HUNEKE-ULRICH ALMOST COMPLETE INTERSECTIONS OF COHEN-MACAULAY TYPE TWO

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Abstract. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, $n \geq 2$ be an integer, X be a $2n{+}1 \times 2n{+}1$ alternating matrix with entries from $\mathfrak{m}$, Y be a $1 \times 2n{+}1$ matrix with entries from $\mathfrak{m}$, I be the ideal $I = I_1(YX)$, and A be the quotient ring R/I. Assume that the grade of I is at least $2n$. (In this case, I is a perfect ideal of grade equal to $2n$ and I is minimally generated by $2n+1$ elements.) We prove that the minimal resolution of A by free $R-$modules is a DG$\Gamma-$algebra. Furthermore, we identify the algebra $\operatorname{Tor}^R_\bullet(A,k)$ and prove that, if R is regular and $\operatorname{char} k = 0$ or $(n+2)/2 \leq \operatorname{char} k$, then the Poincaré series $P^M_A(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}^A_i(M,k) z^i$ is a rational function for every finitely generated $A-$module M. As a consequence, we deduce that if the projective dimension of M is infinite, then, eventually, the betti numbers of M form an increasing sequence with strong exponential growth.

Fix a commutative noetherian local ring $(R, \mathfrak{m}, k)$ and an integer n, with $2 \leq n$. Consider matrices $X_{2n+1\times 2n+1}$ and $Y_{1\times 2n+1}$ with entries from $\mathfrak{m}$. Assume that X is an alternating matrix. Huneke and Ulrich [19] showed that the grade of the ideal $I = I_1(YX)$ is no more than $2n$; furthermore, if the maximum possible grade is attained, then I is a perfect ideal whose grade is exactly one less than its minimal number of generators. (Such ideals are called almost complete intersections.) When R is Gorenstein, the Cohen-Macaulay type of $A = R/I$ (which is defined to be $\dim_k \operatorname{Ext}_R^{\operatorname{depth} A}(k, A)$, and is also equal to $\dim_k \operatorname{Tor}^R_{\operatorname{pd} A}(A, k)$, because I is a perfect ideal) is equal to two (see, for example, Corollary 2.18); which, according to [23], is the smallest possible value. (There do exist almost complete intersection ideals of type two which do not have the form of I, see [32]; such ideals are not considered in the present paper.) Huneke and Ulrich also investigated the linkage history of I. They found that I is in the linkage class of a complete intersection; indeed, I is linked to a hypersurface section of a grade $2n-1$ Gorenstein ideal $I' = I_1(Y'X') + \operatorname{Pf}(X')$ (where X' and Y' have shape $2n \times 2n$ and $1 \times 2n$, respectively, and X' is an alternating matrix); furthermore, I' is linked to a hypersurface section of a grade $2n-2$ almost complete intersection ideal $I'' = I_1(Y''X'')$ (where X'' and Y'' have shape $2n-1 \times 2n-1$ and $1 \times 2n-1$, respectively, and X'' is an alternating matrix). Ideals of the form of I' are known as Huneke-Ulrich deviation two Gorenstein ideals; they are studied extensively in [24, 38, 28]. The interplay

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between the Huneke-Ulrich almost complete intersection ideals and the Huneke-Ulrich deviation two Gorenstein ideals has recently played a role in Beauville's study [11] of hypersurfaces in positive characteristic.

In the present paper we prove, in Corollary 3.21, that the minimal resolution, $\mathbb{M}$, of A by free $R-$modules is a DG$\Gamma-$algebra. We also prove, in Theorem 5.2, that, if R is regular and $\operatorname{char} k = 0$ or $(n+2)/2 \le \operatorname{char} k$, then the Poincaré series

$$P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(M,k) z^i$$

is a rational function for every finitely generated $A-$module M. As a consequence, we deduce, in Corollary 5.3, that if the projective dimension of M is infinite, then, eventually, the betti numbers of M form an increasing sequence with strong exponential growth.

Examples of minimal resolutions which do not support a DG$\Gamma-$structure have been found by Hinič [1], Avramov [3], and Srinivasan [39, 40]. Roughly speaking, there are three ways to put a DG$\Gamma-$structure on the minimal $R-$resolution, $\mathbb{M}$, of A. The first approach is to observe that $\mathbb{M}$ always has a multiplication which satisfies all of the DGΓ axioms, except, it is associative only up to homotopy. If sufficient additional hypotheses are imposed, then every choice of homotopy-associative multiplication is, in fact, associative. This approach works if

- $\operatorname{codim} A \le 3$ [12]; or if
- $\mathbb{M}$ is a graded resolution whose grading satisfies the inequality

  $$d_{a_j} + d_{b_k} + d_{c_\ell} < d_{(a+b+c+1)_i},$$

  for all a, b, c, i, j, k, and ℓ, where $\mathbb{M}_a = \bigoplus_j R(-d_{a_j})$ [41].

The second approach is to prove that if $\mathbb{M}$ is sufficiently short, then a homotopy-associative multiplication can be modified in order to become associative "on the nose." This is the approach of:

- [29, 25] for $\operatorname{codim} A = 4$, and A Gorenstein; and
- [33, 26] for a codimension four almost complete intersection A in which two is a unit.

The third approach is to record an explicit multiplication table for $\mathbb{M}$ and show that it satisfies all of the relevant axioms. This approach works if A is:

- a complete intersection, (in this case, the resolution $\mathbb{M}$ is an exterior algebra);
- one link from a complete intersection [10];
- two links from a complete intersection and is Gorenstein [30];
- a codimension four Gorenstein ring defined by the $(n-1)\times(n-1)$ minors of an $n \times n$ matrix [17];
- a determinantal ring defined by the maximal minors of a matrix in equicharacteristic zero, or the ring defined by I^k, where I is generated by a regular sequence [37]; or
- a Gorenstein ring defined by a Huneke$-$Ulrich deviation two ideal [38].

In section 3 we produce the multiplication table for $\mathbb{M}$, when A is a Huneke-Ulrich almost complete intersection.

For the time being, let A be a quotient of a regular local ring $(R, \mathfrak{m}, k)$, and let $\mathbb{M}$ be the minimal resolution of A by free $R-$modules. If $\mathbb{M}$ is a DG$\Gamma-$algebra, then the machinery of Avramov [2, 5, 9] may be used to convert many interesting and difficult questions about A into questions about the algebra $T_\bullet = \operatorname{Tor}_\bullet^R(A, k)$; for example, $P_A^k(z) = P_R^k(z)P_{T_\bullet}^k(z)$. The algebra $T_\bullet$, although graded-commutative instead of commutative, is in many ways simpler than the original ring A. This philosophy has lead to some striking theorems in the case that A has small codimension or small linking number. If any one of the following conditions hold:

(a) $\operatorname{codim} A \le 3$, or
(b) $\operatorname{codim} A = 4$ and A is Gorenstein, or
(c) A is one link from a complete intersection, or
(d) A is two links from a complete intersection and A is Gorenstein, or
(e) A is an almost complete intersection of codimension four in which two is a unit, or
(f) $(A, \mathfrak{m}, k)$ is a Huneke-Ulrich, deviation two, Gorenstein ring, of codimension $2n-1$, with either $\operatorname{char} k = 0$, or $n - 1 \le \operatorname{char} k$,

then it is shown in [20, 10, 6, 31, 28, 42] that all of the following conclusions hold:

(1) The Poincaré series $P_A^M(z)$ is a rational function for all finitely generated A-modules M.
(2) The Eisenbud Conjecture [13] holds for the ring A. That is, if M is a finitely generated $A-$module whose betti numbers are bounded, then the minimal resolution of M eventually becomes periodic of period at most two.
(3) The betti sequence $\{b_i^A(M)\}$ is eventually nondecreasing for every finitely generated $A-$module M.
(4) The growth of the betti sequence $\{b_i^A(M)\}$ is either polynomially bounded or strongly exponential for every finitely generated $A-$module M.
(5) If R contains the field of rational numbers, then the Herzog Conjecture [18] holds for the ring A. That is, the cotangent modules $T_i(A/R)$ vanish for all large i if and only if A is a complete intersection.

The study of the rationality of Poincaré series has a long and distinguished history; see [36] or the introduction to [10] for a brief synopsis. Gasharov and Peeva [14] found counterexamples to the Eisenbud Conjecture. Questions like (3) and (4) about the asymptotic behavior of betti sequences have been considered at least as far back as [34] and [4]. The present status of these and other, similar, questions may be found in [8]. Ulrich [43] has proved the Herzog Conjecture when A is in the linkage class of a complete intersection; the conjecture remains open for arbitrary rings. (Strictly speaking, as stated, [43] only deals with the weaker version of the Herzog Conjecture, where the vanishing of T_i is required for every $i \ge 2$. But in fact, [43, 2.9 and 1.3] does prove the stronger conjecture (as stated in (5)) for ideals in the linkage class of a complete intersection.)

In each case, (a) – (f), there are three steps to the process:

(i) one proves that the resolution $\mathbb{M}$ is a DG$-$algebra;
(ii) one classifies the Tor$-$algebras $\operatorname{Tor}_\bullet^R(A, k)$; and
(iii) one completes the proof of (1) – (5).

In the present paper we prove that the hypothesis

(g) $(A, \mathfrak{m}, k)$ is a Huneke-Ulrich almost complete intersection of codimension $2n$, with either $\operatorname{char} k = 0$, or $(n+2)/2 \le \operatorname{char} k$

also leads to conclusions (1) – (5). Step (i) is carried out in section 3, step (ii) in section 4, and step (iii) in section 5. The resolution $\mathbb{M}$ is described in section 2; section 1 consists of identities involving pfaffians and binomial coefficients.

Finally, we note that, if A is a Huneke-Ulrich almost complete intersection, then the Poincaré series $P_A^k(z)$ depends on the characteristic of k. In particular, if A is the ring $\mathbb{Z}[X,Y]/I_1(YX)$, where the entries of X and Y are indeterminates, then there is no minimal graded $A-$free resolution of $\mathbb{Z} = \frac{A}{I_1(X)+I_1(Y)}$. This example must be included with the growing list of "determinantal-type" modules whose minimal resolution is characteristic dependent; see, for example, [16] and [35].

1. Preliminary results.

In this paper "ring" means associative ring with 1; furthermore, all rings are either commutative or graded-commutative. We often consider binomial coefficients with negative parameters; consequently, we recall the standard definition and properties of these objects. See [27] or [28] for more details.

Definition 1.1. For integers a and b, the binomial coefficient $\binom{a}{b}$ is defined to be

$$\binom{a}{b} = \begin{cases} \dfrac{a(a-1)\cdots(a-b+1)}{b!} & \text{if } 0 < b, \\ 1 & \text{if } 0 = b, \text{ and} \\ 0 & \text{if } b < 0. \end{cases}$$

Observation 1.2. *If a, b, and c are integers, then the following identities hold:*
(*a*) *if $0 \le a < b$, then $\binom{a}{b} = 0$;*

(b) $$\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b};$$

(c) $$a\binom{a-1}{b-1} = b\binom{a}{b} = (a-b+1)\binom{a}{b-1};$$

(*d*) *if $0 \le a$, then $\binom{a}{b} = \binom{a}{a-b}$;*
(*e*) *if $0 \le b$, then $\binom{-1}{b} = (-1)^b$;*

(f) $$\binom{a}{b} = (-1)^b\binom{b-a-1}{b};$$

(*g*) *if $0 \le a$, then*

$$\sum_{k\in\mathbb{Z}} (-1)^k \binom{b+k}{c+k}\binom{a}{k} = (-1)^a\binom{b}{a+c};$$

(*h*) *if* $0 \le a$, *then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{b-k}{c-k}\binom{a}{k}=\binom{b-a}{c};$$

(i)
$$\sum_{k\in\mathbb{Z}}\binom{b}{c-k}\binom{a}{k}=\binom{a+b}{c}; \quad \textit{and}$$

(j)
$$\binom{a}{a-b}=\binom{a}{b}+(-1)^{a+b}\binom{-b-1}{-a-1}.$$

Proof. Identities (a) – (d) are well known and easy to prove. Identity (f) is an immediate consequence of the definition and (e) follows from (f). The identities (g), (h), and (j) are proved in [27]. Let $L(a,b,c)$ represent the left side of (i). It is clear that

$$L(a+1,b,c)=L(a,b,c)+L(a,b,c-1) \tag{1.3}$$

for all integers a, b, and c. Observe that (i) holds for all a and b whenever $c\le -1$. The identity also holds for all b and c when $a=0$. A quick application of (1.3) shows that (h) holds for all b and c, whenever $0\le a$. A second induction completes the proof. (See Lemma 1.15, if necessary.) □

We also make much use of multilinear and divided power algebra. (More information about these topics may be found in [12, Appendix] or [15].) Let R be a commutative noetherian ring, and F be a free $R-$module of finite rank. Each element of F^* is a graded derivation on $\bigwedge^\bullet F$, and this action gives rise to the $\bigwedge^\bullet F^*$-module structure on $\bigwedge^\bullet F$. The $\bigwedge^\bullet F$-module structure on $\bigwedge^\bullet F^*$ is obtained in an analogous manner. In particular, if $a_i\in\bigwedge^i F$ and $b_j\in\bigwedge^j F^*$, then

$$a_i(b_j)\in\bigwedge^{j-i}F^* \qquad\text{and}\qquad b_j(a_i)\in\bigwedge^{i-j}F.$$

The following well known formulas show some of the interaction between the two module structures.

Proposition 1.4. *Let F be a free module over a commutative noetherian ring R and let $a,b\in\bigwedge^\bullet F$ and $c\in\bigwedge^\bullet F^*$ be homogeneous elements.*

(a) *If* $\deg a=1$, *then*
$$(a(c))\,(b)=a\wedge(c(b))+(-1)^{1+\deg c}c(a\wedge b).$$

(b) *If* $c\in\bigwedge^{\operatorname{rank}F}F^*$, *then*
$$(a(c))\,(b)=(-1)^{\nu}\,(b(c))\,(a),$$
where $\nu=(\operatorname{rank}F-\deg a)(\operatorname{rank}F-\deg b)$.

Note. The value for ν which is given above is correct and is different than the value given in [12].

The first main theorem in this paper states that the resolution $\mathbb{M}$ of section 3 is a DG$\Gamma-$algebra. At this time, we recall most of the definition of this concept. The easiest way to remember the surpressed axioms is to recall that the definition $x^{(q)}=x^q/q!$ gives every graded $\mathbb{Q}-$algebra the structure of a divided power algebra.

Definition 1.5. A DG$\Gamma-$algebra is an associative graded ring $\mathbb{F} = \bigoplus\limits_{i=0}^{\infty} \mathbb{F}_i$, together with a differential $d\colon \mathbb{F} \to \mathbb{F}$, which satisfies:

(a) the multiplication in $\mathbb{F}$ is graded-commutative, that is, $x_i x_j = (-1)^{ij} x_j x_i \in \mathbb{F}_{i+j}$,
(b) $x_i^2 = 0$ for i odd,
(c) d is an $\mathbb{F}_0-$module homomorphism with $d^2 = 0$,
(d) $d(x_i) \in \mathbb{F}_{i-1}$, and
(e) $d(x_i x_j) = d(x_i)x_j + (-1)^i x_i d(x_j)$,

for all $x_i \in \mathbb{F}_i$ and $x_j \in \mathbb{F}_j$. Furthermore, for every homogeneous element x of positive even degree, there is an associated sequence of divided powers $x^{(0)} = 1$, $x^{(1)} = x$, $x^{(2)}$, $x^{(3)}$, $\dots$, which satisfies $\deg x^{(q)} = q \cdot \deg x$ and the other four divided power axioms which are given on [15, page 51] or [12, page 482]. Moreover, the differential structure and the divided power structure of $\mathbb{F}$ are related by

$$d(x^{(q)}) = d(x)x^{(q-1)} \tag{1.6}$$

for all homogeneous $x \in \mathbb{F}$ of positive even degree.

Examples 1.7. (a) If F is a free module of finite rank over the commutative noetherian ring R, then there is a divided power structure on the exterior algebra $\bigwedge^\bullet F$ which makes $(\bigwedge^\bullet F, Y)$ a DG$\Gamma-$algebra for every $Y \in F^*$, see the appendix of [12] for details. We view $\bigwedge^\bullet F$ to be $\bigoplus\limits_{\ell\in\mathbb{Z}} \bigwedge^\ell F$ because we take $\bigwedge^\ell F$ to be zero whenever $\ell < 0$ or $\operatorname{rank} F < \ell$.

(b) If x is a homogeneous cycle in the DG$\Gamma-$algebra $(\mathbb{F}, d)$, then $\widetilde{\mathbb{F}} = \mathbb{F}<X; dX = x>$ is the DG$\Gamma-$algebra which is obtained by "adjoining a divided power variable in order to kill the cycle x." In particular, $\deg X = 1 + \deg x$, and, as a module, $\widetilde{F}$ is free over $\mathbb{F}$ with

$$\widetilde{\mathbb{F}} = \begin{cases} \bigoplus\limits_{0\le q} \mathbb{F}X^{(q)} & \text{if } \deg x \text{ is odd, and} \\ \mathbb{F} \oplus \mathbb{F}X & \text{if } \deg x \text{ is even.} \end{cases}$$

See [15] for more details.

Note 1.8. Let x be a homogeneous element of positive degree in the DG$\Gamma-$algebra $\mathbb{F}$. We view $x^{(q)}$ as an element of $\mathbb{F}$ for any integer q, because we take $x^{(q)}$ to be zero, whenever $q < 0$. In particular, if $y \in \mathbb{F}$ is a homogeneous element with $\deg y = \deg x$, then axioms (1.6),

(a) $(x+y)^{(q)} = \sum\limits_{\ell\in\mathbb{Z}} x^{(q-\ell)} y^{(\ell)}$, and
(b) $x^{(p)}x^{(q)} = \binom{p+q}{p} x^{(p+q)}$,

hold for all integers p and q.

The following pfaffian identity is the main result of [27]. Two variations of this identity play a crucial role in our proof of Theorem 3.8.

Theorem 1.9. *Let R be a commutative noetherian ring, F be a free $R-$module of finite rank, and φ be an element of $\bigwedge^2 F$. Let A, B, C, and d be integers. If $\alpha \in \bigwedge^d F^*$, then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}\alpha(\varphi^{(k)})\wedge\varphi^{(A-k)} = \sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{B-A+k}{C-d+k}\left(\varphi^{(k)}(\alpha)\right)(\varphi^{(A-k)}).$$

Corollary 1.10. *Retain the notation and hypotheses of Theorem 1.9 and let β be an element of $\bigwedge^\bullet F$. If $B = A-1$ and $d \le C$, then*

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}\alpha\left(\varphi^{(k)}\wedge\beta\right)\wedge\varphi^{(A-k)} = (-1)^C\alpha\left(\varphi^{(A)}\wedge\beta\right).$$

Proof. Without loss of generality, we may assume that β is homogeneous. The proof proceeds by induction on $\deg\beta$. If $\deg\beta = 0$, then we may take $\beta = 1$. In this case, Theorem 1.9 shows that the left side of the proposed identity is equal to

$$\sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{k-1}{C-d+k}\left(\varphi^{(k)}(\alpha)\right)\left(\varphi^{(A-k)}\right).$$

The only non-zero term in the above sum occurs when $k=0$. (Indeed, if $k<0$, then $\varphi^{(k)}=0$; if $0<k$, then the binomial coefficient is zero.) Observation 1.2 (e) completes the proof when $\deg\beta=0$. For the general case, let $\beta=\beta_1\wedge\beta'$ for some $\beta_1\in\bigwedge^1 F$ and $\beta'\in\bigwedge^\bullet F$. Apply Proposition 1.4 (a) to see that the left side of the proposed identity is equal to L_1+L_2 for

$$L_1 = (-1)^{d+1}\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}(\beta_1(\alpha))\left(\varphi^{(k)}\wedge\beta'\right)\wedge\varphi^{(A-k)} \quad \text{and}$$

$$L_2 = (-1)^d\beta_1\wedge\sum_{k\in\mathbb{Z}}(-1)^k\binom{B-k}{C-k}\alpha\left(\varphi^{(k)}\wedge\beta'\right)\wedge\varphi^{(A-k)}.$$

The induction applies to both L_1 and L_2 because

$$\deg\beta_1(\alpha) < \deg\alpha \le C \quad \text{and} \deg\beta' < \deg\beta;$$

thus, $L_1 = (-1)^{C+d+1}(\beta_1(\alpha))\left(\varphi^{(A)}\wedge\beta'\right)$ and $L_2 = (-1)^{C+d}\beta_1\wedge\left[\alpha\left(\varphi^{(A)}\wedge\beta'\right)\right]$. Another application of Proposition 1.4 completes the proof. □

Corollary 1.11. *Retain the notation and hypotheses of Theorem 1.9. Let s be a nonnegative integer, let $C_1,\dots,C_s$ and $q_1,\dots,q_s$ be integers, and let $\beta\in\bigwedge^\bullet F$ be a homogeneous element. If*

$$1 + A + \deg\beta + \sum_{i=1}^{s} C_i \le d,$$

then

$$\sum_{k\in\mathbb{Z}}(-1)^k\prod_{i=1}^{s}\binom{k+q_i}{C_i}\left(\varphi^{(k)}(\alpha)\right)\left(\varphi^{(A-k)}\wedge\beta\right)=0. \tag{1.12}$$

Proof. Throughout this proof A and d are fixed. The proof consists of a nested pair of inductions. The outer induction is on $\deg\beta$. We begin with $\deg\beta=0$; in other words, we begin with $\beta=1$.

The inner induction is on s. We first take $s=0$. Apply Theorem 1.9, with $B=A$ and $C=d$, to see that

$$\sum_{k\in\mathbb{Z}}(-1)^k\left(\varphi^{(k)}(\alpha)\right)\left(\varphi^{(A-k)}\right)=\sum_{k\in\mathbb{Z}}(-1)^{d+k}\binom{A-k}{d-k}\alpha\left(\varphi^{(k)}\right)\wedge\varphi^{(A-k)}.$$

The hypothesis $1+A\le d$ ensures that the binomial coefficient is zero whenever $k\le A$. On the other hand, $\varphi^{(A-k)}=0$, whenever $A<k$. Thus, (1.12) holds when $s=\deg\beta=0$.

By induction on s, we now assume that

$$\sum_{k\in\mathbb{Z}}(-1)^k\prod_{i=1}^{s-1}\binom{k+q_i}{C_i}(\varphi^{(k)}(\alpha))(\varphi^{(A-k)})=0, \tag{1.13}$$

whenever $1+A+\sum_{i=1}^{s-1}C_i\le d$. Fix $q_1,\dots,q_{s-1}$ and $C_1,\dots,C_{s-1}$. Let $\Phi:\mathbb{Z}\times\mathbb{Z}\to\bigwedge^\bullet F$ be the function

$$\Phi\begin{bmatrix}q\\C\end{bmatrix}=\sum_{k\in\mathbb{Z}}(-1)^k\binom{k+q}{C}\prod_{i=1}^{s-1}\binom{k+q_i}{C_i}\left(\varphi^{(k)}(\alpha)\right)(\varphi^{(A-k)}).$$

It is clear that

$$\Phi\begin{bmatrix}q\\C\end{bmatrix}=0,\quad\text{whenever } q\in\mathbb{Z}\text{ and } C\le -1.$$

It is also clear that Φ satisfies the Pascal identity of Lemma 1.15. We next show that

$$\Phi\begin{bmatrix}0\\C\end{bmatrix}=0,\quad\text{whenever } 0\le C\le -1-A-\sum_{i=1}^{s-1}C_i+d. \tag{1.14}$$

Consider C_s to be fixed with

$$0\le C_s\le -1-A-\sum_{i=1}^{s-1}C_i+d.$$

Apply (1.13) to the data $A-C_s$, $q_1+C_s,\dots,q_{s-1}+C_s$, $C_1,\dots,C_{s-1}$, and $\varphi^{(C_s)}(\alpha)$. We know that

$$1+(A-C_s)+\sum_{i=1}^{s-1}C_i\le\deg\varphi^{(C_s)}(\alpha);$$

and therefore, we see that

$$\sum_{k\in\mathbb{Z}}(-1)^k\prod_{i=1}^{s-1}\binom{k+C_s+q_i}{C_i}\left(\varphi^{(k)}\left(\varphi^{(C_s)}(\alpha)\right)\right)\left(\varphi^{(A-k-C_s)}\right)=0.$$

A quick index shift yields (1.14). Apply Lemma 1.15 in order to see that equation (1.12) holds whenever $\deg\beta=0$ and $1+A+\sum_{i=1}^{s}C_i\le\deg\alpha$. The inner induction is now complete.

We continue with the outer induction. Henceforth, we assume that $\beta=\beta_1\wedge\beta'$, where $\deg\beta_1=1$. Proposition 1.4 (a) shows that the left side of (1.12) is equal to S_1+S_2, for

$$S_1=(-1)^d\beta_1\wedge\sum_{k\in\mathbb{Z}}(-1)^k\prod_{i=1}^{s}\tbinom{k+q_i}{C_i}\left(\varphi^{(k)}(\alpha)\right)\left(\varphi^{(A-k)}\wedge\beta'\right)\quad\text{and}$$

$$S_2=(-1)^{d+1}\sum_{k\in\mathbb{Z}}(-1)^k\prod_{i=1}^{s}\tbinom{k+q_i}{C_i}\left(\varphi^{(k)}(\beta_1(\alpha))\right)\left(\varphi^{(A-k)}\wedge\beta'\right).$$

The induction hypothesis applies because

$$1+A+\deg\beta'+\sum_{i=1}^{s}C_i\le d-1=\deg\beta_1(\alpha)<\deg\alpha\quad\text{and}\quad\deg\beta'<\deg\beta;$$

therefore, $S_1=S_2=0$ and the proof is complete. □

Lemma 1.15. *Let A be an abelian group and let $\Phi\colon\mathbb{Z}\times\mathbb{Z}\to A$ be a function which satisfies Pascal's identity:*

$$\Phi\begin{bmatrix}q-1\\C-1\end{bmatrix}+\Phi\begin{bmatrix}q-1\\C\end{bmatrix}=\Phi\begin{bmatrix}q\\C\end{bmatrix}$$

for all integers q and C. If there are integers q_0, C_0, and C_1 such that

(a) $\Phi\begin{bmatrix}q_0\\C\end{bmatrix}=0$, *whenever $C_0\le C\le C_1$, and*

(b) $\Phi\begin{bmatrix}q\\C_0\end{bmatrix}=0$ *for all $q\in\mathbb{Z}$,*

then $\Phi\begin{bmatrix}q\\C\end{bmatrix}=0$, *whenever $q\in\mathbb{Z}$ and $C_0\le C\le C_1$.*

Proof. There is nothing to prove if $C_1\le C_0$. Henceforth, we assume that $C_0<C_1$. The proof proceeds by induction on C. We suppose that C is fixed with $C_0<C\le C_1$ and

$$\Phi\begin{bmatrix}q\\C-1\end{bmatrix}=0\quad\text{for all }q\in\mathbb{Z}.\tag{1.16}$$

We must show that $\Phi\begin{bmatrix} q \\ C\end{bmatrix} = 0$ for all $q \in \mathbb{Z}$. Now, we let q be a fixed integer. If $q_0 < q$, then, by induction on q, we assume that

$$\Phi\begin{bmatrix} q-1 \\ C\end{bmatrix} = 0; \tag{1.17}$$

thus, (1.17) and (1.16) give

$$\Phi\begin{bmatrix} q \\ C\end{bmatrix} = \Phi\begin{bmatrix} q-1 \\ C\end{bmatrix} + \Phi\begin{bmatrix} q-1 \\ C-1\end{bmatrix} = 0.$$

If $q < q_0$, then, by induction on q, we assume that

$$\Phi\begin{bmatrix} q+1 \\ C\end{bmatrix} = 0; \tag{1.18}$$

thus, (1.18) and (1.16) give

$$\Phi\begin{bmatrix} q \\ C\end{bmatrix} = \Phi\begin{bmatrix} q+1 \\ C\end{bmatrix} - \Phi\begin{bmatrix} q \\ C-1\end{bmatrix} = 0. \quad \square$$

2. The complexes $\mathbb{F}$ and $\mathbb{M}$.

Data 2.1. Let R be a commutative noetherian ring, $n \geq 2$ be an integer, F be a free $R-$module of rank $2n+1$, $\varphi \in \bigwedge^2 F$, and $Y \in F^*$. Fix orientation elements $\xi \in \bigwedge^{2n+1} F^*$ and $\eta \in \bigwedge^{2n+1} F$ which are compatible in the sense that $\xi(\eta) = (-1)^n$. Let g be the element $Y(\varphi)$ of F, I be the ideal $I_1(g)$ of R, and A be the quotient R/I.

The ideal I of Data 2.1 is, of course, a coordinate free representation of the ideal I of the introduction. For future convenience, we make this identification explicit.

Note 2.2. Let $e_1, \dots, e_{2n+1}$ be a basis for F and let $\varepsilon_1, \dots, \varepsilon_{2n+1}$ be the corresponding dual basis for F^*. It is then natural to choose $\xi = \varepsilon_1 \wedge \dots \wedge \varepsilon_{2n+1}$ and $\eta = e_1 \wedge \dots \wedge e_{2n+1}$. Write $Y = \sum_{i=1}^{2n+1} y_i \varepsilon_i$ and $\varphi = \sum_{1 \leq i < j \leq 2n+1} x_{ij} e_i \wedge e_j$. Let X be the alternating matrix whose entry in row i and column j is x_{ij} whenever $i < j$. Observe that $I_1(g)$ is generated by the entries of the product $[y_1, \dots, y_{2n+1}]X$.

In the present section we record two complexes of free $R-$modules. Each complex is a resolution of A whenever the grade of I is at least $2n$. The complex $\mathbb{F}$ of Definition 2.5 is always infinite, but is easy to manipulate. The complex $\mathbb{M}$ of Definition 2.15 is a finite summand of $\mathbb{F}$; furthermore, $\mathbb{M}$ is a minimal resolution of A whenever the data is chosen in an appropriate manner; see, for example, Corollary 2.17 and Corollary 2.18. A significant amount of information about $\mathbb{M}$ is already available in the literature. The graded modules which comprise $\mathbb{M}$ were calculated in [24] by using the technique of linkage. They were also calculated in [22] by using techniques from multilinear algebra. Furthermore, the main result in [22] is the resolution of the ring which is defined by the ideal $I' = I_1(Y'X')$, where X' is a $2n \times 2n$ alternating matrix and Y' is a $1 \times 2n$ matrix. The ideal I' is never perfect and is not studied in the present paper; nonetheless, it would be possible to modify the resolution in [22] in order to resolve A.

Our plan for resolving A is based on the ideas of [28].

Definition 2.3. Adopt Data 2.1. Let $\mathbb{A}$ and $\mathbb{B}$ be the DG$\Gamma-$algebras

$$\mathbb{A} = \bigwedge^{\bullet} F^*\langle h\rangle \qquad \text{and} \qquad \mathbb{B} = \bigwedge^{\bullet} F\langle\lambda\rangle,$$

where $\bigwedge^{\bullet} F^*$ and $\bigwedge^{\bullet} F$ are exterior algebras and h and λ are divided power variables of degree two. The differential d on $\mathbb{A}$ is given by $d|_{F^*} = g$ and $d(h) = Y$. The differential d on $\mathbb{B}$ is given by $d|_F = Y$ and $d(\lambda) = g$. For each integer t, a map

$$v_t\colon \mathbb{B}_{t-1} = \sum_{p+2q=t-1} \bigwedge^p F\lambda^{(q)} \to \mathbb{A}_t = \sum_{i+2j=t} \bigwedge^i F^* h^{(j)}$$

is defined by

$$v_t(\beta_p\lambda^{(q)}) = \sum_{j\in\mathbb{Z}} (-1)^{j+p} \binom{n+j-p-q-1}{q} \left(\varphi^{(n+j-p-q)} \wedge \beta_p\right)(\xi) h^{(j)}.$$

Proposition 2.4. *In the notation of Definition 2.3,*

(a) *the maps* $\{v_t\colon \mathbb{B}_{t-1} \to \mathbb{A}_t\}$ *form a map of complexes* $v\colon \mathbb{B}[-1] \to \mathbb{A}$, *and*
(b) $v_{2q+1}\left((\varphi-\lambda)^{(q)}\right) = z_{2q+1}$, *where* z_{2q+1} *is the element* $\sum_{j\in\mathbb{Z}} (-1)^j \varphi^{(n+j-q)}(\xi)h^{(j)}$ *of* $\mathbb{A}_{2q+1}$.

Note. The map v was defined in order to make Proposition 2.4 hold. The significance of (b) is explained by the fact that (in the generic case) all of the non-zero homology of $\mathbb{B}$ and $\mathbb{A}_+$ is represented by cycles of the form $(\varphi-\lambda)^{(q)}$ and z_{2q+1}, respectively; see (2.8) and (2.9).

Proof. (a) Fix p and q, with $p+2q = t-1$. Direct calculation gives $(d_t \circ v_t)(\beta_p\lambda^{(q)})$ is equal to $S_1 + S_2$, where

$$S_1 = \sum_{j\in\mathbb{Z}} (-1)^{j+p} \binom{n+j-p-q-1}{q} g\left((\varphi^{(n+j-p-q)} \wedge \beta_p)(\xi)\right) h^{(j)}, \quad \text{and}$$

$$S_2 = \sum_{j\in\mathbb{Z}} (-1)^{j+p+1} \binom{n+j-p-q}{q} Y \wedge \left((\varphi^{(n+j-p-q+1)} \wedge \beta_p)(\xi)\right) h^{(j)}.$$

On the other hand, $(v_{t-1} \circ d_{t-1})(\beta_p\lambda^{(q)}) = S_3 + S_4$, where

$$S_3 = \sum_{j\in\mathbb{Z}} (-1)^{j+p+1} \binom{n+j-p-q}{q} (\varphi^{(n+j-p-q+1)} \wedge Y(\beta_p))(\xi) h^{(j)}, \quad \text{and}$$

$$S_4 = \sum_{j\in\mathbb{Z}} (-1)^{j+p+1} \binom{n+j-p-q-1}{q-1} (\varphi^{(n+j-p-q)} \wedge g \wedge \beta_p)(\xi) h^{(j)}.$$

The module action of $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$, together with (1.6), gives

$$g\left((\varphi^{(n+j-p-q)}\wedge\beta_p)(\xi)\right)=\left(g\wedge\varphi^{(n+j-p-q)}\wedge\beta_p\right)(\xi)=\left(Y(\varphi^{(n+j-p-q+1)})\wedge\beta_p\right)(\xi).$$

It follows that

$$S_1-S_4=\sum_{j\in\mathbb{Z}}(-1)^{j+p}\binom{n+j-p-q}{q}\left(Y(\varphi^{(n+j-p-q+1)})\wedge\beta_p\right)(\xi)h^{(j)},\quad\text{and}$$

$$S_1-S_4-S_3=\sum_{j\in\mathbb{Z}}(-1)^{j+p}\binom{n+j-p-q}{q}\left[Y\left(\varphi^{(n+j-p-q+1)}\wedge\beta_p\right)\right](\xi)h^{(j)}.$$

An application of Proposition 1.4 shows that $S_1-S_3-S_4+S_2=0$; thereby completing the proof that v is a map of complexes.

(b) The formula holds for $q<0$; henceforth, we assume that $0\le q$. Apply Note 1.8, as well as the definition of v, in order to see that $v_{2q+1}\left((\varphi-\lambda)^{(q)}\right)$ is equal to

$$v_{2q+1}\left(\sum_{\ell\in\mathbb{Z}}(-1)^\ell\varphi^{(q-\ell)}\lambda^{(\ell)}\right)=\sum_{\ell,j\in\mathbb{Z}}(-1)^{j+\ell}\binom{n+j-2q+\ell-1}{\ell}\left(\varphi^{(n+j-2q+\ell)}\wedge\varphi^{(q-\ell)}\right)(\xi)h^{(j)}$$

$$=\sum_j(-1)^j\left[\sum_\ell(-1)^\ell\binom{n+j-2q+\ell-1}{\ell}\binom{n+j-q}{q-\ell}\right]\varphi^{(n+j-q)}(\xi)h^{(j)}.$$

Observation 1.2 (h) shows that the expression inside the brackets is equal to 1, whenever $0\le n+j-q$. □

Definition 2.5. In the notation of Proposition 2.4, let $(\mathbb{F},f)$ be the mapping cone of $v\colon\mathbb{B}[-1]\to\mathbb{A}$. In other words, the map

$$f_t\colon\mathbb{F}_t=\mathbb{A}_t\oplus\mathbb{B}_{t-2}\to\mathbb{F}_{t-1}=\mathbb{A}_{t-1}\oplus\mathbb{B}_{t-3}$$

is given by

$$f_t=\begin{bmatrix}d_t&(-1)^{t-1}v_{t-1}\\0&d_{t-2}\end{bmatrix};$$

in particular, if $i+2j=t$ and $p+2q=t-2$, then

$$f_t\begin{bmatrix}\alpha_ih^{(j)}\\\beta_p\lambda^{(q)}\end{bmatrix}=\begin{bmatrix}g(\alpha_i)h^{(j)}+Y\wedge\alpha_ih^{(j-1)}+(-1)^{t-1}v_{t-1}(\beta_p\lambda^{(q)})\\Y(\beta_p)\lambda^{(q)}+g\wedge\beta_p\lambda^{(q-1)}\end{bmatrix}.$$

Note. We write $\mathbb{F}(Y,\varphi)$ or $\mathbb{F}(Y,X)$ whenever we want to emphasize the data of 2.1 or 2.2 which is used in the construction of $\mathbb{F}$.

Proposition 2.6. *If the ideal I of Data 2.1 has grade at least $2n$, then complex $\mathbb{F}$ of Definition 2.5 is acyclic, and $H_0(\mathbb{F}) = A$.*

Proof. Some of the arguments are simplified if we take the data of 2.1 to be generic; moreover, the ideal I is perfect so there is no loss of generality when we assume that

$$R \text{ is the polynomial ring } \mathbb{Z}[y_1, \dots, y_{2n}, \{x_{ij} \mid 1 \le i < j \le 2n\}], \tag{2.7}$$

where the y_i and x_{ij} from Note 2.2 are indeterminates. We show that $\mathbb{F}$ is acyclic by examining the long exact sequence of homology which is associated to a mapping cone of v. In particular, we prove that v induces an isomorphism $H_{i-1}(\mathbb{B}) \to H_i(\mathbb{A})$, for all positive i. With this purpose in mind, we calculate the homology of $\mathbb{B}$ and $\mathbb{A}$. We claim that the homology of $\mathbb{B}$ is given by

$$H_i(\mathbb{B}) \cong \begin{cases} 0, & \text{if } i \text{ is odd, and} \\ \frac{R}{I_1(Y)}, & \text{if } i \ge 0 \text{ is even;} \end{cases} \tag{2.8}$$

furthermore, $H_{2q}(\mathbb{B})$ is generated by $\left[(\varphi - \lambda)^{(q)}\right]$. We also claim that the homology of $\mathbb{A}$ is given by

$$H_i(\mathbb{A}) \cong \begin{cases} A, & \text{if } i = 0, \\ 0, & \text{if } i \ge 2 \text{ is even, and} \\ \frac{R}{I_1(Y)}, & \text{if } i \ge 1 \text{ is odd;} \end{cases} \tag{2.9}$$

furthermore, $H_{2q+1}(\mathbb{A})$ is generated by $[z_{2q+1}]$, where z_{2q+1} is the element of $\mathbb{A}_{2q+1}$ which is defined in Proposition 2.4. Once (2.8) and (2.9) have been established, then Proposition 2.4 completes the proof.

It is clear that $(\varphi - \lambda)^{(q)}$ is a cycle in the DG$\Gamma-$algebra $\mathbb{B}$, because

$$d((\varphi - \lambda)^{(q)}) = d(\varphi - \lambda)(\varphi - \lambda)^{(q-1)},$$

and $d(\varphi - \lambda) = Y(\varphi) - g = 0$. It is now also clear that $z_{2q+1} = v_{2q+1}\left((\varphi - \lambda)^{(q)}\right)$ is a cycle in $\mathbb{A}$.

The proof of (2.8) follows from the fact that $\mathbb{B}$ is the total complex of the following double complex:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^2 F\lambda^{(2)} & \to & \bigwedge^1 F\lambda^{(2)} & \to & \bigwedge^0 F\lambda^{(2)} & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^3 F\lambda^{(1)} & \to & \bigwedge^2 F\lambda^{(1)} & \to & \bigwedge^1 F\lambda^{(1)} & \to & \bigwedge^0 F\lambda^{(1)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^4 F\lambda^{(0)} & \to & \bigwedge^3 F\lambda^{(0)} & \to & \bigwedge^2 F\lambda^{(0)} & \to & \bigwedge^1 F\lambda^{(0)} & \to & \bigwedge^0 F\lambda^{(0)}.
\end{array}$$

The proof of (2.9) follows from Lemma 2.11 because $\mathbb{A}_\ell = (\mathbb{P}^q)_\ell$ for $0 \le \ell \le 2q+1$. □

The hard part of the proof of Proposition 2.6 has been isolated and called Lemma 2.11. The following calculations are well known and are used in our proof of Lemma 2.11.

Lemma 2.10. *Let $X_{2n+1\times 2n+1}$ and $Y=[y_1,\dots,y_{2n+1}]$ be matrices of indeterminates, with X an alternating matrix, and let R be the ring $\mathbb{Z}[X,Y]$. If $g=[g_1,\dots,g_{2n+1}]$ represents the product YX and $\mathbf{p}$ represents the pfaffian of the matrix obtained by deleting the last row and column of X, then*

(a) *$g_1\dots,g_{2n}$ is a regular sequence,*
(b) *$(g_1\dots,g_{2n})\colon g_{2n+1}=(g_1\dots,g_{2n},y_{2n+1},\mathbf{p})$,*
(c) *$(g_1\dots,g_{2n})\colon y_{2n+1}=I_1(g)$, and*
(d) *$(g_1\dots,g_{2n},y_{2n+1})\colon \mathbf{p}=I_1(Y)$*

Proof. Assertion (a) is established in [19] and [24]. For the other assertions, the inclusion $\subseteq$ is obvious and the ideal on the right side is prime. □

Lemma 2.11. *Adopt the notation and hypotheses of Proposition 2.6 and (2.7). For each nonnegative integer q, let $\mathbb{P}^q$ be the subcomplex of $\mathbb{A}$ which is defined by*

$$(\mathbb{P}^q)_\ell=\sum_{j\le q}\bigwedge^{\ell-2j}F^*h^{(j)}.$$

The following statements hold.

(a) *The homology of $\mathbb{P}^q$ is given by*

$$H_i(\mathbb{P}^q)\cong\begin{cases} A & \text{if } i=0,\\ 0 & \text{if } 2\le i \text{ is even},\\ 0 & \text{if } 2q+2\le i,\\ R/I_1(Y) & \text{if } 1\le i\le 2q-1 \text{ and } i \text{ is odd, and}\\ \dfrac{(g_1,\dots,g_{2n},y_{2n+1},\mathbf{p})}{(g_1,\dots,g_{2n})} & \text{if } i=2q+1.\end{cases}$$

(b) *If $0\le \ell\le q-1$, then $[z_{2\ell+1}]$ generates $H_{2\ell+1}(\mathbb{P}^q)$.*
(c) *The homology $H_{2q+1}(\mathbb{P}^q)$ is generated by $[Yh^{(q)}]$ and $[z_{2q+1}]$.*

Proof. Define g_i and $\mathbf{p}$ as in Lemma 2.10. The proof proceeds by induction on q. When $q=0$, $\mathbb{P}^q$ is the Koszul complex, $\bigwedge^\bullet F^*$, on the elements $g_1,\dots,g_{2n+1}$. Since $g_1,\dots,g_{2n}$ form a regular sequence, the standard facts about Koszul complexes yield that $H_i(\mathbb{P}^0)=0$ for $2\le i$ and that

$$H_1(\mathbb{P}^0)\cong\frac{(g_1,\dots,g_{2n})\colon g_{2n+1}}{(g_1,\dots,g_{2n})}=\frac{(g_1,\dots,g_{2n},y_{2n+1},\mathbf{p})}{(g_1,\dots,g_{2n})}.$$

The equality is established in Lemma 2.10 and the isomorphism is induced by

$$\begin{bmatrix} r_1\\ \vdots\\ r_{2n+1}\end{bmatrix}\mapsto r_{2n+1}.$$

In particular, the homology class $[Y]$ in $H_1(\mathbb{P}^0)$ is sent to y_{2n+1} and the homology class $[\varphi^{(n)}(\xi)]$ in $H_1(\mathbb{P}^0)$ is sent to $\pm\mathbf{p}$. The proof for $q=0$ is complete.

We now assume, by induction, that the result holds for some fixed value of q. Observe that $\mathbb{P}^{q+1}$ is the mapping cone of

$$\begin{array}{ccccccccccccc}
\bigwedge^{\bullet} F^* h^{(q+1)} & : & \cdots & \to & \bigwedge^1 F^* h^{(q+1)} & \to & \bigwedge^0 F^* h^{(q+1)} & \to & 0 & \to & \cdots & \to & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{P}^q & : & \cdots & \to & (\mathbb{P}^q)_{2q+2} & \to & (\mathbb{P}^q)_{2q+1} & \to & (\mathbb{P}^q)_{2q} & \to & \cdots & \to & (\mathbb{P}^q)_0.
\end{array}$$

The homology of $\mathbb{P}^q$ is known by induction. The complex $\bigwedge^{\bullet} F^* h^{(q+1)}$ is isomorphic to a shift of $\mathbb{P}^0$; thus, its homology is also known. In particular, $H_{2q+2}(\bigwedge^{\bullet} F^* h^{(q+1)})$ is isomorphic to A and is generated by $[h^{(q+1)}]$; and $H_{2q+3}(\bigwedge^{\bullet} F^* h^{(q+1)})$ is isomorphic to $\frac{(g_1,\dots,g_{2n},y_{2n+1},\mathbf{p})}{(g_1,\dots,g_{2n})}$ and is generated by $[Yh^{(q+1)}]$ and $[\varphi^{(n)}(\xi)h^{(q+1)}]$. The argument is completed by appealing to the long exact sequence of homology which is associated to a mapping cone. There are two critical steps in this calculation. The first involves the exact sequence

$$0 = H_{2q+3}(\mathbb{P}^q) \to H_{2q+3}(\mathbb{P}^{q+1}) \xrightarrow{\delta} H_{2q+3}(\bigwedge^{\bullet} F^* h^{(q+1)}) \to H_{2q+2}(\mathbb{P}^q) = 0.$$

The isomorphism δ is induced by the projection

$$\mathbb{P}^{q+1} \to \bigwedge^{\bullet} F^* h^{(q+1)};$$

and therefore, $\delta([z_{2q+3}]) = \pm[\varphi^{(n)}(\xi)h^{(q+1)}]$ and $\delta([Yh^{(q+1)}]) = [Yh^{(q+1)}]$. The other critical step in our calculation involves the exact sequence

$$0 \to H_{2q+2}(\mathbb{P}^{q+1}) \to H_{2q+2}(\bigwedge^{\bullet} F^* h^{(q+1)}) \to H_{2q+1}(\mathbb{P}^q) \to H_{2q+1}(\mathbb{P}^{q+1}) \to 0.$$

We know that $H_{2q+1}(\mathbb{P}^q)$ is isomorphic to $\frac{(g_1,\dots,g_{2n},y_{2n+1},\mathbf{p})}{(g_1,\dots,g_{2n})}$ and is generated by $[Yh^{(q)}]$ and $[z_{2q+1}]$; furthermore, we also know that $d(h^{(q+1)}) = Yh^{(q)}$ in $\mathbb{A}$. Thus, $H_{2q+1}(\mathbb{P}^{q+1})$ is generated by $[z_{2q+1}]$ and is isomorphic to

$$\frac{(g_1,\dots,g_{2n},y_{2n+1},\mathbf{p})}{(g_1,\dots,g_{2n},y_{2n+1})} \xleftarrow{\cong} \frac{R}{(g_1,\dots,g_{2n},y_{2n+1}) : \mathbf{p}}; \quad \text{and}$$

$$H_{2q+2}(\mathbb{P}^{q+1}) = \ker\left[H_{2q+2}(\bigwedge^{\bullet} F^* h^{(q+1)}) \to H_{2q+1}(\mathbb{P}^q)\right] = \frac{(g_1,\dots,g_{2n}) : y_{2n+1}}{I_1(g)}.$$

Lemma 2.10 yields that $H_{2q+1}(\mathbb{P}^{q+1}) \cong R/I_1(Y)$ and $H_{2q+2}(\mathbb{P}^{q+1}) = 0$. □

We conclude this section by decomposing the complex $\mathbb{F}$ into the direct sum of two complexes $\mathbb{M}$ and $\mathbb{N}$. The complex $\mathbb{M}$ has the same homology as the complex $\mathbb{F}$ and is a minimal resolution when the data is chosen in an appropriate manner; see Corollaries 2.17 and 2.18. The complex $\mathbb{N}$ is split exact. For our purposes in the present section, $\mathbb{N}$ could be any direct sum complement of $\mathbb{M}$ in $\mathbb{F}$; however, when we prove that $\mathbb{M}$ is a DG$\Gamma-$algebra it is necessary that we use a particular representation of $\mathbb{N}$; see, for example, Lemma 3.16.

The next two observations are well known. We use them in our proof of Proposition 2.16.

Observation 2.12. *Let $(\mathbb{F}, f)$ be a complex of modules over a ring R and let $(\mathbb{M}, m)$ be a subcomplex of $\mathbb{F}$. Suppose that each module $\mathbb{F}_t$ decomposes as $\mathbb{M}_t \oplus E_t \oplus E'_t$. For each integer t, let $e_t\colon E_t \to E'_{t-1}$ be the composition*

$$E_t \xrightarrow{\text{inclusion}} \mathbb{F}_t \xrightarrow{f_t} \mathbb{F}_{t-1} \xrightarrow{\text{projection}} E'_{t-1}.$$

If each map e_t is an isomorphism, then the complex $(\mathbb{F}, f)$ is isomorphic to $(\mathbb{F}, f')$, where

$$f'_t\colon \mathbb{F}_t = \mathbb{M}_t \oplus E_t \oplus E'_t \longrightarrow \mathbb{F}_{t-1} = \mathbb{M}_{t-1} \oplus E_{t-1} \oplus E'_{t-1}$$

is given by

$$f'_t = \begin{bmatrix} m_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_t & 0 \end{bmatrix}.$$

Proof. The hypothesis that $(\mathbb{M}, m)$ is a subcomplex of $(\mathbb{F}, f)$ guarantees that $f|_{\mathbb{M}} = m$; and therefore, the map

$$f_t\colon \mathbb{F}_t = \mathbb{M}_t \oplus E_t \oplus E'_t \longrightarrow \mathbb{F}_{t-1} = \mathbb{M}_{t-1} \oplus E_{t-1} \oplus E'_{t-1}$$

may be decomposed as

$$f_t = \begin{bmatrix} m_t & a_t & b_t \\ 0 & c_t & d_t \\ 0 & e_t & g_t \end{bmatrix}.$$

There is no difficulty in checking that $\pi'\colon (\mathbb{F}, f) \to (\mathbb{F}, f')$, with

$$\pi'_t\colon \mathbb{F}_t = \mathbb{M}_t \oplus E_t \oplus E'_t \longrightarrow \mathbb{F}_t = \mathbb{M}_t \oplus E_t \oplus E'_t$$

given by

$$\pi'_t = \begin{bmatrix} 1 & 0 & -a_{t+1}e_{t+1}^{-1} \\ 0 & 1 & -c_{t+1}e_{t+1}^{-1} \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.13}$$

gives the desired isomorphism. $\square$

Observation 2.14. *Let R be a commutative ring and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of $R-$modules. If $\varphi\colon B \to A$ and $\gamma\colon C \to B$ are maps which satisfy $\varphi \circ f = id_A$, $g \circ \gamma = id_C$, and $\varphi \circ \gamma = 0$, then

$$b = \gamma g(b) + f\varphi(b)$$

for all $b \in B$ and

$$0 \longrightarrow C \xrightarrow{\gamma} B \xrightarrow{\varphi} A \longrightarrow 0$$

is a short exact sequence.

Proof. It is easy to see that $b - \gamma g(b) - f\varphi(b)$ is in $\ker g \cap \ker \varphi = 0$. $\square$

Definition 2.15. Let $(\mathbb{F}, f)$ be the complex of Definition 2.5. Fix an integer t. Define $\mathbb{M}_t$ and $\mathbb{N}_t$ to be the submodules

$$\mathbb{M}_t = \sum_{i+j \le n} \bigwedge^i F^* h^{(j)} \quad \bigoplus \quad \sum_{p+q \le n-1} \bigwedge^p F\lambda^{(q)}$$

$$\mathbb{N}_t = \sum_{n+1 \le i+j} \bigwedge^i F^* h^{(j)} \quad \bigoplus \quad \sum_{n \le p+q} \bigwedge^p F\lambda^{(q)}$$

of $\mathbb{F}_t$. (Of course, the parameters i, j, p, and q must satisfy $i + 2j = t$ and $p + 2q = t - 2$.) Define

$$i_t \colon \mathbb{M}_t \to \mathbb{F}_t \quad \text{and} \quad \operatorname{incl}_t \colon \mathbb{N}_t \to \mathbb{F}_t$$

to be the inclusion maps and define

$$\operatorname{proj}_t \colon \mathbb{F}_t \to \mathbb{M}_t \quad \text{and} \quad p_t \colon \mathbb{F}_t \to \mathbb{N}_t$$

to be the natural projection maps; that is, proj_t annihilates $\mathbb{N}_t$, but acts like the identity on $\mathbb{M}_t$, and p_t annihilates $\mathbb{M}_t$, but acts like the identity on $\mathbb{N}_t$. Define $f'_t \colon \mathbb{F}_t \to \mathbb{F}_{t-1}$ by $f'_t = f_t$ on $\mathbb{M}_t$, $f'_t = 0$ on $\mathbb{N}_t \cap \mathbb{A}$, and on $\mathbb{N}_t \cap \mathbb{B}$, f'_t is given by

$$f'_t \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix} = \begin{bmatrix} (-1)^{n+p+1} \beta_p(\xi) h^{(p+q-n)} \\ 0 \end{bmatrix}.$$

Define maps π'_t and π'' from $\mathbb{F}_t$ to $\mathbb{F}_t$ by $\pi' = \pi'' = \text{id}$ on $\mathbb{M} + \mathbb{B}$; however, on $\mathbb{N}_t \cap \mathbb{A}$, π'_t and π''_t are given by

$$\pi'_t \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} - (-1)^i f_{t+1} \begin{bmatrix} 0 \\ \alpha_i(\eta) \lambda^{(i+j-n-1)} \end{bmatrix} \quad \text{and}$$
$$\pi''_t \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} = (-1)^i f_{t+1} \begin{bmatrix} 0 \\ \alpha_i(\eta) \lambda^{(i+j-n-1)} \end{bmatrix}.$$

Define $m_t \colon \mathbb{M}_t \to \mathbb{M}_{t-1}$, $\pi_t \colon \mathbb{F}_t \to \mathbb{M}_t$, $n_t \colon \mathbb{N}_t \to \mathbb{N}_{t-1}$, $\rho_t \colon \mathbb{N}_t \to \mathbb{F}_t$, and $P_t \colon \mathbb{F}_t \to \mathbb{N}_t$ to be the compositions

$$\mathbb{M}_t \xrightarrow{i_t} \mathbb{F}_t \xrightarrow{f_t} \mathbb{F}_{t-1} \xrightarrow{\operatorname{proj}_{t-1}} \mathbb{M}_{t-1},$$

$$\mathbb{F}_t \xrightarrow{\pi'_t} \mathbb{F}_t \xrightarrow{\operatorname{proj}_t} \mathbb{M}_t,$$

$$\mathbb{N}_t \xrightarrow{\operatorname{incl}_t} \mathbb{F}_t \xrightarrow{f'_t} \mathbb{F}_{t-1} \xrightarrow{p_{t-1}} \mathbb{N}_{t-1},$$

$$\mathbb{N}_t \xrightarrow{\operatorname{incl}_t} \mathbb{F}_t \xrightarrow{\pi''_t} \mathbb{F}_t, \quad \text{and}$$

$$\mathbb{F}_t \xrightarrow{\pi'_t} \mathbb{F}_t \xrightarrow{p_t} \mathbb{N}_t,$$

respectively.

Note. We write $\mathbb{M}(Y, \varphi)$ or $\mathbb{M}(Y, X)$ whenever we want to emphasize the data of 2.1 or 2.2 which is used in the construction of $\mathbb{M}$.

Proposition 2.16. *Retain the notation of Definition 2.15.*

(a) *The maps* $\{m_t\colon \mathbb{M}_t \to \mathbb{M}_{t-1}\}$ *form a complex* $(\mathbb{M}, m)$.
(b) *The complex* $(\mathbb{M}, m)$ *is a direct summand of* $(\mathbb{F}, f)$. *In particular,*
  (*i*) $i\colon (\mathbb{M}, m) \to (\mathbb{F}, f)$ *is a map of complexes,*
  (*ii*) $\pi\colon (\mathbb{F}, f) \to (\mathbb{M}, m)$ *is a map of complexes, and*
  (*iii*) $\pi \circ i$ *is the identity map on* $(\mathbb{M}, m)$.
(c) *The complex* $(\mathbb{M}, m)$ *is minimal in the sense that* $m_t(\mathbb{M}_t) \subseteq [I_1(Y) + I_1(\varphi)]\,\mathbb{M}_{t-1}$.
(d) *The maps* $\{n_t\colon \mathbb{N}_t \to \mathbb{N}_{t-1}\}$ *form a complex* $(\mathbb{N}, n)$.
(e) *The complex* $(\mathbb{N}, n)$ *is a direct summand of* $(\mathbb{F}, f)$. *In particular,*
  (*i*) $\rho\colon (\mathbb{N}, n) \to (\mathbb{F}, f)$ *is a map of complexes,*
  (*ii*) $P\colon (\mathbb{F}, f) \to (\mathbb{N}, n)$ *is a map of complexes, and*
  (*iii*) $P \circ \rho$ *is the identity map on* $(\mathbb{N}, n)$.
(f) *The sequences*
  (*i*) $0 \to (\mathbb{N}, n) \xrightarrow{\rho} (\mathbb{F}, f) \xrightarrow{\pi} (\mathbb{M}, m) \to 0$ *and*
  (*ii*) $0 \to (\mathbb{M}, m) \xrightarrow{i} (\mathbb{F}, f) \xrightarrow{P} (\mathbb{N}, n) \to 0$
  are short exact sequences of complexes.
(g) *If* $x_t \in \mathbb{F}_t$, *then* $x_t = i_t\pi_t(x_t) + \rho_t P_t(x_t)$.

The following statements are obtained by combining Propositions 2.16 and 2.6.

Corollary 2.17. *Let* $(R, \mathfrak{m}, k)$ *be a noetherian local ring,* $n \geq 2$ *be an integer, and* $Y_{1\times 2n+1}$ *and* $X_{2n+1\times 2n+1}$ *be matrices with entries from* $\mathfrak{m}$, *with* X *an alternating matrix. Let* I *be the* $R-$*ideal* $I_1(YX)$ *and* A *be the quotient ring* R/I. *If* $2n \leq \operatorname{grade} I$, *then the complex* $\mathbb{M}(Y, X)$ *of Proposition 2.16 is the minimal* $R-$*resolution of* A.

Corollary 2.18. *Let* $R = \bigoplus_{0\leq i} R_i$ *be a graded algebra over the field* $R_0 = k$. *Let* $n \geq 2$ *be an integer, and* $Y_{1\times 2n+1}$ *and* $X_{2n+1\times 2n+1}$ *be matrices with entries from* R_1, *with* X *an alternating matrix. Let* I *be the* $R-$*ideal* $I_1(YX)$ *and* A *be the quotient ring* R/I. *If* $2n \leq \operatorname{grade} I$, *then the complex* $\mathbb{M}(Y, X)$ *of Proposition 2.16 is the minimal graded* $R-$*resolution of* A. *Furthermore,*

$$\mathbb{M}_t = \sum_{(i,j)} R(-(2i+3j))^{\binom{2n+1}{i}} \bigoplus \sum_{(p,q)} R(-(p+3q+n+2))^{\binom{2n+1}{p}},$$

where (i, j) *varies over all pairs of nonnegative integers with* $i+j \leq n$ *and* $i+2j = t$, *and* (p, q) *varies over all pairs of nonnegative integers with* $p + q \leq n - 1$ *and* $p + 2q = t - 2$. *In particular,* $\mathbb{M}_{2n} = R(-3n) \oplus R(-(4n-1))$.

Note. The graded betti numbers of Corollary 2.18 agree with those of Theorem 6.3 of [24] once the typographic error $i+(i+2)/2$ in [24] is corrected to be $i+(i+j)/2$.

Proof of Proposition 2.16. Assertion (b.iii) is obvious. We prove (a) and (b.i) simultaneously, by showing that $f_t(\mathbb{M}_t)$ is contained in $\mathbb{M}_{t-1}$. For this, it suffices to show that $v(\mathbb{M}\cap\mathbb{B}) \subseteq \mathbb{M}$. Suppose that $p+q \leq n-1$. Recall the value of $v(\beta_p\lambda^{(q)})$ from Definition 2.3. If $0 \leq j$ and

$$n+1 \leq j + \deg\left(\varphi^{(n+j-p-q)} \wedge \beta_p\right)(\xi), \tag{2.19}$$

then

$$0 \le n+j-p-q-1 \le q-1 \quad \text{and} \quad \binom{n+j-p-q-1}{q} = 0. \tag{2.20}$$

(b.ii) Let $E_t = \mathbb{N}_t \cap \mathbb{B}$, $E'_t = \mathbb{N}_t \cap \mathbb{A}$, and $e_t \colon E_t \to E'_{t-1}$ be the composition

$$E_t \xrightarrow{\text{inclusion}} \mathbb{F}_t \xrightarrow{f_t} \mathbb{F}_{t-1} \xrightarrow{\text{projection}} E'_{t-1}.$$

We claim that

$$e_t(\beta_p \lambda^{(q)}) = (-1)^{n+p+1} \beta_p(\xi) h^{(p+q-n)}. \tag{2.21}$$

Indeed, e_t is equal to

$$E_t \xrightarrow{\text{inclusion}} \mathbb{B}_{t-2} \xrightarrow{(-1)^{t-1} v_{t-1}} \mathbb{A}_{t-1} \xrightarrow{\text{projection}} E'_{t-1}.$$

Once again, $v(\beta_p \lambda^{(q)})$ is given in Definition 2.3. Notice that the only value of j which corresponds to a non-zero term in E'_{t-1} is $j = p+q-n$. Indeed, if $j < p+q-n$, then $\varphi^{(n+j-p-q)} = 0$; and if $p+q-n < j$ and (2.19) holds, then (2.20) continues to hold.

Now that (2.21) is established, we see that e_t is an isomorphism for all t. The inverse of e_t is $e_t^{-1} \colon E'_{t-1} \to E_t$, with

$$e_t^{-1}(\alpha_i h^{(j)}) = (-1)^i \alpha_i(\eta) \lambda^{(i+j-n-1)}.$$

(Recall, from Proposition 1.4 and Data 2.1, that $(\alpha_i(\eta))(\xi) = (-1)^n \alpha_i$.) Apply Observation 2.12 in order to see that $(\mathbb{F}, f')$ is a complex, $\pi' \colon (\mathbb{F}, f) \to (\mathbb{F}, f')$ is an isomorphism of complexes, and $\pi'' \colon (\mathbb{F}, f') \to (\mathbb{F}, f)$ is the inverse of π'. Observe that

$$f'(\mathbb{M}) \subseteq \mathbb{M} \quad \text{and} \quad f'(\mathbb{N}) \subseteq \mathbb{N}. \tag{2.22}$$

The only interesting case involves the element $b = \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix}$ of $\mathbb{N} \cap \mathbb{B}$. In this case,

$$f'(b) = \begin{bmatrix} (-1)^{n+p+1} \beta_p(\xi) h^{(p+q-n)} \\ 0 \end{bmatrix} \in \mathbb{N},$$

because $n+1 \le n+1+q = \deg \beta_p(\xi) + p + q - n$. It follows that $\operatorname{proj} \colon (\mathbb{F}, f') \to (\mathbb{M}, m)$ is a map of complexes. The proof of (b.ii) is complete because π' and proj are both maps of complexes.

(c) Let x be the element $\begin{bmatrix} \alpha_i h^{(j)} \\ \beta_p \lambda^{(q)} \end{bmatrix}$ of $\mathbb{F}_t$. Every term of $f_t(x)$ is in $[I_1(Y) + I_1(\varphi)]\,\mathbb{F}_{t-1}$, except, possibly, the term which involves

$$\left(\varphi^{(0)} \wedge \beta_p\right)(\xi) h^{(p+q-n)}.$$

If we now take x to be an element of $\mathbb{M}_t$, then $p+q\le n-1$ and $h^{(p+q-n)}=0$.

(d) We saw that $(\mathbb{F},f')$ is a complex; consequently, (d) follows from (2.22).

(e.i) The map ρ is equal to $\pi''\circ \mathrm{incl}$. Use (2.22) to see that $\mathrm{incl}\colon (\mathbb{N},n)\to(\mathbb{F},f')$ is a map of complexes. The proof of (b.ii) shows that $\pi''\colon(\mathbb{F},f')\to(\mathbb{F},f)$ is a map of complexes.

(e.ii) The map P is equal to $p\circ\pi'$. We already saw that $\pi'\colon(\mathbb{F},f)\to(\mathbb{F},f')$ is a map of complexes. It is clear, from (2.22), that $p\colon(\mathbb{F},f')\to(\mathbb{N},n)$ is also a map of complexes.

(e.iii) The map $P\circ\rho\colon(\mathbb{N},n)\to(\mathbb{N},n)$ is the composition:

$$(\mathbb{N},n)\xrightarrow{\mathrm{incl}}(\mathbb{F},f')\xrightarrow{\pi''}(\mathbb{F},f)\xrightarrow{\pi'}(\mathbb{F},f')\xrightarrow{p}(\mathbb{N},n).$$

We know that $\pi'\circ\pi''=\mathrm{id}_{(\mathbb{F},f')}$ and $p\circ\mathrm{incl}=\mathrm{id}_{(\mathbb{N},n)}$.

(f.i) Notice that

$$\pi_t\circ\rho_t=(\mathrm{proj}_t\circ\pi'_t)\circ(\pi''_t\circ\mathrm{incl}_t)=\mathrm{proj}_t\circ(\pi'_t\circ\pi''_t)\circ\mathrm{incl}_t=\mathrm{proj}_t\circ\mathrm{incl}_t=0.$$

Now suppose x is an element of $\mathbb{F}_t$ with $\pi_t(x)=0$. Then,

$$\pi'_t(x)\in\ker\mathrm{proj}_t=\mathbb{N}_t.$$

It follows that

$$\rho_t(\pi'_t(x))=\pi''_t\circ\mathrm{incl}_t\circ\pi'_t(x)=\pi''_t\circ\pi'_t(x)=x.$$

(f.ii) and (g) It is clear that $P\circ i=0$. Apply Observation 2.14. □

3. The complexes $\mathbb{F}$ and $\mathbb{M}$ are DG$\Gamma-$algebras.

In Theorem 3.8 we prove that the complex $\mathbb{F}$ of Definition 2.5 is a DG$\Gamma-$algebra. The proof is based on the ideas of [38], as reformulated in [28]. In particular, we view $\mathbb{B}$ as a left $\mathbb{A}-$module in such a way that $\mathbb{F}$ is the trivial extension, $\mathbb{A}\ltimes\mathbb{B}[-2]$, of the DG$\Gamma-$algebra $\mathbb{A}$ by the graded $\mathbb{A}-$module $\mathbb{B}[-2]$. In other words, the multiplication

$$\mathbb{F}_t\times\mathbb{F}_u\to\mathbb{F}_{t+u}$$

is given by

$$\begin{bmatrix}a_t\\ b_{t-2}\end{bmatrix}\cdot\begin{bmatrix}a_u\\ b_{u-2}\end{bmatrix}=\begin{bmatrix}a_t\cdot a_u\\ a_t\cdot b_{u-2}+(-1)^{tu}a_u\cdot b_{t-2}\end{bmatrix}, \tag{3.1}$$

where the multiplication $a_t\cdot a_u$ takes place in the DG$\Gamma-$algebra $\mathbb{A}$ (see Definition 2.3) and the module multiplication $a_t\cdot b_{u-2}$ and $a_u\cdot b_{t-2}$ is defined in Proposition 3.4. The best feature of this point of view is that we obtain a divided power structure on $\mathbb{F}$ with no additional effort because of the following observation.

Observation 3.2. *Let* $(\mathbb{A}, d)$ *be a* $DG\Gamma-$*algebra and* $\mathbb{C}$ *be a positively graded* $\mathbb{A}-$*module. If there is a differential* D *on the trivial extension* $\mathbb{A}\ltimes\mathbb{C}$*, which satisfies*

(a) $D|_{\mathbb{A}} = d$,
(b) $(\mathbb{A} \ltimes \mathbb{C}, D)$ *is a* $DG-$*algebra, and*
(c) $D(c)c = 0$ *for all homogeneous* $c \in \mathbb{C}$ *of even degree,*

then $(\mathbb{A} \ltimes \mathbb{C}, d)$ *is a* $DG\Gamma-$*algebra.*

Note. If 2 is not a zero divisor in $\mathbb{A} \ltimes \mathbb{C}$, then hypothesis (c) follows from the fact that $0 = D(cc) = 2(Dc)c$.

Proof. If $a + c$ is a homogeneous element of $\mathbb{A} \ltimes \mathbb{C}$ of positive even degree, then define $(a+c)^{(\ell)} = a^{(\ell)} + a^{(\ell-1)}c$. It is straightforward to verify that $\mathbb{A} \ltimes \mathbb{C}$ satisfies all of the necessary axioms. □

The rest of the results in the present section may be proved in the generic situation (that is, when the hypotheses of (2.7) are in effect) and then specialized to an arbitrary situation. Thus, in the course of each proof, we may assume that R is a domain which contains the ring of integers. The DG$\Gamma-$algebra $\mathbb{A}$ of Definition 2.3 is generated as an $R-$algebra by the elements of F^* together with the elements $h^{(j)}$ for $1 \le j$. It follows from the fact $jh^{(j)} = h^{(j-1)}h^{(1)}$, that, if a_t is an element of $\mathbb{A}_t$, then there exists a non-zero integer N for which

(3.3) $\quad Na_t$ is a sum of elements of the form $a_{t-1} \cdot a_1$ and $a_{t-2} \cdot h^{(1)}$

with $a_i \in \mathbb{A}_i$. We will often establish a formula involving elements of $\mathbb{A}$ by first multiplying both sides of the equation by a non-zero integer and then taking advantage of (3.3).

Proposition 3.4. *Adopt the notation of Definition 2.3. Let* α_i *be an element of* $\bigwedge^i F^*$ *and* β_p *be an element of* $\bigwedge^p F$*. The multiplication*

$$\alpha_i \cdot \left(\beta_p\lambda^{(q)}\right) = \sum_{\ell\in\mathbb{Z}}(-1)^{i+\ell}\binom{q+i-1-\ell}{q}\alpha_i\left(\varphi^{(q+i-\ell)} \wedge \beta_p\right)\lambda^{(\ell)} \quad \textit{and}$$

$$h^{(j)}\cdot\left(\beta_p\lambda^{(q)}\right) = \sum_{\ell\in\mathbb{Z}}(-1)^{\ell}\binom{n-p-2q-j+\ell-1}{j}\binom{q+j-1-\ell}{q}\left(\varphi^{(q+j-\ell)} \wedge \beta_p\right)\lambda^{(\ell)}$$

gives $\mathbb{B}$ *the structure of a left* $\mathbb{A}-$*module.*

Proof. We first observe that

$$\alpha_1 \cdot \left(\beta_p\lambda^{(q)}\right) = \alpha_1(\beta_p)\lambda^{(q+1)} - \alpha_1(\varphi^{(q+1)} \wedge \beta_p)\lambda^{(0)} \tag{3.5}$$

for all integers q. Indeed, if $q \le -1$, then both sides of (3.5) are zero. If $0 \le q$, then the only non-zero terms of

$$\sum_{\ell}(-1)^{1+\ell}\binom{q-\ell}{q}\alpha_1\left(\varphi^{(q+1-\ell)} \wedge \beta_p\right)\lambda^{(\ell)}$$

involve $\ell = 0$ and $\ell = q+1$; because, if $\ell < 0$, then $\lambda^{(\ell)} = 0$; if $1 \le \ell \le q$, then $0 \le q-\ell \le q-1$ and $\binom{q-\ell}{q} = 0$; and if $q+2 \le \ell$, then $q+1-\ell \le -1$ and $\varphi^{(q+1-\ell)} = 0$. In a similar manner, we see that

$$(3.6)\quad h^{(1)}\cdot\left(\beta_p\lambda^{(q)}\right) = -(n-p-q-1)\beta_p\lambda^{(q+1)} + (n-p-2q-2)\left(\varphi^{(q+1)}\wedge\beta_p\right)\lambda^{(0)}$$

for all integers q. Apply the trick of (3.3). The proof will be completed once we show that

$$\text{(a)}\qquad \alpha_i\cdot\left(\alpha_1\cdot\left(\beta_p\lambda^{(q)}\right)\right) = (\alpha_i\wedge\alpha_1)\cdot\left(\beta_p\lambda^{(q)}\right),$$

$$\text{(b)}\qquad h^{(j)}\cdot\left(h^{(1)}\cdot\left(\beta_p\lambda^{(q)}\right)\right) = \left(h^{(j)}\cdot h^{(1)}\right)\cdot\left(\beta_p\lambda^{(q)}\right),\quad\text{and}$$

$$\text{(c)}\qquad \alpha_1\cdot\left(h^{(1)}\cdot\left(\beta_p\lambda^{(q)}\right)\right) = h^{(1)}\cdot\left(\alpha_1\cdot\left(\beta_p\lambda^{(q)}\right)\right)$$

for $\alpha_i \in \bigwedge^i F^*$ and $\beta_p \in \bigwedge^p F$. Apply (3.5) to see that the left side of (a) is equal to

$$\alpha_i\cdot\left(\alpha_1(\beta_p)\lambda^{(q+1)} - \alpha_1\left(\varphi^{(q+1)}\wedge\beta_p\right)\lambda^{(0)}\right),$$

which is equal to $S_1 + S_2$ for

$$S_1 = \sum_\ell (-1)^{i+\ell}\binom{q+i-\ell}{q+1}\alpha_i\left(\varphi^{(q+1-i-\ell)}\wedge\alpha_1(\beta_p)\right)\lambda^{(\ell)},\quad\text{and}$$
$$S_2 = -\sum_\ell (-1)^{i+\ell}\alpha_i\left(\varphi^{(i-\ell)}\wedge\alpha_1\left(\varphi^{(q+1)}\wedge\beta_p\right)\right)\lambda^{(\ell)}.$$

The element α_1 of $\bigwedge^1 F^*$ acts like a graded derivation on $\bigwedge^\bullet F$; and therefore,

$$(3.7)\qquad \alpha_1\left(\varphi^{(q+1)}\wedge\beta_p\right) = \alpha_1\left(\varphi^{(1)}\right)\wedge\varphi^{(q)}\wedge\beta_p + \varphi^{(q+1)}\wedge\alpha_1(\beta_p).$$

It follows, from (1.6) and Note 1.8, that $S_2 = S_2' + S_2''$, with

$$S_2' = \sum_\ell (-1)^{i+\ell+1}\binom{q+i-\ell}{q}\alpha_i\left(\alpha_1\left(\varphi^{(q+i-\ell+1)}\right)\wedge\beta_p\right)\lambda^{(\ell)}\quad\text{and}$$
$$S_2'' = \sum_\ell (-1)^{i+\ell+1}\binom{q+i-\ell+1}{q+1}\alpha_i\left(\varphi^{(q+i-\ell+1)}\wedge\alpha_1(\beta_p)\right)\lambda^{(\ell)}.$$

Identity (b) of Observation 1.2 yields

$$S_1 + S_2'' = \sum_\ell (-1)^{i+\ell+1}\binom{q+i-\ell}{q}\alpha_i\left(\varphi^{(q+i-\ell+1)}\wedge\alpha_1(\beta_p)\right)\lambda^{(\ell)}.$$

The module action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$ gives

$$(\alpha_i\wedge\alpha_1)(\beta_{p'}\wedge\beta_p) = \alpha_i(\alpha_1[\beta_{p'}\wedge\beta_p]) = \alpha_i\left(\alpha_1(\beta_{p'})\wedge\beta_p + (-1)^{p'}\beta_{p'}\wedge\alpha_1(\beta_p)\right);$$

therefore,

$$(S_1+S_2'')+S_2' = \sum_\ell (-1)^{i+\ell+1}\binom{q+i-\ell}{q}(\alpha_i\wedge\alpha_1)\left(\varphi^{(q+i-\ell+1)}\wedge\beta_p\right)\lambda^{(\ell)},$$

and (a) has been established.

The right side of (b) is equal to $(j+1)h^{(j+1)}\cdot\left(\beta_p\lambda^{(q)}\right)$, which is equal to

$$\sum_\ell(-1)^\ell N_\ell\left(\varphi^{(q+j+1-\ell)}\wedge\beta_p\right)\lambda^{(\ell)},\quad \text{for}$$

$$N_\ell=(j+1)\binom{n-p-2q-j+\ell-2}{j+1}\binom{q+j-\ell}{q}.$$

Apply (3.6) to see that the left side of (b) is equal to

$$h^{(j)}\cdot\left(-(n-p-q-1)\beta_p\lambda^{(q+1)}+(n-p-2q-2)\left(\varphi^{(q+1)}\wedge\beta_p\right)\lambda^{(0)}\right).$$

It follows that the left side of (b) is equal to

$$\sum_\ell(-1)^\ell M_\ell\left(\varphi^{(q+j+1-\ell)}\wedge\beta_p\right)\lambda^{(\ell)},\quad \text{for}$$

$$M_\ell=\tbinom{n-p-2q-j+\ell-3}{j}\left[-(n-p-q-1)\tbinom{q+j-\ell}{q+1}+(n-p-2q-2)\tbinom{j+q-\ell+1}{q+1}\right].$$

We apply Observation 1.2 to show that $M_\ell=N_\ell$. In particular, identity (b), applied to the last binomial coefficient, followed by identity (c) gives

$$\begin{aligned}M_\ell&=\tbinom{n-p-2q-j+\ell-3}{j}\left[-(q+1)\tbinom{q+j-\ell}{q+1}+(n-p-2q-2)\tbinom{j+q-\ell}{q}\right]\\&=\tbinom{n-p-2q-j+\ell-3}{j}(n-p-2q-2-j+\ell)\tbinom{j+q-\ell}{q}.\end{aligned}$$

One more application of identity (c) yields $M_\ell=N_\ell$ and completes the proof of (b). Straightforward calculations show that both sides of (c) are equal to

$$\begin{aligned}&-(n-p-q-1)\alpha_1(\beta_p)\lambda^{(q+2)}+(n-p-2q-2)\alpha_1\left(\varphi^{(q+1)}\wedge\beta_p\right)\lambda^{(1)}\\&-(n-p-3-2q)(q+1)\alpha_1\left(\varphi^{(q+2)}\wedge\beta_p\right)\lambda^{(0)}.\quad\square\end{aligned}$$

Theorem 3.8. *The complex $\mathbb{F}$ of Definition 2.5 is a DG$\Gamma-$algebra with multiplication given in (3.1) and divided power structure given by*

$$\begin{bmatrix}a_t\\ b_{t-2}\end{bmatrix}^{(\ell)}=\begin{bmatrix}a_t^{(\ell)}\\ a_t^{(\ell-1)}\cdot b_{t-2}\end{bmatrix}$$

for $a_t \in \mathbb{A}_t$ *and* $b_{t-2} \in \mathbb{B}_{t-2}$*, whenever* t *is a positive even integer.*

Proof. Apply Proposition 3.4 and Observation 3.2. It remains to show that $(\mathbb{F}, f)$ satisfies the derivative property. In other words, for X_i in $\mathbb{F}_i$, we must prove that

$$f_t(X_t) \cdot X_u + (-1)^t X_t \cdot f_u(X_u) = f_{t+u}(X_t \cdot X_u). \tag{3.9}$$

When X_t is in $\mathbb{A}$, the result is established by induction on t. In light of (3.3), there are three cases to consider:

$$X_t = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}, \qquad X_u = \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix}, \tag{3.10}$$

$$X_t = \begin{bmatrix} h^{(1)} \\ 0 \end{bmatrix}, \qquad X_u = \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix}, \quad \text{and} \tag{3.11}$$

$$X_t = \begin{bmatrix} 0 \\ \beta_{p'} \lambda^{(q')} \end{bmatrix}, \qquad X_u = \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix}. \tag{3.12}$$

We first consider case (3.10). Let

$$\begin{bmatrix} a_L \\ b_L \end{bmatrix} \text{ be the left side of (3.9), and } \begin{bmatrix} a_R \\ b_R \end{bmatrix} \text{ be the right side of (3.9).} \tag{3.13}$$

Observe that

$$b_L = g(\alpha_1) \wedge \beta_p \lambda^{(q)} - \alpha_1 \cdot \left(Y(\beta_p)\lambda^{(q)} + g \wedge \beta_p \lambda^{(q-1)} \right).$$

Use (3.5) to see that

$$\alpha_1 \cdot \left(Y(\beta_p)\lambda^{(q)} \right) = \alpha_1(Y(\beta_p))\lambda^{(q+1)} - \alpha_1(\varphi^{(q+1)} \wedge Y(\beta_p))\lambda^{(0)} \quad \text{and}$$
$$\alpha_1 \cdot \left((g \wedge \beta_p)\lambda^{(q-1)} \right) = \alpha_1(g \wedge \beta_p)\lambda^{(q)} - \alpha_1 \left(\varphi^{(q)} \wedge g \wedge \beta_p \right) \lambda^{(0)}.$$

At this point, there is no difficulty in checking that

$$b_L = Y\left(\alpha_1(\beta_p)\right)\lambda^{(q+1)} + g \wedge \alpha_1(\beta_p)\lambda^{(q)} - Y\left(\alpha_1 \left(\varphi^{(q+1)} \wedge \beta_p \right) \right) \lambda^{(0)} = b_R.$$

Apply Proposition 1.4 in order to see that

$$a_L = (-1)^p \alpha_1 \wedge v(\beta_p \lambda^{(q)}) = \sum_j (-1)^j \binom{n+j-p-q-1}{q} \left[\alpha_1 \left(\varphi^{(n+j-p-q)} \wedge \beta_p \right) \right](\xi) h^{(j)}.$$

On the other hand, $a_R = S_1 + S_2$, where

$$S_1 = (-1)^p v\left(\alpha_1(\beta_p)\lambda^{(q+1)} \right) \quad \text{and} \quad S_2 = (-1)^{p+1} v\left(\alpha_1 \left(\varphi^{(q+1)} \wedge \beta_p \right) \lambda^{(0)} \right).$$

In other words,

$$S_1 = \sum_j (-1)^{j+1} \binom{n+j-p-q-1}{q+1} \left(\varphi^{(n+j-p-q)} \wedge \alpha_1(\beta_p)\right)(\xi)h^{(j)} \quad \text{and}$$

$$S_2 = \sum_j (-1)^j \left(\varphi^{(n+j-2q-p-1)} \wedge \alpha_1\left(\varphi^{(q+1)} \wedge \beta_p\right)\right)(\xi)h^{(j)}.$$

The reasoning of (3.7) gives $S_2 = S_2' + S_2''$, where

$$S_2' = \sum_j (-1)^j \binom{n+j-q-p-1}{q} \left(\alpha_1\left(\varphi^{(n+j-q-p)}\right) \wedge \beta_p\right)(\xi)h^{(j)} \quad \text{and}$$

$$S_2'' = \sum_j (-1)^j \binom{n+j-q-p}{q+1} \left(\varphi^{(n+j-q-p)} \wedge \alpha_1(\beta_p)\right)(\xi)h^{(j)}.$$

Proceed as in the end of the proof of case (a) in Proposition 3.4 to see that

$$S_1 + S_2'' = \sum_j (-1)^j \binom{n+j-q-p-1}{q} \left(\varphi^{(n+j-q-p)} \wedge \alpha_1(\beta_p)\right)(\xi)h^{(j)},$$

and $a_R = (S_1 + S_2'') + S_2' = a_L$.

We now consider case (3.11). Define a_L, b_L, a_R, and b_R as in (3.13). We know that

$$b_L = Y \cdot \left(\beta_p \lambda^{(q)}\right) + h^{(1)} \cdot \left(Y(\beta_p)\lambda^{(q)} + g \wedge \beta_p \lambda^{(q-1)}\right).$$

Apply (3.5) and (3.6) to see that

$$Y \cdot \left(\beta_p \lambda^{(q)}\right) = Y(\beta_p)\lambda^{(q+1)} - Y\left(\varphi^{(q+1)} \wedge \beta_p\right)\lambda^{(0)},$$

$$h^{(1)} \cdot \left(Y(\beta_p)\lambda^{(q)}\right) = -(n-p-q)Y(\beta_p)\lambda^{(q+1)} + (n-p-2q-1)\varphi^{(q+1)} \wedge Y(\beta_p)\lambda^{(0)}, \quad \text{and}$$

$$h^{(1)} \cdot \left(g \wedge \beta_p \lambda^{(q-1)}\right) = -(n-p-q-1)g \wedge \beta_p \lambda^{(q)} + (n-p-2q-1)(\varphi^{(g)} \wedge \beta_p)\lambda^{(0)}.$$

A straightforward calculation yields that

$$b_L = -(n-p-q-1)\left[Y(\beta_p)\lambda^{(q+1)} + (g \wedge \beta_p)\lambda^{(q)}\right] + (n-p-2q-2)Y\left(\varphi^{(q+1)} \wedge \beta_p\right)\lambda^{(0)} = b_R.$$

Use the fact that $h^{(1)} \cdot h^{(j)} = (j+1)h^{(j+1)}$ in order to see that

$$a_L = (-1)^{p-1}h^{(1)} \cdot v(\beta_p \lambda^{(q)}) = \sum_j (-1)^j j \binom{n+j-p-q-2}{q} \left(\varphi^{(n+j-p-q-1)} \wedge \beta_p\right)(\xi)h^{(j)}.$$

On the other hand,

$$a_R = (-1)^p(n-p-q-1)v\left(\beta_p \lambda^{(q+1)}\right) + (-1)^{p+1}(n-p-2q-2)v\left(\varphi^{(q+1)} \wedge \beta_p \lambda^{(0)}\right).$$

We know that

$$v\left(\beta_p\lambda^{(q+1)}\right)=\sum_j(-1)^{j+p}\binom{n+j-p-q-2}{q+1}\left(\varphi^{(n+j-p-q-1)}\wedge\beta_p\right)(\xi)h^{(j)}.$$

Apply Note 1.8 to see that

$$v\left(\varphi^{(q+1)}\wedge\beta_p\lambda^{(0)}\right)=\sum_j(-1)^{j+p}\binom{n+j-p-q-1}{q+1}\left(\varphi^{(n+j-p-q-1)}\wedge\beta_p\right)(\xi)h^{(j)}.$$

It follows that

$$a_L-a_R=\sum_j(-1)^jN_j\left(\varphi^{(n+j-p-q-1)}\wedge\beta_p\right)(\xi)h^{(j)},\quad\text{for}$$

$$N_j=j\binom{n+j-p-q-2}{q}-(n-p-q-1)\binom{n+j-p-q-2}{q+1}+(n-p-2q-2)\binom{n+j-p-q-1}{q+1}.$$

Apply Observation 1.2 to simplify N_j. In particular, identity (b), applied to the final binomial coefficient, gives

$$N_j=(n+j-p-2q-2)\binom{n+j-p-q-2}{q}-(q+1)\binom{n+j-p-q-2}{q+1},$$

and identity (c) gives $N_j=0$. The proof of (3.9) is complete in case (3.11).

Finally, we consider case (3.12). In this case, equation (3.9) is

$$\begin{bmatrix}(-1)^{p'-1}v\left(\beta_{p'}\lambda^{(q')}\right)\\ b'\end{bmatrix}\cdot\begin{bmatrix}0\\ \beta_p\lambda^{(q)}\end{bmatrix}+(-1)^{p'+p'(p-1)}\begin{bmatrix}(-1)^{p-1}v\left(\beta_p\lambda^{(q)}\right)\\ b\end{bmatrix}\cdot\begin{bmatrix}0\\ \beta_{p'}\lambda^{(q')}\end{bmatrix}=0$$

for appropriate elements b' and b in $\mathbb{B}$; and therefore, it suffices to prove that

$$B-(-1)^{(p-1)(p'-1)}A=0 \tag{3.14}$$

for $A=v\left(\beta_{p'}\lambda^{(q')}\right)\cdot\left(\beta_p\lambda^{(q)}\right)$ and $B=v\left(\beta_p\lambda^{(q)}\right)\cdot\left(\beta_{p'}\lambda^{(q')}\right)$. Definition 2.3 gives

$$B=\sum_j(-1)^{j+p}\binom{n+j-p-q-1}{q}h^{(j)}\cdot\left[\alpha_{(j)}\cdot\left(\beta_{p'}\lambda^{(q')}\right)\right],\quad\text{for}$$

$$\alpha_{(j)}=\left(\varphi^{(n+j-p-q)}\wedge\beta_p\right)(\xi)\in\bigwedge^{2q+p-2j+1}F^*.$$

The multiplication of $\alpha_{(j)}$ on $\mathbb{B}$, as given in Proposition 3.4, yields

$$B=\sum_{j,\ell}(-1)^{1+j+\ell}\tbinom{n+j-p-q-1}{q}\tbinom{q'+2q+p-2j-\ell}{q'}h^{(j)}\cdot\left[\alpha_{(j)}(\varphi^{(q'+2q+p-2j+1-\ell)}\wedge\beta_{p'})\lambda^{(\ell)}\right].$$

The multiplication of $h^{(j)}$ on $\mathbb{B}$ is also given in Proposition 3.4; take p to be

$$\deg \alpha_{(j)}(\varphi^{(q'+2q+p-2j+1-\ell)} \wedge \beta_{p'}) = 2q+p-2j+1+2q'+p'-2\ell,$$

q to be ℓ, and ℓ to be L. It follows that

$$B = \sum_{j,L} (-1)^{1+j+L} \tbinom{n+j-p-q-1}{q} \tbinom{n-2q-p-2q'-p'+j+L-2}{j} B_{j,L} \lambda^{(L)},$$

where

$$B_{j,L} = \sum_{\ell} (-1)^{\ell} \tbinom{q'+2q+p-2j-\ell}{q'} \tbinom{\ell+j-1-L}{\ell} \Big[\alpha_{(j)}(\varphi^{(q'+2q+p-2j+1-\ell)} \wedge \beta_{p'})\Big] \wedge \varphi^{(\ell+j-L)}.$$

Observations 1.2 (i) and (e) show that

$$\binom{J-1}{q'} = \sum_{0 \le u \le q'} (-1)^{u+q'} \binom{J}{u} \tag{3.15}$$

for all integers J. Apply (3.15) and Note 1.8 (b) to see that $B_{j,L}$ is equal to

$$\sum_{0\le u\le q'} \sum_{\ell} (-1)^{q'+u+\ell} \tbinom{\ell+j-1-L}{\ell} \Big[\alpha_{(j)} \Big(\varphi^{(q'-u+2q+p-2j+1-\ell)} \wedge \Big(\varphi^{(u)} \wedge \beta_{p'}\Big)\Big)\Big] \wedge \varphi^{(\ell+j-L)}.$$

We next apply Corollary 1.10. Replace ℓ by $q'-u+2q+p-2j+1-k$; and let A equal $q'-u+2q+p-j-L+1$. Observe that

$$\deg \alpha_{(j)} = 2q+p-2j+1 \le 2q+p-2j+1+(q'-u) = C.$$

It follows that

$$\begin{aligned} B_{j,L} &= \sum_{0\le u\le q'} (-1)^{q'+u} \Big[\alpha_{(j)} \Big(\varphi^{(q'-u+2q+p-j+1-L)} \wedge \Big(\varphi^{(u)} \wedge \beta_{p'}\Big)\Big)\Big] \\ &= \left(\sum_{0\le u\le q'} (-1)^{q'+u} \binom{q'+2q+p-j+1-L}{u}\right) \Big[\alpha_{(j)} \Big(\varphi^{(q'+2q+p-j+1-L)} \wedge \beta_{p'}\Big)\Big]. \end{aligned}$$

One further application of (3.15) gives

$$B_{j,L} = \binom{q'+2q+p-j-L}{q'} \Big[\alpha_{(j)} \Big(\varphi^{(q'+2q+p-j+1-L)} \wedge \beta_{p'}\Big)\Big];$$

and therefore,

$$B = \sum_{j,L} (-1)^{1+j+L} \tbinom{n+j-p-q-1}{q} \tbinom{n-2q-p-2q'-p'+j+L-2}{j} \tbinom{q'+2q+p-j-L}{q'} \alpha_{j,L} \lambda^{(L)}, \text{ for}$$

$$\alpha_{j,L} = \Big[\Big(\varphi^{(n+j-p-q)} \wedge \beta_p\Big)(\xi)\Big] \Big(\varphi^{(q'+2q+p-j+1-L)} \wedge \beta_{p'}\Big).$$

Use symmetery and Proposition 1.4 (b) to see that $(-1)^{(p-1)(p'-1)}A$ is equal to

$$\sum_{j,L}(-1)^{1+j+L}\tbinom{n+j-p'-q'-1}{q'}\tbinom{n-2q'-p'-2q-p+j+L-2}{j}\tbinom{q+2q'+p'-j-L}{q}\alpha'_{j,L}\lambda^{(L)},\text{ for}$$

$$\alpha'_{j,L}=\left[\left(\varphi^{(q+2q'+p'-j+1-L)}\wedge\beta_p\right)(\xi)\right]\left(\varphi^{(n+j-p'-q')}\wedge\beta_{p'}\right).$$

Replace every j in B with $k-n+p+q$ and every j in A with $q+2q'+p'-k+1-L$, in order to write the left side of (3.14) as

$$\sum_{k,L}(-1)^{1+L+k+n+p+q}\tbinom{k-1}{q}\tbinom{q'+q+n-L-k}{q'}N_{k,L}\left[(\varphi^{(k)}\wedge\beta_p)(\xi)\right]\left(\varphi^{(q'+n+q-L+1-k)}\wedge\beta_{p'}\right)\lambda^{(L)},$$

for

$$N_{k,L}=\binom{k-q-2q'-p'+L-2}{k+p+q-n}+(-1)^{p'+L+n+p}\binom{n-q-p-1-k}{q+2q'+p'-k-L+1}.$$

Observation 1.2 (j) guarantees that

$$N_{k,L}=\binom{k+L-q-2q'-p'-2}{n-2q-2q'-p'-p+L-2}.$$

Recall, from Observation 1.2 (f), that

$$\binom{q'+q+n-L-k}{q'}=(-1)^{q'}\binom{k+L-n-q-1}{q'}.$$

Apply Corollary 1.11 with $s=3$, $C_1=q$, $C_2=q'$, $C_3=n-2q-2q'-p-p'+L-2$, and $A=n+q+q'-L+1$. Observe that

$$1+A+C_1+C_2+C_3+\deg\beta_{p'}=2n-p\le 2n+1-p=\deg\beta_p(\xi).$$

Equation 3.14 has been established and the proof is complete. □

One technical result is needed before we can prove that $\mathbb{M}$ is a DG$\Gamma-$algebra.

Lemma 3.16. *Adopt the notation of Definition 2.15. If x and y are elements of $\mathbb{F}$, then $\pi(x\cdot\rho P(y))=0$.*

Proof. Our proof is based on the following three observations:

$$\rho P(\mathbb{F})\subseteq(\mathbb{N}\cap\mathbb{B})+f(\mathbb{N}\cap\mathbb{B}),\tag{3.17}$$

$$\mathbb{F}\cdot(\mathbb{N}\cap\mathbb{B})\subseteq(\mathbb{N}\cap\mathbb{B}),\quad\text{and}\tag{3.18}$$

$$\pi(\mathbb{N}\cap\mathbb{B})=0.\tag{3.19}$$

If we assume the three facts for the time being, then the result follows immediately. Indeed, according to (3.17), we need only show that $\pi(xz)=0$ and $\pi(xf(z))$ for $x\in\mathbb{F}$ and $z\in(\mathbb{N}\cap\mathbb{B})$. On the other hand, $\mathbb{F}$ is a differential algebra so

$$\pi(xf(z))=\pm\pi f(xz)\pm\pi((fx)z).$$

We know from Proposition 2.16 that π is a map of complexes; hence, it suffices to show that $\pi(xz) = 0$ for $x \in \mathbb{F}$ and $z \in (\mathbb{N} \cap \mathbb{B})$, and this is clear from (3.18) and (3.19).

The rest of the proof is devoted to establishing (3.17) – (3.19). Assertion (3.19) holds because $\pi = \operatorname{proj} \circ \pi'$, π' acts like the identity on $\mathbb{B}$, and proj kills $\mathbb{N}$. We next prove (3.17). We know that π' and π'' both act like the identity on $\mathbb{B}$; thus, it is clear that $\rho \circ P(\mathbb{B}) \subseteq (\mathbb{N} \cap \mathbb{B})$. Also, π' acts like the identity on $\mathbb{M}$; so, $P(\mathbb{M} \cap \mathbb{A}) = 0$; and it suffices to apply $\rho \circ P$ to

$$a = \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} \in \mathbb{N}_t.$$

The map π' was defined so that the composition

$$\mathbb{A} \cap \mathbb{N}_t \xrightarrow{\text{inclusion}} \mathbb{F}_t \xrightarrow{\pi'} \mathbb{F}_t \xrightarrow{\text{projection}} \mathbb{A} \cap \mathbb{N}_t$$

is the identity map. (The best way to verify the above claim is to notice that the map π' of Definition 2.15 is described in (2.13).) It follows that $\pi'(a) = a + x$ for some element x of $\mathbb{M} + \mathbb{B}$; therefore,

$$\rho \circ P(a) = \pi'' \circ \operatorname{incl} \circ p \circ \pi'(a) = \pi''(a) + x'$$

for some element x' of $\mathbb{N} \cap \mathbb{B}$, and

$$\rho \circ P(a) = (-1)^i f_{t+1} \begin{bmatrix} 0 \\ \alpha_i(\eta)\lambda^{(i+j-n-1)} \end{bmatrix} + x'.$$

The element

$$\begin{bmatrix} 0 \\ \alpha_i(\eta)\lambda^{(i+j-n-1)} \end{bmatrix}$$

of $\mathbb{F}_{t+1}$ is in $\mathbb{N}$ because

$$n \le n + j = \deg \alpha_i(\eta) + i + j - n - 1.$$

Line (3.17) has been established.

Finally, we prove (3.18). Let

$$x = \begin{bmatrix} \alpha_i h^{(j)} \\ \beta_p \lambda^{(q)} \end{bmatrix} \in \mathbb{F} \qquad \text{and} \qquad y = \begin{bmatrix} 0 \\ \beta_{p'} \lambda^{(q')} \end{bmatrix} \in \mathbb{N}.$$

We know that

$$xy = \begin{bmatrix} 0 \\ \alpha_i h^{(j)} \beta_{p'} \lambda^{(q')} \end{bmatrix}.$$

The proof will be complete when we show that $(\alpha_i h^{(j)}) \cdot \left(\beta_{p'}\lambda^{(q')}\right)$ is an element of $\mathbb{N}$. However, the multiplication in $\mathbb{F}$ is associative, so we apply the trick of (3.3). At this point the result is obvious because (3.5) and (3.6) show that

$$\alpha_1 \cdot \left(\beta_{p'}\lambda^{(q')}\right) = \alpha_1(\beta_{p'})\lambda^{(q'+1)} - \alpha_1(\varphi^{(q'+1)} \wedge \beta_{p'})\lambda^{(0)}, \quad \text{and}$$

$$h^{(1)} \cdot \left(\beta_{p'}\lambda^{(q')}\right) = -(n - p' - q' - 1)\beta_{p'}\lambda^{(q'+1)} + (n - p' - 2q' - 2)\left(\varphi^{(q'+1)} \wedge \beta_{p'}\right)\lambda^{(0)}.$$

We know $n \le p' + q'$. It follows that

$$n \le (p' - 1) + (q' + 1) < p' + q' + 1 \le 2(q' + 1) + p' - 1 < p' + 2(q' + 1). \quad \square$$

Theorem 3.20. *The complex* $(\mathbb{M}, m)$ *of Definition 2.15 is a* $DG\Gamma-$*algebra; furthermore,* $\pi\colon (\mathbb{F}, f) \to (\mathbb{M}, m)$ *is a homomorphism of* $DG\Gamma-$*algebras.*

The following statement is obtained by combining Corollary 2.17 with Theorem 3.20. An analogous statement follows from Corollary 2.18.

Corollary 3.21. *Let* $(R, \mathfrak{m}, k)$ *be a noetherian local ring,* $n \geq 2$ *be an integer, and* $Y_{1\times 2n+1}$ *and* $X_{2n+1\times 2n+1}$ *be matrices with entries from* $\mathfrak{m}$*, with* X *an alternating matrix. Let* I *be the* $R-$*ideal* $I_1(YX)$ *and* A *be the quotient ring* R/I*. If* $2n \leq \operatorname{grade} I$*, then the minimal* $R-$*resolution of* A *is a* $DG\Gamma-$*algebra.*

Proof of Theorem 3.20. Define the multiplication $\times_{\mathbb{M}}\colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}$ by

$$x \times_{\mathbb{M}} y = \pi(i(x) \cdot i(y))$$

for all $x, y \in \mathbb{M}$ (where $\cdot$ represents multiplication in $\mathbb{F}$), and define the divided power structure on $\mathbb{M}$ by

$$x^{(k)_{\mathbb{M}}} = \pi\left((ix)^{(k)}\right)$$

for all homogeneous elements x in $\mathbb{M}$ of even degree. Use Proposition 2.16 (g) and Lemma 3.16 in order to verify that $\mathbb{M}$ and π have all of the necessary properties. For example, we prove

$$\pi(xy) = \pi(x) \times_{\mathbb{M}} \pi(y) \quad \text{for } x \text{ and } y \text{ in } \mathbb{F}, \text{ and} \tag{3.22}$$

$$(x \times_{\mathbb{M}} y) \times_{\mathbb{M}} z = x \times_{\mathbb{M}} (y \times_{\mathbb{M}} z) \quad \text{for } x,\ y, \text{ and } z \text{ in } \mathbb{M}, \text{ and} \tag{3.23}$$

$$k! x^{(k)_{\mathbb{M}}} = \underbrace{x \times_{\mathbb{M}} \cdots \times_{\mathbb{M}} x}_{k \text{ times}} \quad \text{for } x \text{ in } \mathbb{M}. \tag{3.24}$$

Indeed, the right side of (3.22) is equal to

$$\pi([i\pi x] \cdot [i\pi y]) = \pi([x - \rho P(x)] \cdot [y - \rho P(y)]) = \pi(xy).$$

In a similar manner, we see that the right side of (3.23)

$$\pi(i(x) \cdot i\pi[i(y) \cdot i(z)]) = \pi(x \cdot i\pi(yz)) = \pi(x \cdot [yz - \rho P(yz)]) = \pi(x(yz)).$$

The multiplication in $\mathbb{F}$ is associative; therefore, (3.23) is established by symmetry. The left side of (3.24) is equal to

$$k!\pi\left((ix)^{(k)}\right) = \pi\left(k! x^{(k)}\right) = \pi(x^k).$$

Apply (3.22) to see that

$$\pi(x^k) = \underbrace{\pi(x) \times_{\mathbb{M}} \cdots \times_{\mathbb{M}} \pi(x)}_{k \text{ times}},$$

and this is equal to the right side of (3.24) because $x \in \mathbb{M}$. $\square$

4. The algebra $\operatorname{Tor}_\bullet^R(A,k)$.

In the present section

(4.1) $(R,\mathfrak{m},k)$ is a local ring, $n\ge 2$ is an integer, $X^{\text{alt}}_{2n+1\times 2n+1}$ and $Y_{1\times 2n+1}$ are matrices with entries in $\mathfrak{m}$, $I=I_1(YX)$ has grade $2n$, and $A=R/I$.

In Theorem 4.3 we calculate the DG$\Gamma-$algebra $\operatorname{Tor}_\bullet^R(A,k)$.

Lemma 4.2. *Adopt Data 4.1. If* $(\mathbb{M},m)$ *is the DG*$\Gamma-$*algebra* $\mathbb{M}(Y,X)$ *of Theorem 3.20 and "$-$" is the functor* $\_\_\otimes_R k$*, then the multiplication* $\overline{\mathbb{M}}\times\overline{\mathbb{M}}\to\overline{\mathbb{M}}$ *is given by*

$$\begin{bmatrix}\overline{\alpha}_i\overline{h}^{(j)}\\ \overline{\beta}_p\overline{\lambda}^{(q)}\end{bmatrix}\begin{bmatrix}\overline{\alpha}_{i'}\overline{h}^{(j')}\\ \overline{\beta}_{p'}\overline{\lambda}^{(q')}\end{bmatrix}=\operatorname{proj}\begin{bmatrix}\binom{j+j'}{j}\overline{\alpha}_i\wedge\overline{\alpha}_{i'}\overline{h}^{(j+j')}\\ \hline (-1)^j\binom{n-p'-q'-1}{j}\overline{\alpha}_i(\overline{\beta}_{p'})\overline{\lambda}^{(q'+j+i)}\\ +(-1)^{pi'}(-1)^{j'}\binom{n-p-q-1}{j'}\overline{\alpha}_{i'}(\overline{\beta}_p)\overline{\lambda}^{(q+i'+j')}\end{bmatrix},$$

where $\alpha_i\in\bigwedge^i F^*$, $\beta_p\in\bigwedge^p F$, $0\le q,q'$,

$$i+j\le n,\quad i'+j'\le n,\quad p+q\le n-1,\quad p'+q'\le n-1,\quad \text{and}$$

$$\operatorname{proj}\begin{bmatrix}\overline{\alpha}_i\overline{h}^{(j)}\\ 0\end{bmatrix}=\begin{cases}\begin{bmatrix}\overline{\alpha}_i\overline{h}^{(j)}\\ 0\end{bmatrix} & \text{if } i+j\le n,\\ 0 & \text{if } n+1\le i+j,\end{cases}$$
$$\operatorname{proj}\begin{bmatrix}0\\ \overline{\beta}_p\overline{\lambda}^{(q)}\end{bmatrix}=\begin{cases}\begin{bmatrix}0\\ \overline{\beta}_p\overline{\lambda}^{(q)}\end{bmatrix} & \text{if } i+j\le n-1, \text{ and}\\ 0 & \text{if } n\le i+j.\end{cases}$$

Proof. If x and x' are elements of $\mathbb{M}$, then $m\times_{\mathbb{M}}m'=\pi(m\times_{\mathbb{F}}m')$, where $\times_{\mathbb{M}}$ is multiplication in $\mathbb{M}$, $\times_{\mathbb{F}}$ is multiplication in $\mathbb{F}$, and π is given in Definition 2.15. Write $x\equiv x'$ to mean $\overline{x}=\overline{x'}$. Keep in mind that $Y\equiv 0$, $g\equiv 0$, and

$$\varphi^{(j)}\equiv\begin{cases}1 & \text{if } j=0, \text{ and}\\ 0 & \text{for any integer } j \text{ with } j\ne 0.\end{cases}$$

The only interesting calculation involves

$$x=\begin{bmatrix}\alpha_i h^{(j)}\\ 0\end{bmatrix}\quad\text{and}\quad x'=\begin{bmatrix}0\\ \beta_{p'}\lambda^{(q')}\end{bmatrix}.$$

Use Proposition 3.4 and (3.1) to see that

$$x\times_{\mathbb{F}}x'=\begin{bmatrix}0\\ \alpha_i h^{(j)}\cdot\left(\beta_{p'}\lambda^{(q')}\right)\end{bmatrix},$$

$$h^{(j)}\cdot\left(\beta_{p'}\lambda^{(q')}\right)\equiv(-1)^j\binom{n-p'-q'-1}{j}\beta_{p'}\lambda^{(q'+j)},\ \text{and}$$

$$\alpha_i\cdot\left(\beta_{p'}\lambda^{(q'+j)}\right)\equiv\alpha_i(\beta_{p'})\lambda^{(q'+j+i)}.$$

We complete the proof by showing that $\pi\equiv\text{proj}$. It is clear that $\pi|_{\mathbb{M}}$ is the identity map and that $\pi(\mathbb{N}\cap\mathbb{B})=0$. Finally, if $a=\begin{bmatrix}\alpha_ih^{(j)}\\0\end{bmatrix}$ is an element of $\mathbb{N}\cap\mathbb{A}$, then

$$\pi(a)=\pm\,\text{proj}\circ f\begin{bmatrix}0\\\alpha_i(\eta)\lambda^{(i+j-n-1)}\end{bmatrix}\equiv\pm\,\text{proj}\begin{bmatrix}v\left(\alpha_i(\eta)\lambda^{(i+j-n-1)}\right)\\0\end{bmatrix}\equiv\pm\,\text{proj}(a)=0.$$

The map v may be found in Definition 2.3. $\square$

Theorem 4.3. *Adopt the hypotheses of (4.1). Let V be a $k-$vector space of dimension $2n+1$, h be a divided power variable of degree two, $S_\bullet$ be the $DG\Gamma-$algebra*

$$\frac{\bigwedge^\bullet V\langle h\rangle}{\sum_{i=0}^{n+1}\bigwedge^i Vh^{(n+1-i)}},$$

(with differential identically zero), $\mathfrak{A}_\bullet$ be the $S_\bullet$-ideal $\left(\sum_{i=0}^{n}\bigwedge^i Vh^{(n-i)}\right)$, and $N_\bullet$ be the graded left $S_\bullet$-module $(S_\bullet/\mathfrak{A}_\bullet)^*[-2n]$, where $(\cdot)^*$ means $\operatorname{Hom}_k(\cdot,k)$. Then the $DG\Gamma$-algebras $\operatorname{Tor}^R_\bullet(A,k)$ and $S_\bullet\ltimes N_\bullet$ are isomorphic.

Note. The algebra $S_\bullet$ comes equipped with a divided power structure; and therefore, the divided power structure of $S_\bullet\ltimes N_\bullet$ is described in Observation 3.2.

Proof. The $k-$algebra $\operatorname{Tor}^R_\bullet(A,k)$ is equal to $\overline{\mathbb{M}}$ from Lemma 4.2. We begin our study of $S_\bullet\ltimes N_\bullet$ by naming some of the elements of $N_\bullet$; see [28] for more details. If $w_p\in\bigwedge^p V^*$ and q is an integer, then let w_px_q represent the $k-$homomorphism from $S_\bullet/\mathfrak{A}_\bullet$ to k which sends $v_ih^{(j)}$ to

$$\begin{cases}0, & \text{if } p\neq i,\\ 0, & \text{if } q\neq j,\text{ and}\\ w_p(v_i), & \text{if } p=i\text{ and } q=j\end{cases}$$

for $v_i\in\bigwedge^i V$. Observe that w_px_q is a nonzero homomorphism if and only if p and q are nonnegative integers with $p+q\le n-1$. Observe also, that, w_px_q has degree $2n-p-2q$ as an element of $N_\bullet$. It follows that the graded $S_\bullet-$module $N_\bullet$ is equal to $\sum_{d=2}^{2n}N_d$, where $N_d=\sum\bigwedge^p V^*x_q$ and the sum is taken over all pairs of nonnegative integers p and q with $p+q\le n-1$ and $p+2q=2n-d$. We are now able to give a clean description of the module action $S_t\times N_d\to N_{t+d}$. Let i and j be nonnegative integers with $i+j\le n$ and $i+2j=t$; and let p and q be nonnegative integers with $p+q\le n-1$ and $p+2q=2n-d$. If $v_i\in\bigwedge^i V$ and $w_p\in\bigwedge^p V^*$, then

$$v_ih^{(j)}\cdot w_px_q=\binom{q}{j}v_i(w_p)x_{q-j},$$

where $v_i(w_p)$ is the element in $\bigwedge^{p-i}V^*$ which is given by the $\bigwedge^\bullet V-$action on $\bigwedge^\bullet V^*$.

Identify V with $\overline{F^*}$ and V^* with $\overline{F}$. Consider the map $\theta\colon S_\bullet\ltimes N_\bullet\to\overline{\mathbb{M}}$ which is given by

$$\theta\left(v_ih^{(j)}+w_px_q\right)=\operatorname{proj}\begin{bmatrix}v_i\overline{h}^{(j)}\\(-1)^qw_p\overline{\lambda}^{(n-1-q-p)}\end{bmatrix}$$

for $v_i\in\bigwedge^i V$ and $w_p\in\bigwedge^p V^*$. The natural projection $\operatorname{proj}\colon\overline{\mathbb{F}}\to\overline{\mathbb{M}}$ is recalled in Lemma 4.2. It is clear that θ is an isomorphism of graded $k-$vector spaces. The only interesting calculation in showing that θ is a ring homomorphism is

$$\theta(v_ih^{(j)})\cdot\theta(w_px_q)=(-1)^{q+j}\binom{q}{j}\operatorname{proj}\begin{bmatrix}0\\v_i(w_p)\overline{\lambda}^{(n-1-q-p+i+j)}\end{bmatrix}=\theta(v_ih^{(j)}\cdot w_px_q).\quad\square$$

Caution. We use the symbol "w_px_q" to represent an element of $(S_\bullet/\mathfrak{A}_\bullet)^*$; no multiplication of w_p and x_q is involved. (Indeed, no multiplication of w_p and x_q is even defined.)

5. Poincaré series.

In the present section

(5.1) $(R,\mathfrak{m},k)$ is a regular local ring of embedding dimension e, $n\ge 2$ is an integer, $X^{\text{alt}}_{2n+1\times 2n+1}$ and $Y_{1\times 2n+1}$ are matrices with entries in $\mathfrak{m}$, $I=I_1(YX)$ has grade $2n$, and $A=R/I$. The characteristic of k is denoted by $c\ge 0$, and n_0 represents $(n+2)/2$.

The main result in this section is Theorem 5.2, where we prove that the Poincaré series of every $A-$module is a rational function, provided $0=c$ or $n_0\le c$. Corollary 5.3 is concerned with the growth of the betti numbers of modules over A. The remainder of the section contains the calculations which are used in the proof of Theorem 5.2. Some of the arguments in the present section are similar those in section four of [28].

Theorem 5.2. *Adopt the notation of (5.1). Let* $\operatorname{Den}_A(z)$ *be the polynomial*

$$\operatorname{Den}_A(z)=\begin{cases}(1+z)^{2n+1}[(1-z)^{2n+1}-z^3] & \textit{if } c=0 \textit{ or } n+1\le c\\(1+z)^{2n+1}[(1-z)^{2n+1}(1-z^{2c+1}-z^{2c+2})-z^3] & \textit{if } n_0\le c\le n.\end{cases}$$

If $c=0$ *or* $n_0\le c$, *then*

(a) *the Poincaré series* $P^k_A(z)$ *is given by*

$$P^k_A(z)=\frac{(1+z)^e(1+z^3)}{\operatorname{Den}_A(z)},\ \textit{and}$$

(b) $\operatorname{Den}_A(z)P^M_A(z)$ *is a polynomial in* $\mathbb{Z}[z]$ *for every finitely generated* $A-$*module* M.

Remark. When $n = 2$, the above result is contained in Example 3.8 and Corollary 4.2 of [31].

Proof. We saw in Corollary 3.21 that the minimal $R-$resolution of A is a DG$\Gamma-$algebra; therefore, we may apply the technique of [9] which is summarized in [31, section 4]. In Theorem 4.3, we proved that the graded $k-$algebra $\operatorname{Tor}^R_\bullet(R/I,k)$ is isomorphic to the algebra $T_\bullet = S_\bullet \ltimes N_\bullet$. Avramov's Theorem [2, Corollary 3.3] gives

$$P^k_A(z) = P^k_R(z)P^k_{T_\bullet}(z) = (1+z)^e P^k_{T_\bullet}(z).$$

The DG$\Gamma-$algebra $\mathbb{B}$ of (5.6) is obtained from $T_\bullet$ by adjoining $2n+1$ divided power variables of degree two and one divided power variable of degree three. It follows that

$$P^k_{T_\bullet}(z) = \frac{(1+z^3)}{(1-z^2)^{2n+1}} P^k_{\mathbb{B}}(z).$$

In Lemma 5.13 we prove that $\mathbb{B}$ is a Golod DG$\Gamma-$algebra. It follows from [5, Theorem 2.3] that

$$P^k_{\mathbb{B}}(z) = \frac{1}{1 - z\sum\limits_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i};$$

and therefore,

$$P^k_A(z) = \frac{(1+z)^e(1+z^3)}{\operatorname{Den}_A(z)},$$

where

$$\operatorname{Den}_A(z) = (1-z^2)^{2n+1}\left(1 - z(\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i)\right).$$

The homology of $\mathbb{B}$ is calculated in Lemma 5.13. The proof is completed by appealing to [9, Corollary 1.6] or [31, Theorem 4.1]. □

Corollary 5.3. *Take A as in Theorem 5.2. Let M be a finitely generated $A-$module and let b_i be the i^{th} betti number of M; in other words, $b_i = b_i^A(M) = \dim_k \operatorname{Tor}_i^A(M,k)$. If the projective dimension of M is infinite, then*

- (a) *the betti numbers of M exhibit strong exponential growth; that is, there are real numbers M_0 and M_1, with $1 < M_0 \le M_1$, such that $M_0^i \le b_i \le M_1^i$ for all sufficiently large i, and*
- (b) *the betti numbers $\{b_i\}$ form an increasing sequence for all sufficiently large i.*

Proof. According to Theorem 5.2, the Poincaré series $P^M_A(z)$ is a rational function which does not have a pole at 1; consequently, we may apply the technique of [42]. Let $d(z)$ equal $\operatorname{Den}_A(z)/(1+z)^{2n+1}$. If r is a real root of $d(z) = 0$, with $0 < r < 1$, then it suffices to show that

(5.4) $$r \text{ is a root of multiplicity 1, and}$$

(5.5) $$\text{if } z \text{ is a complex number with } |z| = r, \text{ but } z \ne r, \text{ then } d(z) \ne 0.$$

If $c = 0$ or $n+1 \le c$, then the analysis of $d(z) = (1-z)^{2n+1} - z^3$ is straightforward. It is clear that $d'(r) < 0$ for all r. Write $d(z) = h_1(z) - h_2(z)$, for $h_1(z) = (1-z)^{2n+1}$ and $h_2(z) = z^3$. It is easy to see that

$$\left|\frac{h_2(z)}{h_2(r)}\right| = 1 < \left|\frac{h_1(z)}{h_1(r)}\right| \quad \text{for all } z \text{ with } 0 < |z| = r < 1 \text{ and } z \ne r.$$

Conclusion (5.5) now follows readily.

The analysis of $d(z) = (1-z)^{2n+1}(1 - z^{2c+1} - z^{2c+2}) - z^3$ is slightly more complicated. Let r represent a real number with $0 < r < 1$. Write $d'(r) = e(r) - f(r)$ with

$$\begin{aligned} e(r) &= (1-r)^{2n+1}(-(2c+1)r^{2c} - (2c+2)r^{2c+1}) \quad \text{and} \\ f(r) &= (2n+1)(1-r)^{2n}(1 - r^{2c+1} - r^{2c+2}) + 3r^2. \end{aligned}$$

It is clear that $e(r) < 0$. We prove that $d'(r) < 0$ by showing that $0 < f(r)$. Since $r^{2c+2} < r^{2c+1} < r$, we see that $f_0(r) < f(r)$, where

$$f_0(r) = (2n+1)(1-r)^{2n}(1-2r) + 3r^2.$$

If $0 < r < 1/2$, then $0 < 1 - 2r$ and $0 < f_0(r)$. If $1/2 \le r < 1$, then

$$0 < (2n+1)(1-r)^{2n-1} < 1 \quad \text{and} \quad 0 < (2n+1)(1-r)^{2n-1}[(1-r)(1-2r) + 3r^2] < f_0(r).$$

Thus, (5.4) holds. For (5.5), write $d(z) = u(z)[h_1(z) - h_2(z)]$, where $u(z) = (1-z)^{2n+1}$, $h_1(z) = 1 - z^{2c+1}$, and

$$h_2(z) = z^{2c+2}\left(1 + \frac{1}{z^{2c-1}(1-z)^{2n+1}}\right).$$

It is not difficult to see that

$$\left|\frac{h_2(z)}{h_2(r)}\right| < 1 \le \left|\frac{h_1(z)}{h_1(r)}\right| \quad \text{for all } z \text{ with } 0 < |z| = r < 1 \text{ and } z \ne r.$$

Once again, conclusion (5.5) follows readily. □

Remarks. (a) The statement of the above result, and its proof, imitate the work of Li-Chuan Sun. Without Sun's techniques, only the weaker conclusion

$$M_0^i \le \sum_{j=0}^{i} b_j \le M_1^i$$

can be drawn. This weaker conclusion is established by observing that $P_A^M(z)$ is a rational function which does not have a pole at 1. See [6] and [7], or [31, Corollary 5.2] for more details.

(b) Notice that the Eisenbud Conjecture holds for the ring A; indeed, if the betti numbers of M are bounded, then M has finite projective dimension.

(c) One can use Theorem 5.2 to prove that the Herzog Conjecture holds for A; however, A is in the linkage class of a complete intersection; and therefore [43] already yields the Herzog Conjecture for A.

Data 5.6. Let k be a field of characterisic $c \ge 0$, $n \ge 2$ be an integer, and $T_\bullet$ be the graded $k-$algebra $S_\bullet \ltimes N_\bullet$ of Theorem 4.3. Recall that, as a vector space, $T_\bullet$ is generated by elements of have the form $v_ih^{(j)}$ and w_px_q, where $v_i \in \bigwedge^i V$, $w_p \in \bigwedge^p V^*$, and V is a vector space of dimension $2n+1$. The element $v_ih^{(j)}$ is zero if $i<0$, or $j<0$, or $n+1 \le i+j$. The element w_px_q is zero if $p<0$, or $q<0$, or $n \le p+q$. The multiplication in $T_\bullet$ is given by

$$\begin{aligned} &v_ih^{(j)} \cdot v_{i'}h^{(j')} = \tbinom{j+j'}{j} v_i \wedge v_{i'}h^{(j+j')},\\ &v_ih^{(j)} \cdot w_px_q = \tbinom{q}{j} v_i(w_p)x_{q-j}, \quad \text{and}\\ &w_px_q \cdot w_{p'}x_{q'} = 0. \end{aligned}$$

The grading in $T_\bullet$ is given by

$$\deg v_ih^{(j)} = i+2j \qquad \text{and} \qquad \deg w_px_q = 2n-p-2q.$$

Let $(\mathbb{B}, d)$ be the DG$\Gamma-$algebra

$$\mathbb{B} = T_\bullet{<}X_1, \dots, X_{2n+1}, Y\,;\ d(X_i) = e_i,\ d(Y) = h{>},$$

where $e_1, \dots, e_{2n+1}$ is a basis for V over k, the divided power variables $X_1, \dots, X_{2n+1}$ each have degree two, and the divided power variable Y has degree 3.

The rest of this section is devoted to calculating the homology of the complex $\mathbb{B}$. Our first step, in Proposition 5.8, is to decompose $\mathbb{B}$ into a direct sum of subcomplexes.

Definition 5.7. Adopt Data 5.6. For each integer ℓ, let $(X)^{(\ell)}$ be the k-subspace of $\mathbb{B}$ which is generated by

$$\{X_1^{(a_1)} \cdots X_{2n+1}^{(a_{2n+1})} \mid a_1 + \dots + a_{2n+1} = \ell\}.$$

For integers r and m, let $\mathbb{K}^{(r)}_{<m>}$ be the $k-$subspace

$$\left(\bigoplus_i \bigwedge^i V h^{(r-1)} Y (X)^{(m-i)}\right) \oplus \left(\bigoplus_i \bigwedge^i V h^{(r)} (X)^{(m-i)}\right)$$

of $\mathbb{B}$, and let $\mathbb{L}^{(r)}_{<m>}$ be the subspace

$$\left(\bigoplus_i \bigwedge^i V^* x_{n-r} Y (X)^{(m-r+i)}\right) \oplus \left(\bigoplus_i \bigwedge^i V^* x_{n-r-1} (X)^{(m-r+i)}\right)$$

of $\mathbb{B}$.

Proposition 5.8. *Adopt the notation of Definition 5.7.*

(a) *If m and r are integers and d is the differential of $(\mathbb{B},d)$, then $d\left(\mathbb{K}^{(r)}_{<m>}\right)\subseteq \mathbb{K}^{(r)}_{<m>}$ and $d\left(\mathbb{L}^{(r)}_{<m>}\right)\subseteq \mathbb{L}^{(r)}_{<m>}$. In particular, $\left(\mathbb{K}^{(r)}_{<m>},d\right)$ and $\left(\mathbb{L}^{(r)}_{<m>},d\right)$ are subcomplexes of $\mathbb{B}$.*

(b) *The complex $\mathbb{B}$ is equal to the following direct sum of subcomplexes:*

$$\mathbb{B}=\left[\bigoplus_{r=0}^{n+1}\bigoplus_{m=0}^{\infty}\mathbb{K}^{(r)}_{<m>}\right]\oplus\left[\bigoplus_{r=0}^{n}\bigoplus_{m=0}^{\infty}\mathbb{L}^{(r)}_{<m>}\right].$$

Proof. Recall, from (5.6), that

$$\begin{aligned} v_ih^{(j)}d(X_\ell)&=v_i\wedge e_\ell h^{(j)}, & w_px_qd(X_\ell)&=(-1)^pe_\ell(w_p)x_q,\\ v_ih^{(j)}d(Y)&=(j+1)v_ih^{(j+1)},\text{ and} & w_px_qd(Y)&=qw_px_{q-1},\end{aligned}$$

for $v_i\in\bigwedge^i V$ and $w_p\in\bigwedge^p V^*$. Assertion (a) is now established. Assertion (b) is not difficult. □

Note. The complexes $\mathbb{K}^{(r)}_{<m>}$ and $\mathbb{L}^{(r)}_{<m>}$ have been recorded in the proof of Lemma 5.9. An alternate proof of Proposition 5.8 may be obtained by glancing at this record.

Lemma 5.9. *Retain the notation and hypotheses of Definition 5.7. Assume that, either, $c=0$ or $n_0\le c$. Let $B(\mathbb{B})$ and $Z(\mathbb{B})$ represent the boundaries and cycles of $\mathbb{B}$, respectively.*

(1) *Let $\mathbb{V}'$ be the $k-$subspace*

$$\mathbb{V}'=\left(\sum_{i=0}^{n}\bigwedge^{i}Vh^{(n-i)}+\sum_{p=0}^{n-1}\bigwedge^{p}V^*x_{n-p-1}\right)k<X,Y>$$

of $\mathbb{B}$. If $c=0$ or $n+1\le c$, then $Z_+(\mathbb{B})\subseteq\mathbb{V}'+B(\mathbb{B})$.

(2) *For each integer q, with $0\le q\le n$, let $Z^{(q)}$ be the following $k-$subspace of $(\mathbb{B},d)$:*

$$Z^{(q)}=\ker\begin{pmatrix}\bigwedge^{n-q-1}V^*x_qYk<X> & & \bigwedge^{n-q-2}V^*x_qYk<X>\\ \oplus & \xrightarrow{d} & \oplus\\ \bigwedge^{n-q}V^*x_{q-1}k<X> & & \bigwedge^{n-q-1}V^*x_{q-1}k<X>\end{pmatrix}.$$

If $n_0\le c\le n$, and $\mathbb{V}''$ is the $k-$subspace

$$\sum_{i=0}^{n}\bigwedge^{i}Vh^{(n-i)}k<X,Y>+kh^{(c-1)}Y+kh^{(c)}+\sum_{q=0}^{c-2}\sum_{p=0}^{n-1-q}\bigwedge^{p}V^*x_qk<X,Y>$$
$$+\sum_{p=0}^{n-c}\bigwedge^{p}V^*x_{c-1}Yk<X>+\sum_{q=c}^{n}Z^{(q)}$$

of $\mathbb{B}$, then $Z_+(\mathbb{B})\subseteq\mathbb{V}''+B(\mathbb{B})$.

(3) *Let i and m be integers.*

(*a*) *If* $1 \le m$, *then*

$$\dim_k H_i(\mathbb{K}^{(0)}_{<m>}) = \begin{cases} \sum\limits_{\ell=0}^{n}(-1)^{\ell+n}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}, & \textit{if } i=2m-n, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

(*b*) *If* $1 \le r \le n$, *then*

$$\dim_k H_i(\mathbb{K}^{(r)}_{<0>}) = \begin{cases} 1, & \textit{if } i=2c \textit{ and } r=c, \\ 1, & \textit{if } i=2c+1 \textit{ and } r=c, \\ 0, & \textit{otherwise.}\end{cases}$$

(*c*) *If* $1 \le r \le n$ *and* $1 \le m$, *then*

$$\dim_k H_i(\mathbb{K}^{(r)}_{<m>}) = \begin{cases} \binom{2n+1}{n-r+1}\binom{n+m+r-1}{2n}, & \textit{if } i=2m-n+3r, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

(*d*) *If* $0 \le m$, *then*

$$\dim_k H_i(\mathbb{K}^{(n+1)}_{<m>}) = \begin{cases} \binom{2n+m}{2n}, & \textit{if } i=2m+2n+3, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

(*e*) *If* $0 \le m$, *then*

$$\dim_k H_i(\mathbb{L}^{(0)}_{<m>}) = \begin{cases} \binom{2n+m}{2n}, & \textit{if } i=2m+2, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

(*f*) *If* $1 \le r \le n-1$ *and* $0 \le m$, *then*

$$\dim_k H_i(\mathbb{L}^{(r)}_{<m>}) = \begin{cases} \binom{2n+1}{r}\binom{2n+m}{2n}, & \textit{if } i=2m+r+2, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

(*g*) *If* $1 \le m$, *then*

$$\dim_k H_i(\mathbb{L}^{(n)}_{<m>}) = \begin{cases} \sum\limits_{\ell=0}^{n-1}(-1)^{\ell+n-1}\binom{2n+1}{\ell}\binom{n+m+\ell}{2n}, & \textit{if } i=2m+n+2, \textit{ and} \\ 0, & \textit{otherwise.}\end{cases}$$

Proof. We begin by proving part (3); and to that end, we record the complexes $\mathbb{K}^{(r)}_{<m>}$ and $\mathbb{L}^{(r)}_{<m>}$ for $0 \le m$:

$$\begin{aligned}
&\mathbb{K}^{(0)}_{<m>}: \quad 0 \to \textstyle\bigwedge^0 Vh^{(0)}(X)^{(m)} \to \bigwedge^1 Vh^{(0)}(X)^{(m-1)} \to \cdots \to \bigwedge^n Vh^{(0)}(X)^{(m-n)} \to 0,\\
&\mathbb{K}^{(n+1)}_{<m>}: \quad 0 \to \textstyle\bigwedge^0 Vh^{(n)}Y(X)^{(m)} \to 0,\\
&\mathbb{L}^{(0)}_{<m>}: \quad 0 \to \textstyle\bigwedge^0 V^* x_{n-1}(X)^{(m)} \to 0,\\
&\mathbb{L}^{(n)}_{<m>}: \quad 0 \to \textstyle\bigwedge^{n-1} V^* x_0 Y(X)^{(m-1)} \to \bigwedge^{n-2} V^* x_0 Y(X)^{(m-2)} \to \cdots \to \bigwedge^0 V^* x_0 Y(X)^{(m-n)} \to 0,
\end{aligned}$$

if $1 \le r \le n$, then $\mathbb{K}^{(r)}_{<m>}$ is the mapping cone of

$$\begin{array}{ccccccc} 0\to & \bigwedge^0 Vh^{(r-1)}Y(X)^{(m)} & \to & \bigwedge^1 Vh^{(r-1)}Y(X)^{(m-1)} & \to & \dots \\ & \downarrow & & \downarrow & & \\ 0\to & \bigwedge^0 Vh^{(r)}(X)^{(m)} & \to & \bigwedge^1 Vh^{(r)}(X)^{(m-1)} & \to & \dots \end{array}$$

$$\begin{array}{cccccc} \dots\to & \bigwedge^{n-r} Vh^{(r-1)}Y(X)^{(m-n+r)} & \to & \bigwedge^{n-r+1} Vh^{(r-1)}Y(X)^{(m-n+r-1)} & \to 0 \\ & \downarrow & & \downarrow & \\ \dots\to & \bigwedge^{n-r} Vh^{(r)}(X)^{(m-n+r)} & \to & 0 & \to 0, \end{array}$$

and if $1 \le r \le n-1$, then $\mathbb{L}^{(r)}_{<m>}$ is the mapping cone of

$$\begin{array}{cccccccc} 0\to & 0 & \to & \bigwedge^{r-1} V^*x_{n-r}Y(X)^{(m-1)} & \to\cdot\to & \bigwedge^0 V^*x_{n-r}Y(X)^{(m-r)} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \bigwedge^r V^*x_{n-r-1}(X)^{(m)} & \to & \bigwedge^{r-1} V^*x_{n-r-1}(X)^{(m-1)} & \to\cdot\to & \bigwedge^0 V^*x_{n-r-1}(X)^{(m-r)} & \to 0. \end{array}$$

The horizontal maps are the "partial derivative with respect to X", and the vertical maps are the "partial derivative with respect to Y". The key to the proof is the following observation.

Claim. The nonzero homology of each horizontal strand of each $\mathbb{K}^{(r)}_{<m>}$ occurs at the right hand side of the strand and the nonzero homology of each horizontal strand of each $\mathbb{L}^{(r)}_{<m>}$ occurs at the left hand side of the strand.

Proof of claim. Let $\mathbb{E}$ represent the DG$\Gamma-$algebra

$$\bigwedge_k^{\bullet} V{<}X_1,\dots,X_{2n+1}; dX_i = e_i{>},$$

where $\bigwedge^\bullet V$ is the exterior algebra on the $2n+1$ dimensional vector space $V = \bigoplus_{i=1}^{2n+1} ke_i$, the differential on $\bigwedge^\bullet V$ is identically zero, and each of the divided power variables X_i has degree two. It is well known (see, for example, [21, Theorem 5.2]) that $\mathbb{E}$ is acyclic. It follows that the subcomplex

$$\mathbb{E}^{(\ell)}:\quad 0\to \bigwedge^0 V(X)^{(\ell)} \longrightarrow \bigwedge^1 V(X)^{(\ell-1)} \longrightarrow \dots \longrightarrow \bigwedge^{2n} V(X)^{(\ell-2n)} \longrightarrow \bigwedge^{2n+1} V(X)^{(\ell-2n-1)} \longrightarrow 0$$

of $\mathbb{E}$ is exact for every integer ℓ, except $\ell = 0$. For each integer ℓ, consider the complex

$$\widetilde{\mathbb{E}}^{(\ell)}:\quad 0\to \bigwedge^{2n+1} V^*(X)^{(\ell)} \to \bigwedge^{2n} V^*(X)^{(\ell-1)} \to \dots \to \bigwedge^{0} V^*(X)^{(\ell-2n-1)} \to 0,$$

where the differential

$$\bigwedge^a V^*(X)^{(b)} \to \bigwedge^{a-1} V^*(X)^{(b-1)}$$

is given by

$$w_a X_1^{(b_1)} \cdots X_{2n+1}^{(b_{2n+1})} \mapsto \sum_{i=1}^{2n+1} e_i(w_a) X_1^{(b_1)} \cdots X_i^{(b_i-1)} \cdots X_{2n+1}^{(b_{2n+1})}.$$

It is easy to see that $\mathbb{E}^{(\ell)}$ is isomorphic to $\widetilde{\mathbb{E}}^{(\ell)}$. Each horizontal strand of each $\mathbb{K}^{(r)}_{<m>}$ is isomorphic to a quotient of some $\mathbb{E}^{(\ell)}$. Each horizontal strand of each $\mathbb{L}^{(r)}_{<m>}$ is isomorphic to a subcomplex of some $\widetilde{\mathbb{E}}^{(\ell)}$ with $0 < \ell$. The claim is now established.

As we verify (3), we use the well known combinatorial fact

$$\dim_k (X)^{(\ell)} = \binom{2n+\ell}{2n},$$

which holds for all integers ℓ, with $0 \le 2n+\ell$. Assertions (a) and (c)–(g) are all now clear. Assertion (b) is also obvious because

$$h^{(r-1)}Y \mapsto rh^{(r)}$$

is an isomorphism, unless c divides r. However, in the second case, c must equal r because $n_0 \le c \le r \le n$.

Assertion (1) is obvious now that (3) has been established. The proof of (2) is also straightforward. Sometimes we were quite generous as we placed elements in $\mathbb{V}''$. For example, $1 \cdot x_0$ is a boundary and therefore is not used in the proof of $Z_+(\mathbb{B}) \subseteq \mathbb{V}'' + B(\mathbb{B})$; nonetheless, we placed $1 \cdot x_0$ in $\mathbb{V}''$ because it is used in the proof of Lemma 5.13. On the other hand, at other times we were quite stingy. For example, the vector space $\mathbb{V}'$ of part (1) contains all of

$$\bigwedge^{n-q-1} V^* x_q Y k{<}X{>} + \bigwedge^{n-q} V^* x_{q-1} k{<}X{>} \tag{5.10}$$

for every q; however, if $c \le q$, then we have allowed $\mathbb{V}''$ to contain only those elements of (5.10) which are cycles in $\mathbb{B}$. Once again, the proof of Lemma 5.13 dictated the need for frugality. □

The next calculation is used in our proof that $\mathbb{B}$ is Golod when $n_0 \le c \le n$.

Lemma 5.11. *Retain the notation and hypotheses of Definition 5.7. Let $\mathbb{V}''$ be the $k-$subspace of $\mathbb{B}$ which is described in Lemma 5.9. For integers ℓ and q, let $u = u[\ell, q]$ be the integer $n + 2\ell + 2c + 2 - q$, and let $L[\ell, q]$ and $M[\ell, q]$ be the $k-$subspaces*

$$L[\ell,q] = \ker\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_u \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u-1}\right) \quad \text{and}$$
$$M[\ell,q] = \ker\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+1} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u}\right)$$

of $\mathbb{B}$*. If* $0 \le \ell$ *and* $n_0 \le c \le q \le n$*, then* $L[\ell,q]+M[\ell,q] \subseteq d\mathbb{V}''$*.*

Proof. Consider the subcomplex

$$\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+2} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+1} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u-1} \tag{5.12}$$

of $\mathbb{L}^{(n+c-q)}_{<\ell+c>}$; in other words, complex (5.12) is the same as

$$\begin{matrix} \bigwedge^{n-q+1} V^* x_{q-c} Y(X)^{(\ell+1)} \\ \oplus \\ \bigwedge^{n-q+2} V^* x_{q-c-1}(X)^{(\ell+2)} \end{matrix} \xrightarrow{d} \begin{matrix} \bigwedge^{n-q} V^* x_{q-c} Y(X)^{(\ell)} \\ \oplus \\ \bigwedge^{n-q+1} V^* x_{q-c-1}(X)^{(\ell+1)} \end{matrix} \xrightarrow{d}$$

$$\begin{matrix} \bigwedge^{n-q-1} V^* x_{q-c} Y(X)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-c-1}(X)^{(\ell)} \end{matrix} \xrightarrow{d} \begin{matrix} \bigwedge^{n-q-2} V^* x_{q-c} Y(X)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-c-1}(X)^{(\ell-1)}. \end{matrix}$$

Lemma 5.9 shows that the homology of $\mathbb{L}^{(n+c-q)}_{<\ell+c>}$ is concentrated in degree

$$i = 2(\ell+c)+(n+c-q)+2.$$

Observe that $u < u+1 < i$. We conclude that (5.12) is exact. It follows that

$$M[\ell,q] \subseteq d\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+2} \quad \text{and} \quad L[\ell,q] \subseteq d\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+1}.$$

On the other hand, the hypothesis $n_0 \le c \le q \le n$ ensures that

$$q-c-1 \le q-c \le c-2; \quad \text{thus,}$$

$$\left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+2} + \left(\mathbb{L}^{(n+c-q)}_{<\ell+c>}\right)_{u+1} \subseteq \sum_{q'=0}^{c-2} \sum_{p=0}^{n-1-q'} \bigwedge^{p} V^* x_{q'} k<X,Y> \subseteq \mathbb{V}'',$$

and $L[\ell,q]+M[\ell,q] \subseteq d\mathbb{V}''$. □

Lemma 5.13. *Retain the notation and hypotheses of Definition 5.7. If* $c=0$ *or* $n_0 \le c$*, then* $\mathbb{B}$ *is a Golod algebra, and*

$$\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = \begin{cases} \dfrac{z^2}{(1-z)^{2n+1}} & \text{if } 0=c \text{ or } n+1 \le c \\[2ex] z^{2c}+z^{2c+1}+\dfrac{z^2}{(1-z)^{2n+1}} & \text{if } n_0 \le c \le n. \end{cases}$$

Proof. Proposition 5.8 shows that

$$\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = S_1+S_2+S_3+S_4+S_5+S_6,$$

where

$$S_1=\sum_{i\in\mathbb{Z}}\sum_{m=1}^{\infty}\dim H_i(\mathbb{K}^{(0)}_{<m>})z^i,\qquad S_4=\sum_{i\in\mathbb{Z}}\sum_{m=0}^{\infty}\dim H_i(\mathbb{L}^{(0)}_{<m>})z^i,$$

$$S_2=\sum_{i\in\mathbb{Z}}\sum_{r=1}^{n}\sum_{m=0}^{\infty}\dim H_i(\mathbb{K}^{(r)}_{<m>})z^i,\quad S_5=\sum_{i\in\mathbb{Z}}\sum_{r=1}^{n-1}\sum_{m=0}^{\infty}\dim H_i(\mathbb{L}^{(r)}_{<m>})z^i,\ \text{and}$$

$$S_3=\sum_{i\in\mathbb{Z}}\sum_{m=0}^{\infty}\dim H_i(\mathbb{K}^{(n+1)}_{<m>})z^i,\qquad S_6=\sum_{i\in\mathbb{Z}}\sum_{m=1}^{\infty}\dim H_i(\mathbb{L}^{(n)}_{<m>})z^i.$$

Notice that the complex $\mathbb{K}^{(0)}_{<0>}$ is $0\to\mathbb{B}_0\to 0$; and therefore, it does not contribute to $H_i(\mathbb{B})$ for $1\le i$. Notice also, that the complex $\mathbb{L}^{(n)}_{<m>}$ is equal to zero when $m\le 0$. The homology of each complex $\mathbb{K}^{(r)}_{<m>}$ and $\mathbb{L}^{(r)}_{<m>}$ has been calculated in Lemma 5.9. In particular, $S_2=\varepsilon\left(z^{2c}+z^{2c+1}\right)+S_2'$, where

$$S_2'=\sum_{i\in\mathbb{Z}}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\dim H_i(\mathbb{K}^{(r)}_{<m>})z^i,\quad\text{and}\quad \varepsilon=\begin{cases}0, & \text{if } 0=c \text{ or } n+1\le c,\\ 1, & \text{if } n_0\le c\le n.\end{cases}$$

We apply the identity

$$\sum_{m=a-b}^{\infty}\binom{m+b}{a}z^{2m}=\frac{z^{2(a-b)}}{(1-z^2)^{a+1}},$$

which holds for all integers a and b provided $0\le a$. It follows that

$$S_1=\sum_{m=1}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell+n}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}z^{2m-n}$$

$$S_2'=\sum_{m=1}^{\infty}\sum_{r=1}^{n}\binom{2n+1}{n-r+1}\binom{n+m+r-1}{2n}z^{2m-n+3r}=\frac{\sum_{r=1}^{n}\binom{2n+1}{n-r+1}z^{n+r+2}}{(1-z^2)^{2n+1}}$$

$$S_3=\sum_{m=0}^{\infty}\binom{2n+m}{2n}z^{2m+2n+3}=\frac{z^{2n+3}}{(1-z^2)^{2n+1}}$$

$$S_4=\sum_{m=0}^{\infty}\binom{2n+m}{2n}z^{2m+2}=\frac{z^2}{(1-z^2)^{2n+1}}$$

$$S_5=\sum_{m=0}^{\infty}\sum_{r=1}^{n-1}\binom{2n+1}{r}\binom{2n+m}{2n}z^{2m+r+2}=\frac{\sum_{r=1}^{n-1}\binom{2n+1}{r}z^{r+2}}{(1-z^2)^{2n+1}}$$

$$S_6=\sum_{m=1}^{\infty}\sum_{\ell=0}^{n-1}(-1)^{\ell+n-1}\binom{2n+1}{\ell}\binom{n+m+\ell}{2n}z^{2m+n+2}$$

Observe that

$$S_2'+S_5=\frac{z^2(1+z)^{2n+1}-z^2-\binom{2n+1}{n}z^{n+2}-z^{2n+3}}{(1-z^2)^{2n+1}}.$$

Replace m with $m-n-1$ and ℓ with $2n+1-\ell$ to see that

$$S_6=\sum_{m=n+2}^{\infty}\sum_{\ell=n+2}^{2n+1}(-1)^{\ell+n}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}z^{2m-n}.$$

If $1\le m\le n+1$ and $n+2\le \ell\le 2n+1$, then $0\le 2n+m-\ell\le 2n-1$ and $\binom{2n+m-\ell}{2n}=0$. It follows that

$$S_6=\sum_{m=1}^{\infty}\sum_{\ell=n+2}^{2n+1}(-1)^{\ell+n}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}z^{2m-n}.$$

Thus, S_1+S_6 is equal to

$$\sum_{m=1}^{\infty}(-1)^n\left[\sum_{\ell=0}^{2n+1}(-1)^{\ell}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}+(-1)^n\binom{2n+1}{n+1}\binom{n+m-1}{2n}\right]z^{2m-n}.$$

We know that $0\le 2n+m-\ell$; therefore,

$$\binom{2n+m-\ell}{2n}=\binom{2n+m-\ell}{m-\ell},$$

and Observation 1.2 (h) yields that

$$\sum_{\ell=0}^{2n+1}(-1)^{\ell}\binom{2n+1}{\ell}\binom{2n+m-\ell}{2n}=\binom{m-1}{m},$$

which is zero when $1\le m$. We conclude that

$$S_1+S_6=\sum_{m=1}^{\infty}\binom{2n+1}{n+1}\binom{n+m-1}{2n}z^{2m-n}=\frac{\binom{2n+1}{n+1}z^{n+2}}{(1-z^2)^{2n+1}}$$

and $\sum\limits_{i=1}^{\infty}\dim_k H_i(\mathbb{B})z^i$ has been calculated.

To show that $\mathbb{B}$ is a Golod algebra we exhibit a $k-$subspace $\mathbb{V}$ of $\mathbb{B}_+$ such that

$$Z_+(\mathbb{B})\subseteq \mathbb{V}+B(\mathbb{B})\quad\text{and}\tag{5.14}$$

$$\mathbb{V}^2\subseteq d\mathbb{V};\tag{5.15}$$

and then we apply [10, Lemma 5.7] or [31, Lemma 2.6]. We first assume that $c=0$ or $n+1\le c$. Let $\mathbb{V}$ be the subspace $\mathbb{V}'$ of part (1) of Lemma 5.9. We know that condition (5.14) holds. It is apparent that $\mathbb{V}^2=0$; and therefore, condition (5.15) also holds.

Henceforth, we assume that $n_0\le c\le n$. Let $\mathbb{V}$ be the subspace $\mathbb{V}''$ of part (2) of Lemma 5.9. We know that condition (5.14) holds. The hypothesis $n_0\le c$ ensures that $h^{(c-1)}\cdot h^{(c)}=0$. This hypothesis also ensures that $h^{(c-1)}w_px_q$ is equal to zero, whenever $c \le q \le n-1$. Recall, also, that Y has degree 3; thus $Y^2 = 0$. It follows that

$$\mathbb{V}^2 = h^{(c-1)} Y Z^{(c)} + h^{(c)} \cdot \sum_{q=c}^{n} Z^{(q)}.$$

Fix an integer q, with $c \le q \le n$. We next prove that $h^{(c)} Z^{(q)} \subseteq d\mathbb{V}$. Let $\ell \ge 0$ be an integer, and let $u = u[\ell, q]$ and $L = L[\ell, q]$ be the integer and vector space, respectively, of Lemma 5.11. The element $h^{(c)}$ of $\mathbb{B}$ is a cycle; and therefore, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z^{(q)}_{u-2c} & \longrightarrow & \begin{array}{c}\bigwedge^{n-q-1} V^* x_q Y(X)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-1}(X)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-q-2} V^* x_q Y(X)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-1}(X)^{(\ell-1)}\end{array} \\
 & & \Big\downarrow{\scriptstyle h^{(c)}} & & \Big\downarrow{\scriptstyle h^{(c)}} & & \Big\downarrow{\scriptstyle h^{(c)}} \\
0 & \longrightarrow & L & \longrightarrow & \begin{array}{c}\bigwedge^{n-q-1} V^* x_{q-c} Y(X)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-c-1}(X)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-q-2} V^* x_{q-c} Y(X)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-c-1}(X)^{(\ell-1)}\end{array}
\end{array}$$

commutes and has exact rows, where all of the vertical maps are multiplication by $h^{(c)}$. It follows that $(h^{(c)} Z^{(q)})_u \subseteq L$. Lemma 5.11 guarantees that $L \subseteq d\mathbb{V}$. Since ℓ is an arbitrary non-negative integer, we conclude that $h^{(c)} Z^{(q)} \subseteq d\mathbb{V}$. The proof that $h^{(c-1)} Y Z^{(c)}$ is contained in $d\mathbb{V}$ is very similar. This time, we let $u = u[\ell, c]$ and $M = M[\ell, c]$ for some $\ell \ge 0$. The element $h^{(c-1)} Y$ is a cycle of $\mathbb{B}$; and therefore, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z^{(c)}_{u-2c} & \longrightarrow & \begin{array}{c}\bigwedge^{n-c-1} V^* x_c Y(X)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-c} V^* x_{c-1}(X)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-c-2} V^* x_c Y(X)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-c-1} V^* x_{c-1}(X)^{(\ell-1)}\end{array} \\
 & & \Big\downarrow{\scriptstyle h^{(c-1)}Y} & & \Big\downarrow{\scriptstyle h^{(c-1)}Y} & & \Big\downarrow{\scriptstyle h^{(c-1)}Y} \\
0 & \longrightarrow & M & \longrightarrow & \bigwedge^{n-c} V^* x_0 Y(X)^{(\ell)} & \xrightarrow{d} & \bigwedge^{n-c-1} V^* x_0 Y(X)^{(\ell-1)}
\end{array}$$

also commutes and has exact rows. Thus, $(h^{(c-1)} Y Z^{(c)})_{u+1} \subseteq M$. Once again, Lemma 5.11 ensures that $M \subseteq d\mathbb{V}$ and we let $\ell \ge 0$ vary in order to see that $h^{(c-1)} Y Z^{(c)}$ is contained in $d\mathbb{V}$. Condition (5.15) has been established and the proof is complete. $\square$

Remark. The above proof fails when $2 \le c \le (n+1)/2$, because, in this case, $h^{(c-1)} Y \cdot h^{(c)}$, which is equal to $\binom{2c-1}{c} h^{(2c-1)} Y$, is not a boundary in $\mathbb{B}$; and therefore, it is not in $d\mathbb{V}$ for any choice of $\mathbb{V}$. This observation makes it very likely that $\mathbb{B}$ is not Golod. We do not know what form Theorem 5.2 takes under the present hypothesis on c.

References

1. L. Avramov, *On the Hopf algebra of a local ring*, Appendix by V. Hinič, Izv. Akad. Nauk. SSSR Ser. Mat. **38** (1974), 253–277; English transl. in Math. USSR-Izv. **8** (1974), 259–284.
2. L. Avramov, *Small homomorphisms of local rings*, J. Alg. **50** (1978), 400–453.

3. L. Avramov, *Obstructions to the existence of multiplicative structures on minimal free resolutions*, Amer. J. Math. **103** (1981), 1–31.
4. L. Avramov, *Local algebra and rational homotopy*, Astérisque **113-114** (1984), 15–43.
5. L. Avramov, *Golod homomorphisms*, Algebra, Algebraic Topology and their Interactions, Lecture Notes in Mathematics **1183**, Springer Verlag, Berlin Heidelberg New York, 1986, pp. 59–78.
6. L. Avramov, *Homological asymptotics of modules over local rings*, Commutative Algebra, Mathematical Sciences Research Institute Publications **15**, Springer Verlag, Berlin Heidelberg New York, 1989, pp. 33–62.
7. L. Avramov, *Modules of finite virtual projective dimension*, Invent. math. **96** (1989), 71–101.
8. L. Avramov, *Problems on infinite free resolutions*, Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990), Res. Notes. Math., 2, Jones and Bartlett, Boston, MA, 1992, pp. 3–23.
9. L. Avramov, *Local rings over which all modules have rational Poincaré series*, preprint, (Purdue University, 1992).
10. L. Avramov, A. Kustin, and M. Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Alg. **118** (1988), 162–204.
11. A. Beauville, *Sur les hypersurfaces dont les sections hyperplanes sont à module constant*, Appendix "*Ideals with a regular sequence as syzygy*" by D. Eisenbud and C. Huneke, The Grothendieck Festschrift I, Progress in Mathematics **86**, Birkhäuser, Boston, 1990, pp. 121–133.
12. D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.
13. D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), 35–64.
14. V. Gasharov and I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), 569–580.
15. T. Gulliksen and G. Levin, *Homology of local rings*, Queen's papers in pure and applied mathematics **20**, Queen's University, Kingston, Ontario, 1969.
16. M. Hashimoto, *Determinantal ideals without minimal free resolutions*, Nagoya Math. J. **118** (1990), 203–216.
17. J. Herzog, *Komplexe, Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift, Regensburg, 1973.
18. J. Herzog, *Homological properties of the module of differentials*, Atas da 6^a escola de álgebra Recife, Coleção Atas. Soc. Brasileira de Mat. **14**, Rio de Janeiro, 1981, pp. 33–64.
19. C. Huneke and B. Ulrich, *Divisor class groups and deformations*, Amer. J. Math. **107** (1985), 1265–1303.
20. C. Jacobsson, A. Kustin, and M. Miller, *The Poincaré series of a codimension four Gorenstein ring is rational*, J. Pure Appl. Algebra **38** (1985), 255–275.
21. T. Józefiak, *Tate resolutions for commutative graded algebras over a local ring*, Fund. Math. **74** (1972), 209-231.
22. S. Kim, *Projective resolutions of generic order ideals*, Ph. D. thesis, University of Illinois, Urbana, 1988.
23. E. Kunz, *Almost complete intersections are not Gorenstein rings*, J. Alg. **28** (1974), 111–115.
24. A. Kustin, *The minimal free resolutions of the Huneke-Ulrich deviation two Gorenstein ideals*, J. Alg. **100** (1986), 265–304.
25. A. Kustin, *Gorenstein algebras of codimension four and characteristic two*, Comm. in Alg. **15** (1987), 2417–2429.
26. A. Kustin, *The minimal resolution of a codimension four almost complete intersection is a DG-algebra*, J. Alg. (to appear).
27. A. Kustin, *Pfaffian identities, with applications to free resolutions, DG−algebras, and algebras with straightening law*, Proceedings of the Summer Research Conference on Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island (to appear).
28. A. Kustin, *The deviation two Gorenstein rings of Huneke and Ulrich*, preprint (University of South Carolina, 1992).

29. A. Kustin and M. Miller, *Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four*, Math. Zeit. **173** (1980), 171–184.
30. A. Kustin and M. Miller, *Multiplicative structure on resolutions of algebras defined by Herzog ideals*, J. London Math. Soc. (2) **28** (1983), 247–260.
31. A. Kustin and S. Palmer, *The Poincaré series of every finitely generated module over a codimension four almost complete intersection is a rational function*, J. Pure Appl. Algebra (to appear).
32. E. Lopez, *Licci Gorenstein ideals of deviation two*, Ph. D. thesis, Michigan State University, 1988.
33. S. Palmer, *Algebra structures on resolutions of rings defined by grade four almost complete intersections*, J. Alg. (to appear).
34. M. Ramras, *Sequences of Betti numbers*, J. Alg. **66** (1980), 193–204.
35. J. Roberts and J. Weyman, *A short proof of a theorem of M. Hashimoto*, J. Alg. **134** (1990), 144–156.
36. J.-E. Roos, *On the use of graded Lie algebras in the theory of local rings*, Commutative Algebra: Durham 1981, Lond. Math. Soc. **72**, Cambridge University Press, Cambridge, 1982, pp. 204–230.
37. H. Srinivasan, *Algebra structures on some canonical resolutions*, J. Alg. **122** (1989), 150–187.
38. H. Srinivasan, *Minimal algebra resolutions for cyclic modules defined by Huneke-Ulrich ideals*, J. Alg. **137** (1991), 433–472.
39. H. Srinivasan, *The non-existence of a minimal algebra resolution despite the vanishing of Avramov obstructions*, J. Alg. **146** (1992), 251-266.
40. H. Srinivasan, *A grade five Gorenstein cyclic module with no minimal algebra resolution*, preprint (University of Missouri, 1992).
41. H. Srinivasan, *Algebra structures for graded free resolutions*, Proceedings of the Summer Research Conference on Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island (to appear).
42. L.-C. Sun, *Growth of Betti numbers of modules over local rings of small embedding codimension or small linking number*, preprint (Purdue University, 1993).
43. B. Ulrich, *Vanishing of cotangent functors*, Math. Zeit. **196** (1987), 463–484.

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