represent $\bigwedge^{q_1} E^* \otimes \cdots \otimes \bigwedge^{q_g} E^*$.
(c) For each integer a, let $N(a;\boldsymbol{q})$ be the free module $S_a E^* \otimes \bigwedge^{\boldsymbol{q}} E^*$.
(d) The differential δ_k carries $N(a;\boldsymbol{b}_-,b_k,\boldsymbol{b}_+)$ to $N(a+1;\boldsymbol{b}_-,b_k-1,\boldsymbol{b}_+)$ by way of the following composition:

$$N(a,\boldsymbol{b}) \xrightarrow{1\otimes\Delta\otimes 1} S_aE^* \otimes \bigwedge^{\boldsymbol{b}_-} E^* \otimes E^* \otimes \bigwedge^{b_k-1} E^* \otimes \bigwedge^{\boldsymbol{b}_+} E^* \xrightarrow{1\otimes\rho\otimes 1}$$

$$S_aE^* \otimes E^* \otimes \bigwedge^{\boldsymbol{b}_-} E^* \otimes \bigwedge^{b_k-1} E^* \otimes \bigwedge^{\boldsymbol{b}_+} E^* \xrightarrow{\text{mult}\otimes 1} N(a+1;\boldsymbol{b}_-,b_k-1,\boldsymbol{b}_+),$$

for $\boldsymbol{b}_- = (b_1,\dots,b_{k-1})$ and $\boldsymbol{b}_+ = (b_{k+1},\dots,b_g)$. The rearrangement map

$$\rho\colon \bigwedge^{\boldsymbol{b}_-} E^* \otimes E^* \to E^* \otimes \bigwedge^{\boldsymbol{b}_-} E^*$$

sends $V \otimes v$ to $(-1)^{|\boldsymbol{b}_-|} v \otimes V$, where $V \in \bigwedge^{\boldsymbol{b}_-} E^*$ and $v \in E^*$; and

$$\text{mult}\colon S_aE^* \otimes E^* \to S_{a+1}E^*$$

is multiplication in the symmetric algebra $S_\bullet E^*$.

Definition 2.14. Let $\mathbb{K}[E^*,g]$ be the generalized Koszul complex

$$\cdots \to \mathbb{K}_j \to \mathbb{K}_{j-1} \to \cdots ,$$

where $\mathbb{K}_j$ is the free module $\mathbb{K}_j = \bigoplus N(a;\boldsymbol{b})$. The sum is taken over all integers a and all g-tuples of integers, $\boldsymbol{b}$, with $|\boldsymbol{b}| = j$. The differential of $\mathbb{K}[E^*,g]$ is $\sum_{k=1}^{g} \delta_k$.

Definition 2.15. For each integer P and each g-tuple $\boldsymbol{q}$, let $\mathbb{K}[E^*,g,P;\boldsymbol{q}]$ be the following subcomplex of $\mathbb{K}[E^*,g]$:

$$\mathbb{K}[E^*,g,P;\boldsymbol{q}] = \bigoplus N(a;\boldsymbol{b}).$$

The sum is taken over all $g+1$ tuples $(a;\boldsymbol{b})$ with $a+|\boldsymbol{b}| = P$, and $b_i \le q_i$ for $1 \le i \le g$.

Observation 2.16. *If P and Q are integers, then*

$$\mathbb{N}(P,Q) \cong \bigoplus \mathbb{K}[E^*,g,P;\boldsymbol{q}],$$

where the sum is taken over all g-tuples $\boldsymbol{q}$ with $|\boldsymbol{q}| = Q$

Proof. For each g-tuple of integers $\boldsymbol{q} = (q_1,\dots,q_g)$ with $|\boldsymbol{q}| = Q$, let $x^{\boldsymbol{q}}$ be the monomial $x_1^{q_1}\cdots x_g^{q_g}$, of degree Q, in $S_\bullet G$, let $\mathbb{N}[x^{\boldsymbol{q}}]$ represent the subcomplex which consists of all elements of $\mathbb{N}(P,Q)$ which involve exactly q_1 x_1's, q_2 x_2's, etc. Observe that $\mathbb{N}(P,Q)$ is equal to the direct sum of the subcomplexes $\oplus\mathbb{N}[x^{\boldsymbol{q}}]$, as $x^{\boldsymbol{q}}$ varies over the monomials of degree Q. It is not difficult to see that $\mathbb{N}[x^{\boldsymbol{q}}]$ is isomorphic to the subcomplex $\mathbb{K}[E^*,g,P;\boldsymbol{q}]$ of $\mathbb{K}[E^*,g]$. Indeed, the isomorphism

$$\Xi\colon \mathbb{K}[E^*,g,P;\boldsymbol{q}] \to \mathbb{N}[x^{\boldsymbol{q}}]$$

sends the element $V\otimes V_1\otimes\ldots\otimes V_g$ in $N(a;\boldsymbol{b})$ to the element

$$V\otimes x_1^{q_1-b_1}\cdots x_g^{q_g-b_g}\otimes (V_1\bowtie x_1^{(b_1)})\wedge\ldots\wedge(V_g\bowtie x_g^{(b_g)}) \in \mathcal{N}(a,Q-|\boldsymbol{b}|,|\boldsymbol{b}|). \quad \square$$

Lemma 2.17. *If for $i=p$ or $i=p+2$, then $\mathrm{H}_i(\mathfrak{C}^{0,p})$ and $\mathrm{H}_i(\mathfrak{C}^{e+g,p+g})$ are both zero.*

Proof. We prove

$$\bigwedge^{p+g}(E^*\otimes G) \xrightarrow{\Gamma\circ\omega} \mathcal{N}(g,0,p) \to \mathcal{N}(g+1,1,p-1) \tag{2.18}$$

is exact. A similar argument takes care of $\mathrm{H}_p(\mathfrak{C}^{0,p})$. Duality completes the proof.

We know, from Observation 2.16 that Ξ gives an isomorphism from

$$\sum_{\boldsymbol{p}}\left(N(g;\boldsymbol{p}) \xrightarrow{\sum\delta_k} \sum_k N(g+1;\boldsymbol{p}-\boldsymbol{\varepsilon}_k)\right)$$

to $\mathcal{N}(g,0,p)\to\mathcal{N}(g+1,1,p-1)$, where $\sum\limits_{\boldsymbol{p}}$ is taken over all g-tuples $\boldsymbol{p}$ with $|\boldsymbol{p}| = p$. Fix $\boldsymbol{p}$ with $|\boldsymbol{p}| = p$. Lemma 2.20 tells us that Ξ gives a map of complexes from

$$N(0;\boldsymbol{p}+\sum\boldsymbol{\varepsilon}_k) \xrightarrow{\delta_1\circ\cdots\circ\delta_g} N(g,\boldsymbol{p}) \to \sum N(g+1;\boldsymbol{p}-\boldsymbol{\varepsilon}_k) \tag{2.19}$$

to (2.18). Lemma 2.22 shows that (2.19) is exact. $\square$

Lemma 2.20. *If $\boldsymbol{p}$ is a g-tuple, with $|\boldsymbol{p}| = p$, then the diagram*

$$\begin{array}{ccc} N(0;p_1+1,\dots,p_g+1) & \xrightarrow{\delta_1\circ\cdots\circ\delta_g} & N(g;\boldsymbol{p}) \\ \Xi\big\downarrow & & \Xi\big\downarrow \\ \bigwedge^{p+g}(E^*\otimes G) & & \\ \omega_{E\otimes G^*}\big\downarrow & & \\ \bigwedge^{eg-p-g}(E\otimes G^*) & \xrightarrow{\Gamma} & \mathcal{N}(g,0,p) \end{array}$$

commutes up to sign.

Proof. Take $T = U\otimes 1\otimes Z \in \mathcal{M}(g,0,p)$ and $S \in N(0;p_1+1,\dots,p_g+1)$. We compare

$$\mathcal{A} = [(\Xi\circ\delta_1\circ\cdots\circ\delta_g)(S)](T) \quad \text{and} \quad \mathcal{B} = \left[\Gamma\left([\Xi(S)](\omega_{E\otimes G^*})\right)\right](T).$$

We see that $\mathcal{A} = \pm Z\left((U\otimes 1\otimes 1)\left[(\Xi\circ\delta_1\circ\cdots\circ\delta_g)(S)\right]\right)$. It is clear that

$$(U\otimes 1\otimes 1)\left[(\Xi\circ\delta_1\circ\cdots\circ\delta_g)(S)\right] = \pm\left((U\bowtie\omega_{G^*})\circ\Xi\right)(S). \tag{2.21}$$

Indeed, one may test this formula at $U = u^{(g)}$ for $u\in E$ and $S = V_1\otimes\ldots\otimes V_g$. In this case, each side of (2.21) is equal to $\pm u(V_1)\bowtie x_1^{(p_1)}\wedge\ldots\wedge u(V_g)\bowtie x_g^{(p_g)}$. It follows that $\mathcal{A}$ is equal to

$$\begin{aligned} &\pm\left[(U\bowtie\omega_{G^*})\wedge Z\right](\Xi(S)) = \pm[\gamma(T)](\Xi(S)) \\ &= \pm\left[\gamma(T)\wedge\left([\Xi(S)](\omega_{E\otimes G^*})\right)\right](\omega_{E^*\otimes G}), \end{aligned}$$

and this is equal to $\pm\mathcal{B}$. □

Lemma 2.22. *Fix an integer w and a g-tuple of integers $\boldsymbol{q}$. Let $\boldsymbol{\varepsilon}_k$ be the g-tuple $(0,\dots,0,1,0\dots,0)$ with the 1 in position k. Each sum is taken over all k, with $1\le k\le g$.*

(a) *If $g\le w$, then the complex*

$$N(w-g;\boldsymbol{q}+\textstyle\sum\boldsymbol{\varepsilon}_k) \xrightarrow{\delta_1\circ\cdots\circ\delta_g} N(w;\boldsymbol{q}) \xrightarrow{\sum\delta_k} \sum N(w+1;\boldsymbol{q}-\boldsymbol{\varepsilon}_k)$$

is exact.

(b) *If $1\le w$, then the complex*

$$\textstyle\sum N(w-1;\boldsymbol{q}+\boldsymbol{\varepsilon}_k) \xrightarrow{\sum\delta_k} N(w;\boldsymbol{q}) \xrightarrow{\delta_1\circ\cdots\circ\delta_g} N(w+g;\boldsymbol{q}-\sum\boldsymbol{\varepsilon}_k)$$

is exact.

Remark. Assertion (a) is false if $w\le g-1$ and assertion (b) is false if $w=0$, since the complexes

$$0\to N(w;q_1,0,\dots,0)\to N(w+1;q_1-1,0,\dots,0)$$

and $0 \to N(0;0,\dots,0) \to 0$ are not exact.

Proof. The result holds when $g=1$. We complete the proof by induction on g and q_g. We assume that the results hold at $g-1$ for all $q_1,\dots,q_{g-1}$ and at g and q_g-1 for all $q_1,\dots,q_{g-1}$.

We now prove that (b) holds at g and q_g. We start with $A\in N(w;q_1,\dots,q_g)$, $1\le w$, and $\delta_1\circ\dots\circ\delta_g(A)=0$. Apply (a) at q_g-1 to $\delta_2\circ\dots\circ\delta_g(A)$ to find B with $\delta_1\circ\dots\circ\delta_g(B)=\delta_2\circ\dots\circ\delta_g(A)$. Observe that $A\pm\delta_1(B)\in\ker\delta_2\circ\dots\circ\delta_g$ since $\delta_i\delta_j+\delta_j\delta_i=0$. Use assertion (b) at $g-1$ to complete the argument.

We now prove (a) at g and q_g. We start with $A\in N(w;q_1,\dots,q_g)$, $g\le w$, and $\delta_k A=0$ for all k. Apply (a) at $g-1$. We see that $\delta_k(A)=0$ for $1\le k\le g-1$ and $g-1\le w$. So, there exists B with $\delta_1\circ\dots\circ\delta_{g-1}B=A$. Use (b) at q_g to find $B_1,\dots,B_g$ with $\sum\delta_k(B_k)=B$. It follows that $A=\delta_1\circ\dots\delta_g(B_g)$, as desired. □

At this point we know that the homology of the the complex $\mathfrak{C}^{0,p}$ is free and is concentrated in one position. The next result is the final step in our proof of Theorem 2.1.

Lemma 2.23. *If p is an integer, then the complexes $\mathfrak{C}^{0,p}$ and $\mathfrak{C}^{e+g,p+g}$ each have Euler charateristic equal to zero.*

Proof. Remark 2.10 shows that it suffices to prove the result for $\mathfrak{C}^{0,p}$. Let $p'=\alpha-1-p$. We prove that $\binom{eg}{p+e}=A+B$, for

$$A=\sum_{i=0}^{p'}(-1)^i\operatorname{rank}\mathcal{M}(g+i,i,p'-i),\ \text{and}$$

$$B=\sum_{i=0}^{p}(-1)^i\operatorname{rank}\mathcal{N}(i,e+i,p-i).$$

It is not difficult to see that

$$A=\sum_{0\le i}(-1)^i\binom{g+i+e-1}{e-1}\binom{g+i-1}{g-1}\binom{eg}{g+p+e+i}\ \text{and}$$

$$B=\sum_{0\le i}(-1)^i\binom{e+i-1}{i}\binom{g+e+i-1}{e+i}\binom{eg}{p-i}.$$

The binomial coefficient $\binom{a}{b}$ is defined for all integers a and b; see [8] or [9] for more details and elementary results. We have $A=A_1+A_2$, with

$$A_1=\sum_{i\in\mathbb{Z}}(-1)^i\binom{g+i+e-1}{e-1}\binom{g+i-1}{g-1}\binom{eg}{g+p+e+i}\ \text{and}$$

$$A_2=-\sum_{i\le -1}(-1)^i\binom{g+i+e-1}{e-1}\binom{g+i-1}{g-1}\binom{eg}{g+p+e+i}.$$

Apply Lemma 2.24 with m replaced by $g+p+e+i$, a by eg, z by $-p-e-1$, p by $g-1$, w by $-p-1$, and c by $e-1$, to get

$$A_1=(-1)^{p'}\sum_{\ell\in\mathbb{Z}}\binom{-p-e-1}{g-1-\ell}\binom{eg}{\ell}\binom{-p-1+\ell}{e-1-eg+\ell}.$$

A term in A_1 is zero unless $0 \le g-1-\ell$ and $0 \le e-1-eg+\ell$. That is, the non-zero terms of A_1 have

$$(g-1)+1 \le e(g-1)+1 = eg-e+1 \le \ell \le g-1;$$

hence, A_1 is zero. In A_2, if $-g+1 \le i \le -1$, then the middle binomial coefficient is zero; if $i=-g$, then the resulting term in A_2 is $\binom{eg}{p+e}$; if $-g-e+1 \le i \le -g-1$, then the first binomial coefficient is zero. It follows that $A = \binom{eg}{p+e} + A'$, for

$$A' = -\sum_{i \le -g-e} (-1)^i \binom{g+i+e-1}{e-1}\binom{g+i-1}{g-1}\binom{eg}{g+p+e+i}.$$

Use the identity, $\binom{a}{b} = (-1)^b\binom{b-a-1}{b}$, which holds for all integers a and b, to see that

$$A' = \sum_{i \le -g-e} (-1)^{i+e+g+1} \binom{-g-i-1}{e-1}\binom{-i-1}{g-1}\binom{eg}{g+p+e+i}.$$

The parameters $-g-i-1$ and $-i-1$ are non-negative; thus,

$$A' = \sum_{i \le -g-e} (-1)^{i+e+g+1} \binom{-g-i-1}{-g-i-e}\binom{-i-1}{-i-g}\binom{eg}{g+p+e+i}.$$

Replace i by $-e-g-j$ to get $A'+B=0$, and $A+B=\binom{eg}{p+e}$, as desired. $\square$

Lemma 2.24. *Let a, z, p, w, and c be integers with $0 \le a$. Then*

$$\sum_{m\in\mathbb{Z}} (-1)^m \binom{a}{m}\binom{m+z}{p}\binom{m+w}{c} = (-1)^a \sum_{\ell\in\mathbb{Z}} \binom{z}{p-\ell}\binom{a}{\ell}\binom{w+\ell}{c-a+\ell}.$$

Proof. If $a=0$, then both sides of the equation equal $\binom{z}{p}\binom{w}{c}$. If $1 \le a$, then Pascal's identity, which holds for all integers, gives that the left side is T_1+T_2, with

$$T_1 = \sum_{m\in\mathbb{Z}} (-1)^m \binom{a-1}{m}\binom{m+z}{p}\binom{m+w}{c} \quad \text{and}$$
$$T_2 = \sum_{m\in\mathbb{Z}} (-1)^m \binom{a-1}{m-1}\binom{m+z}{p}\binom{m+w}{c}.$$

Induction on a gives

$$T_1 = (-1)^{a-1} \sum_{\ell\in\mathbb{Z}} \binom{z}{p-\ell}\binom{a-1}{\ell}\binom{w+\ell}{c-a+\ell+1} \quad \text{and}$$
$$T_2 = (-1)^a \sum_{\ell\in\mathbb{Z}} \binom{z+1}{p-\ell}\binom{a-1}{\ell}\binom{w+1+\ell}{c-a+\ell+1}.$$

Apply Pascal's identity again to write $T_2 = T_2' + T_2''$, where

$$T_2' = (-1)^a \sum_{\ell \in \mathbb{Z}} \binom{z}{p-\ell}\binom{a-1}{\ell}\binom{w+1+\ell}{c-a+\ell+1} \text{ and}$$

$$\begin{aligned} T_2'' &= (-1)^a \sum_{\ell \in \mathbb{Z}} \binom{z}{p-\ell-1}\binom{a-1}{\ell}\binom{w+1+\ell}{c-a+\ell+1} \\ &= (-1)^a \sum_{L \in \mathbb{Z}} \binom{z}{p-L}\binom{a-1}{L-1}\binom{w+L}{c-a+L}. \end{aligned}$$

Apply Pascal's identity to the third binomial coefficient to see that $T_1 + T_2'$ is

$$(-1)^a \sum_{\ell \in \mathbb{Z}} \binom{z}{p-\ell}\binom{a-1}{\ell}\binom{w+\ell}{c-a+\ell}.$$

Finally, we apply Pascal's identity to the middle binomial coefficient in $(T_1+T_2')+T_2''$ to complete the proof. □

Proof of Theorem 2.1. It suffices to prove the result when $R = \mathbb{Z}$; the general case follows by way of a base change. Fix p and p'. Let $\mathfrak{C} = \mathfrak{C}^{0,p}$. Lemma 2.6 shows that

$$0 \to \mathrm{H}_{\mathcal{M}}(g,0,p') \to \textstyle\bigwedge^{p'+g}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(0,e,p) \to 0 \tag{2.25}$$

is a complex of free modules which is homologically equivalent to $\mathfrak{C}$. Lemma 2.17 shows that

$$\textstyle\bigwedge^{p'+g}(E \otimes G^*) \to \mathrm{H}_{\mathcal{N}}(0,e,p)$$

is a surjection; hence, a split surjection. The map

$$\mathcal{M}(g,0,p') \to \textstyle\bigwedge^{p'+g}(E \otimes G^*) \quad \text{is dual to} \quad \textstyle\bigwedge^{p+e}(E \otimes G^*) \to \mathcal{N}(g,0,p');$$

so, $\mathrm{H}_{\mathcal{M}}(g,0,p') \to \bigwedge^{p'+g}(E \otimes G^*)$ in (2.25) is a split injection. It follows that the homology of (2.25) is concentrated in the middle and is equal to $\mathrm{H}_{p+1}(\mathfrak{C})$. Furthermore, the complexes

$$0 \to \mathrm{H}_{p+1}(\mathfrak{C}) \to 0,$$

(2.25), and $\mathfrak{C}$ all are complexes of free modules with the same Euler characteristic, which by Lemma 2.23 is zero. A dual calculation may be applied to $\mathfrak{C}^{e+g,p+g}$. □

One by-product of our proof of Theorem 2.1 is the following result which is used as the base case in [11] where an explicit quasi-isomorphism

$$\mathbb{M}(P,Q) \to \mathbb{N}(eg-e-P, eg-g-Q)[\alpha-1-P-Q]$$

is given, provided $1-e \le P-Q \le g-1$. This quasi-isomorphism yields an alternate proof of Theorem 1.1 (f).

Corollary 2.26. *For each integer p, the complexes $\mathfrak{C}^{0,p}$ and $\mathfrak{C}^{e+g,p+g}$ are split exact.*

3. Splittable complexes.

In some calculations it is advantageous to assume that the base ring is $\mathbb{Z}$. The results of the present section enable us to pass conclusions to arbitrary base rings. The results of this section are well-known and/or obvious; it is convenient to have a careful record of them.

Definition 3.1. The complex $\mathbb{L}$ is *splittable* if $\mathbb{L}$ is the direct sum of two subcomplexes $\mathbb{L}'$ and $\mathbb{L}''$, with $\mathbb{L}'$ split exact, and the differential on $\mathbb{L}''$ identically zero.

Proposition 3.2. *If $\mathbb{F}$ is a bounded complex of projective modules, then $\mathbb{F}$ is splittable if and only if $\mathrm{H}_j(\mathbb{F})$ is projective for all j.*

Proof. Suppose $\mathbb{F}$ is equal to the direct sum of the two subcomplexes $\mathbb{L}'$ and $\mathbb{L}''$ with $\mathbb{L}'$ split exact and the differential on $\mathbb{L}''$ identically zero. The short exact sequence of complexes $0 \to \mathbb{L}' \to \mathbb{F} \to \mathbb{L}'' \to 0$ shows that $\mathrm{H}_j(\mathbb{F})$ is equal to the projective module L_j'' for all j. We prove the converse by induction on the least integer i with $\mathrm{H}_i \neq 0$. We know that

$$0 \to \operatorname{im} d_i \to F_{i-1} \xrightarrow{d_{i-1}} F_{i-2} \xrightarrow{d_{i-2}} \dots$$

is a split exact complex; hence, in particular, $\operatorname{im} d_i$ is a projective module. We know that

$$0 \to \ker d_i \to F_i \to \operatorname{im} d_i \to 0$$

is a split exact sequence. Let $\gamma\colon \operatorname{im} d_i \to F_i$ be a retract of $d_i : F_i \twoheadrightarrow \operatorname{im} d_i$. We have $F_i = \ker d_i \oplus \operatorname{im} \gamma$. We know that

$$0 \to \operatorname{im} d_{i+1} \to \ker d_i \to \mathrm{H}_i(\mathbb{F}) \to 0$$

is exact with the two right most modules projective; hence, this short exact sequence splits and $\operatorname{im} d_{i+1}$ is a projective module. Let $\gamma'\colon \mathrm{H}_i(\mathbb{F}) \to \ker d_i$ be a retract of $\ker d_i \twoheadrightarrow \mathrm{H}_i(\mathbb{F})$. We have

$$F_i = \ker d_i \oplus \operatorname{im} \gamma = \operatorname{im} d_{i+1} \oplus \operatorname{im} \gamma' \oplus \operatorname{im} \gamma.$$

Observe that $\mathbb{F}$ is the direct sum of the following three complexes. Each is a complex of projective modules.

$$\begin{array}{l} \cdots \to F_{i+2} \to F_{i+1} \to \operatorname{im} d_{i+1} \to 0 \\ \qquad\qquad\qquad\qquad 0 \to \operatorname{im} \gamma \to F_{i-1} \to F_{i-2} \to \dots \\ \qquad\qquad\qquad\qquad 0 \to \operatorname{im} \gamma' \to 0 \end{array}$$

The bottom complex has zero differential. The middle complex is split exact. The induction hypothesis applies to the top complex because its i^{th} homology is zero. Thus, each complex is splittable and the direct sum of splittable complexes is splittable. □

Corollary 3.3. *Let $\mathbb{F}$ be a bounded complex of projective R-modules and S be an R-algebra.*

(a) *If $\mathbb{F}$ is splittable, then $\mathbb{F}^*$ and $\mathbb{F} \otimes_R S$ are splittable.*

(b) *If $\mathrm{H}_j(\mathbb{F})$ is free for all j, then $\mathrm{H}_j(\mathbb{F}^*)$ and $\mathrm{H}_j(\mathbb{F} \otimes_R S)$ are free for all j.*

Proof. In each case $\mathbb{F}$ is the direct sum of complexes $\mathbb{L}'$ and $\mathbb{L}''$, with $\mathbb{L}'$ split exact and the differential of $\mathbb{L}''$ identically zero. It is clear that $\mathbb{L}'^*$ and $\mathbb{L}' \otimes S$ are split exact and that $\mathbb{L}''^*$ and $\mathbb{L}'' \otimes S$ have zero differential. Furthermore, if L''_j is free, then so are L''^*_j and $L''_j \otimes S$. □

Observation 3.4. *Let $\mathbb{F}$ be a bounded complex of projective modules. If $\mathrm{H}_j(\mathbb{F})$ is zero for $j \le i$, then the complex $0 \to \operatorname{coker}(d^*_{i+1}) \xrightarrow{d^*_{i+2}} F^*_{i+2} \xrightarrow{d^*_{i+3}} F^*_{i+3} \to \dots$ is a complex of projective modules which has the same homology as $\mathbb{F}^*$.*

Proof. This result is well known. We record only a few details from the proof. The hypothesis ensures the existence of homotopy maps $s_j \colon F_j \to F_{j+1}$, which satisfy $d_{j+1} \circ s_j + s_{j-1} \circ d_j$ is the identity on F_j, for $j \le i$. One may form the dual of these equations. In particular, $d^*_{i+1} \circ s^*_i \circ d^*_{i+1} = d^*_{i+1}$. It follows that F^*_{i+1} is equal to the direct sum $\operatorname{im} d^*_{i+1} \oplus \ker d^*_{i+1} \circ s^*_i$; and therefore, $\operatorname{coker} d^*_{i+1}$ is a projective module. □

In the next result we apply Observation 3.4 to the complexes $\mathbb{N}(P,Q)$ and $\mathbb{M}(P,Q)$ of Definition 2.4. The hypothesis in (a) concerns the right side of $\mathbb{N}(P,Q)$ and the conclusion is about the left side of $\mathbb{M}(P,Q)$.

Corollary 3.5. *Fix integers m, n, and p.*

(a) *If $\mathrm{H}_{\mathcal{N}}(m+q,n+q,p-q) = 0$ for all positive q, then $\mathrm{H}_{\mathcal{M}}(m+q,n+q,p-q) = 0$ for all positive q. Furthermore, if the base ring is $\mathbb{Z}$, then $\mathrm{H}_{\mathcal{M}}(m,n,p)$ is a free module.*

(b) *If $\mathrm{H}_{\mathcal{M}}(m-q,n-q,p+q) = 0$ for all positive q, then $\mathrm{H}_{\mathcal{N}}(m-q,n-q,p+q) = 0$ for all positive q. Furthermore, if the base ring is $\mathbb{Z}$, then $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is a free module.*

Proof. The statements are dual to one another. We prove (a). Let $P = m+p$ and $Q = n+p$. We are told that the homology of the complex $\mathbb{N}(P,Q)$ is zero at all positions to the right of $\mathcal{N}(m,n,p)$. Apply Observation 3.4 to see that the complex

$$0 \to \frac{\mathcal{M}(m,n,p)}{\operatorname{im}(\mathcal{M}(m+1,n+1,p-1))} \to \mathcal{M}(m-1,n-1,p+1) \to \dots$$

is a complex of projective modules which has the same homology as the complex $\mathbb{M}(P,Q)$. If the base ring is $\mathbb{Z}$, then $H_{\mathcal{M}}(m,n,p)$ is a submodule of a finitely generated projective $\mathbb{Z}$-module; and is therefore free. □

Corollary 3.6. *If the complexes $\mathbb{N}(P,Q)$ and $\mathbb{M}(P,Q)$ both have all of their homology concentrated in position (m,n,p), then the homology of $\mathbb{N}(P,Q)$ and $\mathbb{M}(P,Q)$ is free.*

Proof. In light of Corollary 3.3, we may assume that the base ring is $\mathbb{Z}$. We know that $\mathrm{H}_{\mathcal{N}}(m+q,n+q,p-q) = 0$ for all positive q. Corollary 3.5 reminds us that $\mathrm{H}_{\mathcal{M}}(m,n,p)$ is a free module; thus, all of the homology of $\mathbb{M}(P,Q)$ is free and Corollary 3.3 guarantees that the homology of $\mathbb{N}(P,Q)$ is also free. □

4. The case $e = 2$.

In this section we use the Eagon-Northcott and Buchsbaum-Rim complexes to calculate the homology of the complex (0.1) when $e = 2$. The case $g = 2$ may be treated in analogous manner. Once one knows that the homology of the complex $\mathbb{N}(P,Q)$ is free, then one also knows that the homology of $\mathbb{M}(P,Q)$ is also free by Corollary 3.3.

Theorem 4.1. *If $e = 2$, and P and Q are integers with $Q-2 \le P$, then the complex $\mathbb{N}(P,Q)$ has free homology.*

(a) *If $P = Q-2$, then the homology of $\mathbb{N}(P,Q)$ is concentrated in position $(0,2,P)$ and has rank $\binom{2g}{P+2} - (P+3)\binom{g}{P+2}$.*
(b) *If $Q-1 \le P \le 2Q-2$, then the homology of $\mathbb{N}(P,Q)$ is concentrated in position $(P-Q+1,1,Q-1)$ and has rank $(2Q-1-P)\binom{g}{Q}$.*
(c) *If $P = 2Q-1$, then $\mathbb{N}(P,Q)$ is split exact.*
(d) *If $2Q \le P$, then the homology of $\mathbb{N}(P,Q)$ is concentrated in position $(P-Q,0,Q)$ and has rank $(P-2Q+1)\binom{g}{Q}$.*

Proof. Theorem 2.1, together with (d), takes care of (a). Henceforth, the parameters (P,Q) satisfy $-1 \le P-Q$. The p^{th} homology of $\mathbb{N}(P,Q)$ is also called $\mathrm{H}_{\mathcal{N}}(P-p,Q-p,p)$ and, by (1.4), this homology is equal to $\mathrm{Tor}^{\mathcal{P}}_{p,Q}(M_{P-Q},R)$, in the notation which is given at the beginning of section 1. The ring $T = \mathcal{P}/I_2(\varphi)$ is defined by the 2×2 minors of a $2\times g$ matrix of indeterminates. The minimal homogeneous $\mathcal{P}$ resolution $\mathcal{C}^\ell$ of M_ℓ is known (see, for example, [5, Thm. A2.10]) for all ℓ with $-1 \le \ell$. Let $\widetilde{E} = E\otimes_R \mathcal{P}$, $\widetilde{G} = G\otimes_R\mathcal{P}$, and $\widetilde{\varphi}\colon \widetilde{G}\to\widetilde{E}$ be the map which is defined by

$$(v\otimes 1)[\widetilde{\varphi}(x\otimes 1)] = v\otimes x \in \mathcal{P}$$

for all $v\in E^*$ and $x\in G$. Notice that the matrix for $\widetilde{\varphi}$, with respect to the bases $x_1\otimes 1,\dots,x_g\otimes 1$ and $u_1\otimes 1, u_2\otimes 1$, for $\widetilde{G}$ and $\widetilde{E}$, respectively, is the $2\times g$ matrix Z of Observation 0.6. The entry of Z in row i column j is the indeterminate z_{ij} which is equal to the element $v_i\otimes x_j$ of $\mathcal{P} = S_\bullet(E^*\otimes G)$, where v_1, v_2 is the basis for E^* which is dual to the basis u_1,u_2 for E. The resolution $\mathcal{C}^\ell$ is

$$0\to D_{g-1}\widetilde{E}^*\otimes {\textstyle\bigwedge^g}\widetilde{G}\to\cdots\to D_0\widetilde{E}^*\otimes{\textstyle\bigwedge^1}\widetilde{G},\quad \text{for } \ell=-1,$$

$$\begin{gathered}0\to D_{g-\ell-2}\widetilde{E}^*\otimes{\textstyle\bigwedge^g}\widetilde{G}\to\dots\\ \cdots\to D_0\widetilde{E}^*\otimes{\textstyle\bigwedge^{\ell+2}}\widetilde{G}\to S_0\widetilde{E}\otimes{\textstyle\bigwedge^\ell}\widetilde{G}\to\cdots\to S_\ell\widetilde{E}\otimes{\textstyle\bigwedge^0}\widetilde{G},\quad\text{for } 0\le\ell\le g-1,\end{gathered}$$

$$0\to S_{\ell-g}\widetilde{E}\otimes{\textstyle\bigwedge^g}\widetilde{G}\to\cdots\to S_\ell\widetilde{E}\otimes{\textstyle\bigwedge^0}\widetilde{G},\quad\text{for } g-1\le\ell.$$

The augmentation map $\mathcal{P}\to M_0 = T$ is the map φ of Observation 0.6. The augmentation map $\widetilde{E} = E\otimes_R\mathcal{P}\to M_1$ is

$$u\otimes f\mapsto (u(\omega_{E^*})\otimes 1)\cdot\varphi(f).$$

The multiplication takes place in $S = S_\bullet(E^*)\otimes S_\bullet G$. Recall that M_1 is the submodule $\sum_n S_{n+1}E^*\otimes S_nG$ of S. If $2\le\ell$, then the augmentation map from $S_\ell\widetilde{E}\to M_\ell$ is induced by the augmentation from $\mathcal{C}^1$ to M_1. The augmentation $\widetilde{G} = G\otimes_R\mathcal{P}\to M_{-1}$ is $x\otimes f\mapsto(1\otimes x)\cdot\varphi(f)$.

Each map in each $\mathcal{C}^\ell$, with $-1 \le \ell$, is linear everywhere, except

$$D_0\widetilde{E}^* \otimes \bigwedge^{\ell+2} \widetilde{G} \to S_0\widetilde{E} \otimes \bigwedge^{\ell} \widetilde{G}$$

has degree two. In other words, if $\mathcal{C}^\ell$ is $\cdots \to C_1 \to C_0$, with $C_p = \bigoplus_Q \mathcal{P}(-Q)^{\beta_{p,Q}}$, then the summand $S_{\ell-p}\widetilde{E} \otimes \bigwedge^p \widetilde{G}$ of $\mathcal{C}^\ell$ is $\mathcal{P}(-p)^{\beta_{p,p}}$, and $D_{Q-\ell-2}\widetilde{E}^* \otimes \bigwedge^Q \widetilde{G}$ is $\mathcal{P}(-Q)^{\beta_{Q-1,Q}}$. This grading holds for $\ell = -1$ because the generators $1 \otimes x_j$ of M_{-1} all have degree 1. See, for example, the proof of Theorem 1.1 (d).

We see that differential in each complex $\mathcal{C}^\ell \otimes_{\mathcal{P}} R$ is identically equal to zero. Also, $\widetilde{E} \otimes_{\mathcal{P}} R = E$ and $\widetilde{G} \otimes_{\mathcal{P}} R = G$. It is clear that if $-1 \le P-Q$, then $\operatorname{Tor}^{\mathcal{P}}_{p,Q}(M_{P-Q}, R)$ is the free R-module

$$\begin{cases} S_{P-Q-p}E \otimes \bigwedge^p G, & \text{if } p = Q \text{ and } p \le P-Q, \\ D_{2Q-P-2}E^* \otimes \bigwedge^Q G, & \text{if } p = Q-1 \text{ and } P-Q+1 \le p, \\ 0, & \text{otherwise.} \quad \square \end{cases}$$

5. Vanishing results.

Corollaries 5.1 and 5.2 and Theorem 5.7 are all used in [10]. Theorem 5.4 is the main new result in the section. Everything else is a consequence of Theorem 1.1 or 2.1.

Corollary 5.1.
(a) *Assume $1 - e \le P - Q \le g - 1$. If either $eg - g + 1 \le Q$ or $eg - e + 1 \le P$, then $\mathbb{M}(P,Q)$ and $\mathbb{N}(P,Q)$ are split exact.*
(b) *If $Q = e+P$, then $\mathbb{M}(P,Q)$ and $\mathbb{N}(P,Q)$ have free homology which is concentrated in position $(0,e,P)$; furthermore, if $eg+1 \le Q$, then $\mathbb{M}(P,Q)$ and $\mathbb{N}(P,Q)$ are split exact.*
(c) *If $g + Q = P$, then the extended complex $\widetilde{\mathbb{M}}(P,Q)$, which is defined to be*

$$0 \to \mathcal{M}(g+Q,Q,0) \to \cdots \to \mathcal{M}(g,0,Q) \xrightarrow{\gamma} \bigwedge^{g+Q}(E \otimes G^*) \to 0,$$

has free homology which is concentrated in the position of $\bigwedge^{g+Q}(E \otimes G^*)$; furthermore, if $\alpha \le Q$, then $\widetilde{\mathbb{M}}(P,Q)$ is split exact.

Proof. Theorem 1.1 (f) and Remark 2.5 tell us that if $1 - e \le P - Q \le g - 1$, then

$$\mathrm{H}_i(\mathbb{N}(P,Q)) \cong \mathrm{H}_{i'}(\mathbb{M}(P',Q')),$$

provided $P + P' = eg - e$, $Q + Q' = eg - g$, and $i' = i + eg - P - Q$. Assertion (a) is now obvious. The other two assertions follow immediately from Theorem 2.1. $\square$

Corollary 5.2. *Fix integers m, n, and p. Let $P = m + p$ and $Q = n + p$. Assume that $1 - e \le P - Q \le g - 1$. Consider the truncation*

$$\mathfrak{T}: \quad 0 \to \mathcal{M}(P,Q,0) \to \cdots \to \mathcal{M}(m,n,p) \to 0$$

of $\mathbb{M}(P,Q)$. *If* $g-1 \le m$ *or* $e-1 \le n$, *then* $\mathfrak{T}$ *has free homology concentrated in position* (m,n,p).

Proof. In light of Corollary 3.3, we may take the base ring to be $\mathbb{Z}$. Apply Theorem 1.1 (f) to the complex $\mathbb{N}(P,Q)$ to see that

$$\mathrm{H}_{\mathcal{N}}(m+q,n+q,p-q) \cong \mathrm{H}_{\mathcal{M}}(g-1-m-q,e-1-n-q,\alpha-p+q)$$

for all integers q. If q is positive, then the hypothesis on the size of m or n ensures that $\mathrm{H}_{\mathcal{N}}(m+q,n+q,p-q)$ is zero. Apply Corollary 3.5 (a). □

The idea behind the decomposition of $\mathbb{M}(P,Q)$ in 5.3, is similar to the idea behind $\mathbb{N}(P,Q) = \oplus \mathbb{N}[x^{\boldsymbol{q}}]$, from Observation 2.16.

Notation 5.3. Fix bases

$$x_1,\dots,x_g \text{ for } G;\ y_1,\dots,y_g \text{ for } G^*;\ u_1,\dots,u_e \text{ for } E;\text{ and } v_1,\dots,v_e \text{ for } E^*,$$

with $\{x_i\}$ dual to $\{y_j\}$, and $\{u_i\}$ dual to $\{v_j\}$. Let N be a monomial of degree $m+p$ in $u_1,\dots,u_e$ and let M be a monomial of degree $n+p$ in $y_1,\dots,y_g$. If $N = u_1^{\ell_1}\cdots u_e^{\ell_e}$ and $M = y_1^{\lambda_1}\cdots y_g^{\lambda_g}$, then define $\mathcal{M}(m,n,p)|_{N,M}$ to be the submodule of $\mathcal{M}(m,n,p)$ which consists of those elements which are homogeneous of degree ℓ_i in u_i and of degree λ_j in y_j for all i and all j. The submodules $\mathcal{M}(m,n,p)|_N$ and $\mathcal{M}(m,n,p)|_M$, homogeneous in just the $\{u_i\}$ or just the $\{y_j\}$, are defined in an analogous manner. The differential of $\mathbb{M}$ is homogeneous in the u's and y's; and therefore, the complex $\mathbb{M}(P,Q)$ naturally decomposes into a direct sum of subcomplexes $\mathbb{M}(P,Q)|_{N,M}$, where the sum is taken over all monomials N and M of degree P and Q, respectively. Take $\mathrm{H}_{\mathcal{M}}(m,n,p)|_{N,M}$ to mean the homology of the complex $\mathbb{M}(m+p,n+p)|_{N,M}$ at $\mathcal{M}(m,n,p)|_{N,M}$. We see that $\mathrm{H}_{\mathcal{M}}(m,n,p)$ is equal to $\bigoplus_{N,M} \mathrm{H}_{\mathcal{M}}(m,n,p)|_{N,M}$.

Theorem 5.4. *Let* m, n, *and* p *be integers.*
(a) *If* $ng-n+1 \le p$, *then* $\mathrm{H}_{\mathcal{M}}(0,n,p)=0$.
(b) *If* $me-m+1 \le p$, *then* $\mathrm{H}_{\mathcal{M}}(m,0,p)=0$.
(c) *If* $e+g-2 \le p$, *then* $\mathrm{H}_{\mathcal{M}}(1,1,p)=0$.

Proof. We prove (b) and (c) using the same set up, in (c) we take $m=1$. Let $N = u_1^{\ell_1}\cdots u_e^{\ell_e}$ be a monomial of degree $p+m$. There is an index i with $e\ell_i \le p+m$. For notational convenience, we say that $i=e$. Let ℓ represent ℓ_e, $\bar N = N/u_e$, and $\hat N = N/u_e^{\ell}$. Consider the short exact sequence of complexes

$$\begin{array}{ccccccc}
\dots \longrightarrow & \hat{\mathcal{M}}(m+1,1,p-\ell)|_{\hat N}\otimes \bigwedge^{\ell} G^* & \longrightarrow & \hat{\mathcal{M}}(m,0,p-\ell)|_{\hat N}\otimes \bigwedge^{\ell} G^* & \longrightarrow & 0 \\
& \downarrow & & \downarrow{\scriptstyle \mathrm{incl}} & & \\
\dots \longrightarrow & \mathcal{M}(m+1,1,p-1)|_N & \longrightarrow & \mathcal{M}(m,0,p)|_N & \longrightarrow & 0 \\
& \downarrow{\scriptstyle v_e} & & \downarrow{\scriptstyle v_e} & & \\
\dots \longrightarrow & \mathcal{M}(m,1,p-1)|_{\bar N} & \longrightarrow & \mathcal{M}(m-1,0,p)|_{\bar N} & \longrightarrow & 0,
\end{array}$$

where each module $\hat{\mathcal{M}}$ is formed using the free module $\hat E = Ru_1 \oplus \dots \oplus Ru_{e-1}$ in place of E. The map "v_e" represents $v_e \otimes 1 \otimes 1$ and the map "incl" sends $T\otimes w$ to $T \wedge [u_e^{(\ell)} \bowtie w]$.

The assertion in (b) is obvious if $e=1$ or $m=0$. Henceforth, we take $2\le e$ and $1\le m$. Induction on m ensures that $\mathrm{H}_{\mathcal{M}}(m-1,0,p)=0$. Observe that

$$m(e-1)-m+1\le p-\ell.$$

There is nothing to prove if $\ell\le m$. If $m+1\le \ell$, then

$$m(e-1)-m+1\le (\ell-1)(e-1)-m+1=(\ell e-m)-\ell+(2-e)\le p-\ell.$$

Induction on e gives $\mathrm{H}_{\hat{\mathcal{M}}}(m,0,p-\ell)=0$. The long exact sequence of homology yields that $\mathrm{H}_{\mathcal{M}}(m,0,p)|_N=0$. The proof of (b) is complete. Assertion (a) follows by symmetry. We prove (c). If $g=1$, then the result is obvious. Henceforth, we assume that $2\le g$. The statement is symmetric in e and g; so, without loss of generality, we may assume $g\le e$. Consider the short exact sequence of complexes which is analogous the the above short exact sequence but has $\mathcal{M}(1,1,p)|_N$ in the place of $\mathcal{M}(m+1,1,p-1)|_N$. Apply (a) to see that $\mathrm{H}_{\mathcal{M}}(0,1,p)=0$. Observe that

$$(e-1)+g-2\le p-\ell.$$

There is nothing to prove if $\ell\le 1$. If $2\le \ell$, then

$$e+g-3+\ell\le 2e-3+\ell\le \ell e-1\le p.$$

The middle inequality holds because $0\le(\ell-2)(e-1)$. We may now apply induction to see that $\mathrm{H}_{\hat{\mathcal{M}}}(1,1,p-\ell)=0$. The long exact sequence of homology yields that $\mathrm{H}_{\mathcal{M}}(1,1,p)|_N=0$. □

Remark. The constraints in Theorem 5.4 are necessary, at least in the base cases. In (a), if $g=2$, $n=e-1$, and $ng-n=p$, then Theorem 1.1 (f) shows $\mathrm{H}_{\mathcal{M}}(0,n,p)\cong \mathrm{H}_{\mathcal{N}}(1,0,0)\neq 0$. In (c), if $g=2$ and $e+g-3=p$, then $\mathrm{H}_{\mathcal{M}}(1,1,p)$ is isomorphic to $\mathrm{H}_{\mathcal{N}}(0,e-2,0)\neq 0$.

Observation 5.5. *If $\mathbb{M}(P,Q)$ has free homology when* $\operatorname{rank}E=P$ *and* $\operatorname{rank}G=Q$, *then $\mathbb{M}(P,Q)$ has free homology for all e and g. If $\mathrm{H}_{\mathcal{M}}(m,n,p)=0$ when* $\operatorname{rank}E=m+p$ *and* $\operatorname{rank}G=n+p$, *then $\mathrm{H}_{\mathcal{M}}(m,n,p)=0$ for all e and g.*

Proof. We saw in 5.3 that $\mathbb{M}(P,Q)$ is a direct sum of subcomplexes of the form $\mathbb{M}(P,Q)|_{N,M}$ where N is a monomial of degree P in $S_\bullet E$ and and M is a monomial of degree Q in $S_\bullet G^*$. □

Observation 5.6. *For each integer p, there are isomorphisms*

$$\mathrm{H}_{\mathcal{M}}(0,1,p)\cong S_pE\otimes {\textstyle\bigwedge^{p+1}} G^*,\qquad \mathrm{H}_{\mathcal{M}}(1,0,p)\cong {\textstyle\bigwedge^{p+1}} E\otimes S_pG^*,$$
$$\mathrm{H}_{\mathcal{N}}(0,1,p)\cong D_pE^*\otimes {\textstyle\bigwedge^{p+1}} G,\quad \textit{and}\quad \mathrm{H}_{\mathcal{N}}(1,0,p)\cong {\textstyle\bigwedge^{p+1}} E^*\otimes D_pG.$$

Proof. We establish the isomorphism which is listed first. The others follow by symmetry and duality. We define a homomorphism $\rho\colon \mathcal{M}(0,1,p)\to S_pE\otimes\bigwedge^{p+1}G^*$

by describing the action of $\rho(1\otimes y\otimes Z)$, for $y\in G^*$ and $Z\in\bigwedge^p(E\otimes G^*)$, on the arbitrary element $V\otimes X$ of $D_pE^*\otimes\bigwedge^{p+1}G$. We define

$$\rho(1\otimes y\otimes Z)[V\otimes X]=Z[V\bowtie y(X)]\in R.$$

It is easy to see that

$$\mathcal{M}(1,2,p-1)\to\mathcal{M}(0,1,p)\xrightarrow{\rho}S_pE\otimes\textstyle\bigwedge^{p+1}G^*\to 0$$

is a complex. Also, in the language of 5.3, it is easy to see that

$$\rho(1\otimes y_{i_{p+1}}\otimes(u_{j_1}\otimes y_{i_1})\wedge\ldots\wedge(u_{j_p}\otimes y_{i_p}))=(u_{j_1}\cdots u_{j_p})\otimes(y_{i_1}\wedge\ldots\wedge y_{i_{p+1}}),$$

since

$$(u\otimes y)[V\bowtie X']=u(V)\bowtie y(X'),$$

for $u\in E$, $y\in G^*$, $V\in D_pE^*$, and $X'\in\bigwedge^p G$. The most recent claim may be verified by checking it at $V=v^{(p)}$ as v varies over E^*. It follows that ρ is surjective. Finally, it is not difficult to see that $\mathcal{M}(0,1,p)/\operatorname{im}(\mathcal{M}(1,2,p-1))$ is generated by

$$\{1\otimes y_{i_{p+1}}\otimes(u_{j_1}\otimes y_{i_1})\wedge\ldots\wedge(u_{j_p}\otimes y_{i_p})\mid i_1<\cdots<i_{p+1}\quad\text{and}\quad j_1\le\cdots\le j_p\}.$$

Thus, ρ carries a generating set for $\mathrm{H}_{\mathcal{M}}(0,1,p)$ onto a basis for $S_pE\otimes\bigwedge^{p+1}G^*$; and therefore, ρ induces the desired isomorphism. □

In the next result we gather a large amount of information about the homology of the complexes $\mathbb{M}(P,Q)$ and $\mathbb{N}(P,Q)$ into one place. Notice we need to prove only one eighth of what we state. First of all, as soon as we prove that $\mathbb{M}(P,Q)$ has free homology, then $\mathbb{N}(P,Q)$ automatically also has free homology by Corollary 3.3. Secondly, a valid statement remains valid if the roles of P and Q are reversed, along with the roles of e and g. Finally, as soon as we identify the homology of $\mathbb{M}(P,Q)$, then by Theorem 1.1 (f), we have also identified the homology of $\mathbb{N}(eg-e-P,eg-g-Q)$. Indeed, an alternate method for proving Theorem 1.1 (f) is to exhibit a quasi-isomorphism

$$M\colon\mathbb{M}(P,Q)\to\mathbb{N}(eg-e-P,eg-g-Q)[\alpha-1-P-Q].$$

This approach is carried out in [11].

Theorem 5.7. *Let $\mathbb{M}=\mathbb{M}(P,Q)$ and $\mathbb{N}=\mathbb{N}(P,Q)$ for integers P and Q which satisfy $1-e\le P-Q\le g-1$.*

(1) *If (P,Q) is equal to $(1,1)$ or $(eg-e-1,eg-g-1)$, then $\mathbb{M}$ and $\mathbb{N}$ are split exact.*
(2) *If $2\le P=Q\le 4$, then $\mathbb{M}$ and $\mathbb{N}$ have free homology concentrated at spot $(1,1,P-1)$.*
(3) *If $2\le eg-e-P=eg-g-Q\le 4$, then $\mathbb{M}$ and $\mathbb{N}$ have free homology concentrated at spot $(g-2,e-2,P-g+2)$.*
(4) *If $2Q-1\le P$, then $\mathbb{M}$ and $\mathbb{N}$ have free homology concentrated at spot $(P-Q,0,Q)$.*

(5) *If* $2P-1\le Q$, *then* $\mathbb{M}$ *and* $\mathbb{N}$ *have free homology concentrated at spot* $(0,Q-P,P)$.
(6) *If* $eg-2g+e-1\le 2Q-P$, *then* $\mathbb{M}$ *and* $\mathbb{N}$ *have free homology concentrated at spot* $(P-Q-1+e,e-1,Q+1-e)$.
(7) *If* $eg-2e+g-1\le 2P-Q$, *then* $\mathbb{M}$ *and* $\mathbb{N}$ *have free homology concentrated at spot* $(g-1,Q-P+g-1,P-g+1)$.
(8) *If* $(P,Q)=(3,4)$ *or* $(eg-e-3,eg-g-4)$, *then* $\mathbb{M}$ *and* $\mathbb{N}$ *have free homology. The non-zero homology modules have rank*

$$\binom{g}{4}\binom{e+2}{3} \text{ at spot } (0,1,3) \text{ or } (g-1,e-2,\alpha-3) \text{ and}$$

$$\frac{g}{2}\binom{e}{3}\binom{g+1}{3} \text{ at spot } (1,2,2) \text{ or } (g-2,e-3,\alpha-2).$$

(9) *If* $(P,Q)=(4,3)$ *or* $(eg-e-4,eg-g-3)$, *then* $\mathbb{M}$ *and* $\mathbb{N}$ *have free homology. The non-zero homology modules have rank*

$$\binom{e}{4}\binom{g+2}{3} \text{ at spot } (1,0,3) \text{ or } (g-2,e-1,\alpha-3) \text{ and}$$

$$\frac{e}{2}\binom{g}{3}\binom{e+1}{3} \text{ at spot } (2,1,2) \text{ or } (g-3,e-2,\alpha-2).$$

Proof. Assertion (1) is obvious. We prove (4). Assume first that $2Q\le P$. Notice $Q\le P-Q$; so, Observation 5.5 shows that it suffices to establish the result for $(e',g')=(P,P-Q)$. In this case, $P=Q+g'$, so Theorem 2.1 tells us that the homology of $\mathbb{N}(P,Q)$ and $\mathbb{M}(P,Q)$ is concentrated in position $(P-Q,0,Q)$. Now assume that $2Q-1=P$. It suffices to establish the result for $(e',g')=(P,Q)$. In this case, Corollary 5.2 applies since $\mathbb{M}(P,Q)$ is

$$0\to\mathcal{M}(2Q-1,Q,0)\to\cdots\to\mathcal{M}(Q-1,0,Q)\to 0,$$

with $Q-1=g'-1$. The proof of (4) is complete. Assertion (6) follows from (4) by way of Theorem 1.1 (f). The same technique yields (5); hence also (7).

We now prove (2). According to Observation 5.5, it suffices to prove the assertion for $e=g=P=Q$. If $P=Q=2$, then assertion (6) applies. If $P=Q=3$, then Theorem 1.1 (f) yields

$$\mathrm{H}_{\mathcal{M}}(3,3,0)=0,\quad \mathrm{H}_{\mathcal{M}}(2,2,1)\cong\mathrm{H}_{\mathcal{N}}(0,0,3),\quad \mathrm{H}_{\mathcal{M}}(1,1,2)\cong\mathrm{H}_{\mathcal{N}}(1,1,2),$$

$$\mathrm{H}_{\mathcal{M}}(0,0,3)\cong\mathrm{H}_{\mathcal{N}}(2,2,1),\quad \text{and}\quad \mathrm{H}_{\mathcal{N}}(3,3,0)=0.$$

Theorem 5.4 tells us that $\mathrm{H}_{\mathcal{M}}(0,0,3)=0$; so, Corollary 3.5 tells us that $\mathrm{H}_{\mathcal{N}}(0,0,3)$ is also zero. The complexes $\mathbb{M}(3,3)$ and $\mathbb{N}(3,3)$ both have all of their homology concentrated in position $(1,1,2)$, so Corollary 3.6 completes the proof in the present case. If $P=Q=4$, then it is clear that $\mathrm{H}_{\mathcal{M}}(0,0,4)$ is zero. In light of Corollary

3.3, we may take the base ring to be $\mathbb{Z}$. Theorem 1.1 (f) shows that $\mathrm{H}_{\mathcal{M}}(4,4,0)=0$, $\mathrm{H}_{\mathcal{M}}(3,3,1)\cong\mathrm{H}_{\mathcal{N}}(0,0,8)$,

$$\mathrm{H}_{\mathcal{M}}(2,2,2)\cong\mathrm{H}_{\mathcal{N}}(1,1,7),\quad\text{and}\quad\mathrm{H}_{\mathcal{M}}(1,1,3)\cong\mathrm{H}_{\mathcal{N}}(2,2,6).$$

On the other hand, Theorem 5.4 ensures that

$$\mathrm{H}_{\mathcal{M}}(0,0,8)=0\quad\text{and}\quad\mathrm{H}_{\mathcal{M}}(1,1,7)=0;$$

thus, Corollary 3.5 guarantees that $\mathrm{H}_{\mathcal{N}}(0,0,8)=0$, $\mathrm{H}_{\mathcal{N}}(1,1,7)=0$, and $\mathrm{H}_{\mathcal{N}}(2,2,6)$ is a free module. The proof of (2) is complete; assertion (3) follows by way of Theorem 1.1 (f).

For (8) and (9) it suffices to study the homology of $\mathbb{M}(4,3)$ when the base ring is $\mathbb{Z}$. Observation 5.6 tells us that $\mathrm{H}_{\mathcal{M}}(1,0,3)$ is free of rank $\binom{e}{4}\binom{g+2}{3}$. Apply Observation 5.5. Theorem 1.1 (f), with $e=4$ and $g=3$, yields $\mathrm{H}_{\mathcal{M}}(4,3,0)$ and $\mathrm{H}_{\mathcal{M}}(3,2,1)$ are zero, and that $\mathrm{H}_{\mathcal{M}}(2,1,2)$ is isomorphic to the free module $\mathrm{H}_{\mathcal{N}}(0,2,4)$. To compute the rank of $\mathrm{H}_{\mathcal{M}}(2,1,2)$, use the fact that $\mathbb{M}(P,Q)$ and $\mathrm{H}(\mathbb{M}(P,Q))$ have the same Euler characteristic. □

Remark. The proof of Theorem 5.7 shows that assertions (4) and (5) continue to hold even if the hypothesis $1-e\le P-Q\le g-1$ is not satisfied.

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