THE DEVIATION TWO GORENSTEIN RINGS OF HUNEKE AND ULRICH

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Abstract. Let $(R, \mathfrak{m}, k)$ be a regular local ring, $n \geq 3$ be an integer, X be a $2n \times 2n$ alternating matrix with entries from $\mathfrak{m}$, Y be a $1 \times 2n$ matrix with entries from $\mathfrak{m}$, I be the ideal $I = I_1(YX) + \operatorname{Pf}(X)$, and A be the quotient ring R/I. Assume that the grade of I is at least $2n-1$. (In this case, I is a Gorenstein ideal of grade equal to $2n-1$ and I is minimally generated by $2n+1$ elements.) Assume, also, that either $\operatorname{char} k = 0$, or else, $(n+1)/2 \leq \operatorname{char} k$. We prove that the Poincaré series $P_A^M(z)$, which is equal to $\sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(M,k) z^i$, is a rational function for all finitely generated $A-$modules M. As a consequence, we prove that if the projective dimension of M is infinite, then, eventually, the betti numbers of M form an increasing sequence with strong exponential growth.

Fix a commutative noetherian local ring $(R, \mathfrak{m}, k)$ and an integer n, with $3 \leq n$. Consider matrices $X_{2n\times 2n}$ and $Y_{1\times 2n}$ with entries from $\mathfrak{m}$. Assume that X is an alternating matrix. The ideal $I = I_1(YX) + \operatorname{Pf}(X)$ was first studied by Huneke and Ulrich in [14]. They showed that the grade of I is no more than $2n-1$; furthermore, if the maximum possible grade is attained, then I is a perfect Gorenstein ideal of deviation two (that is, the minimal number of generators of I is 2 more than $\operatorname{grade} I$). Huneke and Ulrich also investigated the linkage history of I. They found that I is in the linkage class of a complete intersection; indeed, in the generic case, I is linked to a hypersurface section of a grade $2n-2$ almost complete intersection ideal $I' = I_1(Y'X')$ (where X' and Y' have shape $2n-1 \times 2n-1$ and $1 \times 2n-1$, respectively, and X' is an alternating matrix); furthermore, I' is linked to a hypersurface section of a grade $2n-3$ Gorenstein ideal $I'' = I_1(Y''X'') + \operatorname{Pf}(X'')$ (where X'' and Y'' have shape $2n-2 \times 2n-2$ and $1 \times 2n-2$, respectively, and X'' is an alternating matrix).

The minimal $R-$resolution $\mathbb{M}$ of $A = R/I$ was found in [17]. Srinivasan [24] proved $\mathbb{M}$ is a DG$\Gamma-$algebra; and therefore, the machinery of Avramov [1, 2, 5] may be used to convert many interesting and difficult questions about A into questions about the algebra $T_\bullet = \operatorname{Tor}_\bullet^R(A,k)$. In particular, if R is a regular local ring, then $P_A^k(z) = P_R^k(z) P_{T_\bullet}^k(z)$, where $P_A^M(z)$ is the Poincaré series

$$P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(M,k) z^i$$

of the $A-$module M. This philosophy has lead to some striking theorems in the case that a noetherian local ring A has small codimension or small linking number. If any one of the following conditions hold:

(a) $\operatorname{codim} A \leq 3$, or
(b) $\operatorname{codim} A = 4$ and A is Gorenstein, or
(c) A is one link from a complete intersection, or
(d) A is two links from a complete intersection and A is Gorenstein, or
(e) A is an almost complete intersection of codimension four in which two is a unit,

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then it is shown in [15, 6, 3, 20] that all of the following conclusions hold:

(1) The Poincaré series $P_A^M(z)$ is a rational function for all finitely generated A-modules M.
(2) The Eisenbud Conjecture [8] holds for the ring A. That is, if M is a finitely generated $A-$module whose betti numbers are bounded, then the minimal resolution of M eventually becomes periodic of period at most two.
(3) If R contains the field of rational numbers, then the Herzog Conjecture [12] holds for the ring A. That is, the cotangent modules $T_i(A/R)$ vanish for all large i if and only if A is a complete intersection.

The study of the rationality of Poincaré series has a long and distinguished history; see [23] or the introduction to [6] for a brief synopsis. Gasharov and Peeva [9] found counterexamples to the Eisenbud Conjecture. Ulrich [26, 2.9 and 1.3] has proved the Herzog Conjecture when A is in the linkage class of a complete intersection; the conjecture remains open for arbitrary rings.

The main result in the present paper is Theorem 4.2, where we prove that conclusion (1) also holds under the hypothesis

(f) $(A,\mathfrak{m},k)$ is a Huneke-Ulrich, deviation two, Gorenstein ring, of codimension $2n-1$, with either $\operatorname{char} k = 0$, or $(n+1)/2 \le \operatorname{char} k$.

A strong form of conclusion (2) for the rings in (f) is established in Corollary 4.3. The rings of (f) are all in the linkage class of a complete intersection; and therefore, conclusion (3) is already known to hold for them by [26]. Recently, (indeed after the first version of the present paper was completed), it was found (see [19]) that conclusion (1) also holds under the presence of hypothesis

(g) $(A,\mathfrak{m},k)$ is a Huneke-Ulrich almost complete intersection of codimension $2n$, with either $\operatorname{char} k = 0$, or $(n+2)/2 \le \operatorname{char} k$.

For all of the rings of (a) – (d) and most of the rings of (e), the proof of conclusion (1) has included the proof (see [6]) that there exists a complete intersection C and a Golod homomorphism $C \twoheadrightarrow A$. This statement is false for the rings of (f). (The obstruction $T_2^2 \not\subseteq T_1T_3$ can be quickly read from the algebra $T_\bullet = \operatorname{Tor}_\bullet^R(R/I,\, k)$.) The key new technique is supplied by [5].

Finally, it is worth noting that the Poincaré series $P_A^k(z)$ of a Huneke-Ulrich deviation two Gorenstein ring A depends on the characteristic of k. In particular, if $A = \mathbb{Z}[X,Y]/(I_1(YX) + \operatorname{Pf}(X))$, where the entries of X and Y are indeterminates, then there is no minimal graded $A-$free resolution of $\mathbb{Z} = \frac{A}{I_1(X)+I_1(Y)}$. This example must be included with the growing list of "determinantal type" modules whose minimal resolution is characteristic dependent; see, for example, [11] and [22].

Section 1 introduces some notation and conventions. Section 2 is concerned with the $R-$resolution of the Huneke-Ulrich, deviation two, Gorenstein ring $A = R/I$. We calculate $\operatorname{Tor}_\bullet^R(A,\, k)$ in section 3. The statement and proof of the main theorem are contained in section 4.

1. Preliminary results.

If $\mathbb{E}$ is a DG$\Gamma-$algebra and e is a homogeneous element of $\mathbb{E}$ of odd degree, then $\mathbb{E}{<}X\,;\, dX = e{>}$ represents the DG$\Gamma-$algebra $\bigoplus_{0\le\ell} \mathbb{E}X^{(\ell)}$. We adopt the convention that $X^{(\ell)} = 0$ in $\mathbb{E}{<}X{>}$, for all $\ell < 0$. More information about divided power algebras may be found in [10].

We sometimes consider binomial coefficients with negative parameters; consequently, we now recall the standard definition and properties of these objects.

Definition 1.1. For integers i and m, the binomial coefficient $\binom{m}{i}$ is defined to be

$$\binom{m}{i} = \begin{cases} \dfrac{m(m-1)\cdots(m-i+1)}{i!} & \text{if } 0 < i, \\ 1 & \text{if } 0 = i, \text{ and} \\ 0 & \text{if } i < 0. \end{cases}$$

Observation 1.2. (*a*) *If* $0 \le m < i$, *then* $\binom{m}{i} = 0$.

(b) *For all integers* i *and* m,

$$\binom{m}{i-1} + \binom{m}{i} = \binom{m+1}{i}.$$

(c) *If* i *and* m *are integers with* $0 \le m$, *then* $\binom{m}{i} = \binom{m}{m-i}$.

(d) *If* i *is a nonnegative integer, then* $\binom{-1}{i} = (-1)^i$.

Lemma 1.3. *Let* A, B, *and* C *be integers. If* $0 \le A$, *then*

(a)
$$\sum_{K\in\mathbb{Z}} (-1)^K \binom{B+K}{C+K}\binom{A}{K} = (-1)^A \binom{B}{A+C}, \text{ and}$$

(b)
$$\sum_{K\in\mathbb{Z}} (-1)^K \binom{B-K}{C-K}\binom{A}{K} = \binom{B-A}{C}.$$

Proof. The proof of (a) proceeds by induction on A. Replace K with $A-K$ in order to deduce (b) from (a). A complete proof appears in [18]. □

2. The minimal algebra resolution.

Data 2.1. Let R be a commutative noetherian ring, $n \ge 3$ be an integer, F be a free $R-$module of rank $2n$, $\varphi \in \bigwedge^2 F$, and $Y \in F^*$. Fix orientation elements $\xi \in \bigwedge^{2n} F^*$ and $\eta \in \bigwedge^{2n} F$ which are compatible in the sense that $\xi(\eta) = (-1)^n$. Let g be the element $Y(\varphi)$ of F, $\mathbf{p}$ be the element $\varphi^{(n)}(\xi)$ of R, I be the ideal $I_1(g) + (\mathbf{p})$ of R and A be the quotient R/I. Assume that grade $I \ge 2n-1$.

We use the divided power structure on $\bigwedge^\bullet F$, the $\bigwedge^\bullet F^*-$module structure on $\bigwedge^\bullet F$, and the $\bigwedge^\bullet F-$module structure on $\bigwedge^\bullet F^*$. In particular, if $\beta_p \in \bigwedge^p F$ and $\alpha_i \in \bigwedge^i F^*$, then

$$\beta_p(\alpha_i) \in \bigwedge^{i-p} F^* \qquad \text{and} \qquad \alpha_i(\beta_p) \in \bigwedge^{p-i} F.$$

More information about multilinear algebra and divided power algebra may be found in [7] or [10]. The ideal I of Data 2.1 is, of course, a coordinate free representation of the ideal I of the introduction. For future convenience, we make this identification explicit.

Note 2.2. Let $e_1, \dots, e_{2n}$ be a basis for F and let $\varepsilon_1, \dots, \varepsilon_{2n}$ be the corresponding dual basis for F^*. It is then natural to choose $\xi = \varepsilon_1 \wedge \dots \wedge \varepsilon_{2n}$ and $\eta = e_1 \wedge \dots \wedge e_{2n}$. Write $Y = \sum_{i=1}^{2n} y_i\varepsilon_i$ and $\varphi = \sum_{1\le i<j\le 2n} x_{ij} e_i \wedge e_j$. Let X be the alternating matrix whose entry in row i and column j is x_{ij} whenever $i < j$. It is now easy to see that $I_1(g)$ is generated by the entries of the product $[y_1, \dots, y_{2n}]X$, and $\mathbf{p}$ is $(-1)^n$ times the pfaffian of X.

The minimal $R-$resolution of A was found in [17]. Srinivasan [24] proved that this resolution is a DG$\Gamma-$algebra. In the present section we reformulate Srinivasan's work and give a new proof that the complex $\mathbb{F}$ of Theorem 2.4 is acyclic.

Definition 2.3. Adopt Data 2.1. Let $\mathbb{A}$ and $\mathbb{B}$ be the DG$\Gamma-$algebras

$$\mathbb{A} = \bigwedge^\bullet F^*\langle h\rangle \qquad \text{and} \qquad \mathbb{B} = \bigwedge^\bullet F\langle\lambda\rangle,$$

where $\bigwedge^\bullet F^*$ and $\bigwedge^\bullet F$ are exterior algebras and h and λ are divided power variables of degree two. The differential d on $\mathbb{A}$ is given by $d|_{F^*} = g$ and $d(h) = Y$. The differential d on $\mathbb{B}$ is given by $d|_F = Y$ and $d(\lambda) = g$.

Theorem 2.4. ([24]) *Adopt the notation of Definition 2.3.*

(a) *There exists a map of complexes $v\colon \mathbb{B}\to\mathbb{A}$, such that the mapping cone $(\mathbb{F},f)$ of v is an $R-$resolution of A. In particular, the differential $f_t\colon \mathbb{F}_t=\mathbb{A}_t\oplus\mathbb{B}_{t-1}\to\mathbb{F}_{t-1}=\mathbb{A}_{t-1}\oplus\mathbb{B}_{t-2}$ is given by*

$$f_t=\begin{bmatrix} d_t & (-1)^{\frac{(t-1)(t-2)}{2}}v_{t-1}\\ 0 & (-1)^{t-1}d_{t-1}\end{bmatrix}.$$

(b) *There exists a right $\mathbb{A}-$module structure on $\mathbb{B}$ such that $\mathbb{F}=\mathbb{A}\ltimes\mathbb{B}[-1]$ is a DG$\Gamma-$algebra. In other words, the multiplication $\mathbb{F}_t\otimes\mathbb{F}_u\to\mathbb{F}_{t+u}$ is given by*

$$\begin{bmatrix} a_t\\ b_{t-1}\end{bmatrix}\cdot\begin{bmatrix} a_u\\ b_{u-1}\end{bmatrix}=\begin{bmatrix} a_ta_u\\ b_{t-1}a_u+(-1)^{tu}b_{u-1}a_t\end{bmatrix},$$

and the divided power structure on $\mathbb{F}$ is given by

$$\begin{bmatrix} a_t\\ b_{t-1}\end{bmatrix}^{(\ell)}=\begin{bmatrix} a_t^{(\ell)}\\ b_{t-1}a_t^{(\ell-1)}\end{bmatrix}$$

for $a_i\in\mathbb{A}_i$, $b_i\in\mathbb{B}_i$, and $\ell\in\mathbb{Z}$.

(c) *The subcomplex*

$$\mathbb{M}=\left(\left[\sum_{i+j\le n-1}\bigwedge^i F^*h^{(j)}\right]\bigoplus\left[\sum_{p+q\le n-1}\bigwedge^p F\lambda^{(q)}\right]\ ;\ f|_{\mathbb{M}}\right)$$

is the minimal resolution of A.

(d) *The resolution $\mathbb{M}$ is a DG$\Gamma-$algebra and there is a projection $\pi\colon\mathbb{F}\to\mathbb{M}$ which is a homomorphism of DG$\Gamma-$algebras.*

Parts (a), (b), and (d) of Theorem 2.4 all guarantee the existence of certain maps. For the sake of completeness we record those maps here; however, the proof that these maps do what they are supposed to do is quite involved and may be found in [24]. (An alternate proof, which uses the same notation as is used here, but applies to a slightly different situation, may be found in [19].)

(2.5) The map $v_t\colon\mathbb{B}_t\to\mathbb{A}_t$ is defined by

$$v_t\left(\beta_p\lambda^{(q)}\right)=(-1)^{p+n+1}\sum_{j\in\mathbb{Z}}(-1)^j\binom{n-p-q+j-1}{q}\left(\varphi^{(n-p-q+j)}\wedge\beta_p\right)(\xi)\,h^{(j)}$$

for $\beta_p\in\bigwedge^p F$ and $p+2q=t$.

(2.6) Fix elements $\beta_p\in\bigwedge^p F$ and $\alpha_i\in\bigwedge^i F^*$. The $\mathbb{A}-$module structure on $\mathbb{B}$,

$$\mathbb{B}_t\otimes\mathbb{A}_u\to\mathbb{B}_{t+u},$$

is given by:

$$\left(\beta_p\lambda^{(q)}\right)\alpha_i=(-1)^{\frac{i(i+1)}{2}}\sum_{\ell\in\mathbb{Z}}(-1)^\ell\binom{q+i-1-\ell}{q}\alpha_i\left(\beta_p\wedge\varphi^{(q+i-\ell)}\right)\lambda^{(\ell)}$$

and

$$\left(\beta_p\lambda^{(q)}\right)h^{(j)}=\sum_{\ell\in\mathbb{Z}}(-1)^{j+\ell}\binom{n+\ell-p-2q-1-j}{j}\binom{q+j-1-\ell}{q}\beta_p\wedge\varphi^{(q+j-\ell)}\lambda^{(\ell)}.$$

(2.7) Fix elements $\alpha_i \in \bigwedge^i F^*$ and $\beta_p \in \bigwedge^p F$ and fix integers j and q with $i+2j=t$ and $p+2q=t-1$. Let $\operatorname{proj}_t\colon \mathbb{F}_t \to \mathbb{M}_t$ be the natural projection; that is,

$$\operatorname{proj}_t \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} = \begin{cases} \begin{bmatrix} \alpha_i h^{(j)} \\ \\ 0 \end{bmatrix} & \text{if } i+j \le n-1 \\ \\ \quad 0 & \text{if } n \le i+j; \end{cases}$$

and

$$\operatorname{proj}_t \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ \\ \beta_p \lambda^{(q)} \end{bmatrix} & \text{if } p+q \le n-1 \\ \\ \quad 0 & \text{if } n \le p+q. \end{cases}$$

The map $\pi_t\colon \mathbb{F}_t \to \mathbb{M}_t$ is defined by

$$\pi_t \begin{bmatrix} \alpha_i h^{(j)} \\ \beta_p \lambda^{(q)} \end{bmatrix} = \operatorname{proj}_t \left(\begin{bmatrix} \alpha_i h^{(j)} \\ \beta_p \lambda^{(q)} \end{bmatrix} + (-1)^{\frac{t(t-1)}{2}+n} f_{t+1} \begin{bmatrix} 0 \\ \alpha_i(\eta) \lambda^{(i+j-n)} \end{bmatrix} \right).$$

Observation 2.9 (together with the long exact sequence of homology associated to a mapping cone) gives a new proof of part (a) of Theorem 2.4. The proof of this result in [24] is "the exactness of $\mathbb{F}$ follows once we identify it as the complex of [17]." Identifying the coordinate free complex $\mathbb{F}$ with the coordinate dependent complex in [17] is thoroughly unpleasant; furthermore, the proof in [17] is quite awkward. The present proof is much more natural. Some of the arguments are simplified if we take the data of 2.1 to be generic; moreover, the ideal I is perfect so there is no loss of generality when we assume that

(2.8) $\qquad R$ is the polynomial ring $\mathbb{Z}[y_1, \dots, y_{2n}, \{x_{ij} \mid 1 \le i < j \le 2n\}]$,

for y_i and x_{ij} as described in Note 2.2.

Observation 2.9. *Adopt the notation and hypotheses of Definition 2.3 and (2.8).*

(*a*) *The sequence*

$$0 \to H_0(\mathbb{B}) \xrightarrow{v_0} H_0(\mathbb{A}) \to R/I \to 0$$

is exact.

(*b*) *The homology of* $\mathbb{B}_+$ *is given by*

$$H_i(\mathbb{B}) \cong \begin{cases} 0, & \text{if } i \text{ is odd, and} \\ \frac{R}{I_1(Y)}, & \text{if } i \ge 2 \text{ is even;} \end{cases}$$

furthermore, $H_{2\ell}(\mathbb{B})$ *is generated by* $\left[(\varphi-\lambda)^{(\ell)}\right]$.

(*c*) *The homology of* $\mathbb{A}_+$ *is given by*

$$H_i(\mathbb{A}) \cong \begin{cases} 0, & \text{if } i \text{ is odd, and} \\ \frac{R}{I_1(Y)}, & \text{if } i \ge 2 \text{ is even;} \end{cases}$$

furthermore, $H_{2\ell}(\mathbb{A})$ *is generated by* $[z_{2\ell}]$, *where*

$$z_{2\ell} = \sum_{J=0}^{\ell} (-1)^J \varphi^{(n-\ell+J)}(\xi) h^{(J)} \in \mathbb{A}_{2\ell}.$$

(*d*) *The map v induces an isomorphism $H_+(\mathbb{B}) \to H_+(\mathbb{A})$; in particular,*

$$v_{2\ell}\left((\varphi-\lambda)^{(\ell)}\right) = (-1)^{n+1} z_{2\ell}.$$

Proof. To prove (a) it suffices to show that $I_1(g)\colon \mathbf{p} \subseteq I_1(Y)$; but this is clear because $I_1(Y)$ is a prime ideal. It is clear that $(\varphi-\lambda)^{(\ell)}$ is a cycle in the DG$\Gamma-$algebra $\mathbb{B}$, because

$$d((\varphi-\lambda)^{(\ell)}) = d(\varphi-\lambda)(\varphi-\lambda)^{(\ell-1)},$$

and $d(\varphi-\lambda) = Y(\varphi) - g = 0$. A straightforward calculation shows that $z_{2\ell}$ is a cycle in $\mathbb{A}$. Observe that

$$d(z_{2\ell}) = \sum_{J=0}^{\ell}(-1)^J\left[g\left(\varphi^{(n-\ell+J)}(\xi)\right) - \varphi^{(n-\ell+J+1)}(\xi)\wedge Y\right]h^{(J)}.$$

Let A be the fixed integer $n-\ell+J$. The module action of $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$ gives

$$g\left(\varphi^{(A)}(\xi)\right) = [Y(\varphi)]\left(\varphi^{(A)}(\xi)\right) = \left(Y(\varphi)\wedge\varphi^{(A)}\right)(\xi) = \left(Y(\varphi^{(A+1)})\right)(\xi).$$

Recall that the measuring identity [7, Proposition A.3]

$$(a(c))(b) = a\wedge c(b) + (-1)^{1+\deg c}c(a\wedge b)$$

holds for all homogeneous elements $c \in \bigwedge^\bullet F$ and $a, b \in \bigwedge^\bullet F^*$, with $\deg a = 1$. It follows that

$$g\left(\varphi^{(A)}(\xi)\right) = \left(Y(\varphi^{(A+1)})\right)(\xi) = Y\wedge(\varphi^{(A+1)}(\xi));$$

and therefore, $d(z_{2\ell}) = 0$.

The proof of (b) follows from the fact that $\mathbb{B}$ is the total complex of the following double complex:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^2 F\lambda^{(2)} & \to & \bigwedge^1 F\lambda^{(2)} & \to & \bigwedge^0 F\lambda^{(2)} & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^3 F\lambda^{(1)} & \to & \bigwedge^2 F\lambda^{(1)} & \to & \bigwedge^1 F\lambda^{(1)} & \to & \bigwedge^0 F\lambda^{(1)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \to & \bigwedge^4 F\lambda^{(0)} & \to & \bigwedge^3 F\lambda^{(0)} & \to & \bigwedge^2 F\lambda^{(0)} & \to & \bigwedge^1 F\lambda^{(0)} & \to & \bigwedge^0 F\lambda^{(0)}.
\end{array}$$

Part (c) follows from Lemma 2.10 because $\mathbb{A}_\ell = (\mathbb{P}^q)_\ell$ for $0 \le \ell \le 2q+1$. To prove (d), use the axioms of divided powers and the definition of v_t in order to see that

$$v_{2\ell}\left((\varphi-\lambda)^{(\ell)}\right) = \sum_{q=0}^{\ell}(-1)^q v_{2\ell}\left(\varphi^{(\ell-q)}\lambda^{(q)}\right)$$

$$= \sum_{q=0}^{\ell}(-1)^{q+n+1}\sum_{j\in\mathbb{Z}}(-1)^j\binom{n-2\ell+q+j-1}{q}\left(\varphi^{(n-2\ell+q+j)}\wedge\varphi^{(\ell-q)}\right)(\xi)h^{(j)}.$$

For all integers a and b the identity

$$\varphi^{(a)}\wedge\varphi^{(b)} = \binom{a+b}{a}\varphi^{(a+b)}$$

holds. It follows that $v_{2\ell}\left((\varphi-\lambda)^{(\ell)}\right)$ is equal to

$$\sum_{j\in\mathbb{Z}}(-1)^{j+n+1}\left[\sum_{0\le q\le\ell}(-1)^q\binom{n-2\ell+q+j-1}{q}\binom{n-\ell+j}{\ell-q}\right]\varphi^{(n-\ell+j)}(\xi)h^{(j)}.$$

Apply Lemma 1.3 in order to see that the sum inside the brackets is equal to 1, whenever $0 \le n-\ell+j$. The conclusion now follows. $\square$

Lemma 2.10. *Adopt the notation and hypotheses of Observation 2.9. For each nonnegative integer q, let $\mathbb{P}^q$ be the subcomplex of $\mathbb{A}$ which is defined by*

$$(\mathbb{P}^q)_\ell = \sum_{j\le q} \bigwedge^{\ell-2j} F^* h^{(j)}.$$

The following statements hold.

(a) *The holomolgy of $\mathbb{P}^q$ is given by*

$$H_i(\mathbb{P}^q) \cong \begin{cases} 0 & \text{if } i \text{ is odd and } i < 2q+1, \\ 0 & \text{if } 2q+1 \le i, \\ R/I_1(g) & \text{if } i = 0, \\ R/I_1(Y) & \text{if } 2 \le i \le 2q \text{ and } i \text{ is even, and} \\ R/I & \text{if } i = 2q+1. \end{cases}$$

(b) *If $1 \le \ell \le q$, then $[z_{2\ell}]$ generates $H_{2\ell}(\mathbb{P}^q)$.*

(c) *The homology $H_{2q+1}(\mathbb{P}^q)$ is generated by $[Yh^{(q)}]$.*

Proof. The proof proceeds by induction on q. When $q = 0$, $\mathbb{P}^q$ is the Koszul complex, $\bigwedge^\bullet F^*$, on the entries $g_1, \dots, g_{2n}$ of the product $[y_1, \dots, y_{2n}]X$. It is known (see [14] or [17]) that $g_1, \dots, g_{2n-1}$ form a regular sequence. The standard facts about Koszul complexes now yield that $H_i(\mathbb{P}^0) = 0$ for $2 \le i$ and that

$$H_1(\mathbb{P}^0) \cong \frac{(g_1, \dots, g_{2n-1}) : g_{2n}}{(g_1, \dots, g_{2n-1})}.$$

The above isomorphism is induced by

$$\begin{bmatrix} r_1 \\ \vdots \\ r_{2n} \end{bmatrix} \mapsto r_{2n};$$

in particular, the homology class $[Y]$ in $H_1(\mathbb{P}^0)$ is sent to y_{2n}. The proof for $q = 0$ is complete because [14] and [17] show that

$$(g_1, \dots, g_{2n-1}) : g_{2n} = (g_1, \dots, g_{2n-1}, y_{2n}) \quad \text{and} \quad (g_1, \dots, g_{2n-1}) : y_{2n} = I.$$

(In each case the inclusion $\subseteq$ is obvious and the ideal on the right side is prime.)

We now assume, by induction, that the result holds for some fixed value of q. Observe that $\mathbb{P}^{q+1}$ is the mapping cone of

$$\begin{array}{ccccccccccccc} \bigwedge^\bullet F^* h^{(q+1)} & : & \cdots & \to & (\bigwedge^1 F^*)h^{(q+1)} & \to & (\bigwedge^0 F^*)h^{(q+1)} & \to & 0 & \to & \cdots & \to & 0 \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathbb{P}^q & : & \cdots & \to & (\mathbb{P}^q)_{2q+2} & \to & (\mathbb{P}^q)_{2q+1} & \to & (\mathbb{P}^q)_{2q} & \to & \cdots & \to & (\mathbb{P}^q)_0. \end{array}$$

The homology of $\mathbb{P}^q$ is known by induction. The complex $\bigwedge^\bullet F^* h^{(q+1)}$ is isomorphic to a shift of $\mathbb{P}^0$; thus, its homology is also known. In particular, $H_{2q+2}(\bigwedge^\bullet F^* h^{(q+1)})$ is isomorphic to $R/I_1(g)$ and is generated by $[h^{(q+1)}]$; and $H_{2q+3}(\bigwedge^\bullet F^* h^{(q+1)})$ is isomorphic to $R/I_1(Y)$ and is generated by $[Yh^{(q+1)}]$. The argument is completed by appealing to the long exact sequence of homology which is associated to a mapping cone. The critical step in this calculation involves the exact sequence

$$0 \to H_{2q+2}(\mathbb{P}^{q+1}) \xrightarrow{\delta} H_{2q+2}(\bigwedge^\bullet F^* h^{(q+1)}) \to H_{2q+1}(\mathbb{P}^q) \to H_{2q+1}(\mathbb{P}^{q+1}) \to 0. \tag{2.11}$$

We know that $H_{2q+1}(\mathbb{P}^q)$ is isomorphic to R/I and is generated by $[Yh^{(q)}]$; furthermore, we also know that $d(h^{(q+1)}) = Yh^{(q)}$ in $\mathbb{A}$. Thus, $H_{2q+1}(\mathbb{P}^{q+1}) = 0$ and $H_{2q+2}(\mathbb{P}^{q+1}) \cong K$, where

$$K = \ker\left(H_{2q+2}(\bigwedge^{\bullet} F^* h^{(q+1)}) \to H_{2q+1}(\mathbb{P}^q)\right).$$

It is clear that $K \cong I/I_1(g)$. Recall that $I = I_1(g) + (\mathbf{p})$. It follows that K is generated by $[\mathbf{p}h^{(q+1)}]$, and that K is isomorphic to $R/I_1(g)\colon \mathbf{p}$. In the proof of Observation 2.9 we saw that $I_1(g)\colon \mathbf{p} = I_1(Y)$, and that z_{2q+2} is a cycle in $\mathbb{P}^{q+1}$. The map δ in (2.11) is induced by the projection

$$\mathbb{P}^{q+1} \to \bigwedge^{\bullet} F^* h^{(q+1)};$$

and therefore, $\delta([z_{2q+2}]) = \pm[\mathbf{p}h^{(q+1)}]$. We conlcude that $H_{2q+2}(\mathbb{P}^{q+1})$ is isomorphic to $R/I_1(Y)$ and is generated by $[z_{2q+2}]$. $\square$

3. The algebra $\mathrm{Tor}_{\bullet}^R(A,k)$.

In the present section

(3.1) $(R,\mathfrak{m},k)$ is a local ring, $n \geq 3$ is an integer, $X^{\mathrm{alt}}_{2n\times 2n}$ and $Y_{1\times 2n}$ are matrices with entries in $\mathfrak{m}$, $I = I_1(YX) + \mathrm{Pf}(X)$ has grade $2n-1$, and $A = R/I$.

In Theorem 3.4 we calculate the graded $k-$algebra $\mathrm{Tor}_{\bullet}^R(A,k)$. (The analogous calculation in [24] is not correct.)

Definition 3.2. Let V be a $k-$vector space of dimension $2n$, h be a divided power variable of degree two, $S_{\bullet}$ be the graded $k-$algebra

$$\frac{\bigwedge^{\bullet} V\langle h\rangle}{\sum_{i=0}^{n} \bigwedge^{i} V h^{(n-i)}},$$

$N_{\bullet}$ be the graded left $S_{\bullet}-$module $S^*_{\bullet}[-(2n-1)]$, where $S^*_{\bullet} = \mathrm{Hom}_k(S_{\bullet},k)$, and $T_{\bullet}$ be the graded $k-$algebra $S_{\bullet} \ltimes N_{\bullet}$.

Notes. (a) The multiplication in $T_{\bullet}$ has been defined so that

$$(s+n)(s'+n') = ss' + sn' + (-1)^{(\deg n)(\deg s')}s'n$$

for homogeneous elements $s, s' \in S_{\bullet}$ and $n, n' \in N_{\bullet}$.

(b) The $S_{\bullet}-$action on $S^*_{\bullet}$ is given by $(s\psi)(s') = \psi(s's)$ for all $s, s' \in S_{\bullet}$ and all $\psi \in S^*_{\bullet}$.

(c) If $w_p \in \bigwedge^p V^*$ and q is an integer, then let $w_p x_q$ represent the $k-$homomorphism from $S_{\bullet}$ to k which sends $v_i h^{(j)}$ to

$$\begin{cases} 0, & \text{if } p \neq i, \\ 0, & \text{if } q \neq j, \text{ and} \\ w_p(v_i), & \text{if } p = i \text{ and } q = j \end{cases}$$

for $v_i \in \bigwedge^i V$. Observe that $w_p x_q$ is a nonzero homomorphism when p and q are nonnegative integers with $p+q \leq n-1$. Observe also, that, $w_p x_q$ has degree $2n-1-p-2q$ as an element of $N_{\bullet}$. It

follows that the graded $S_\bullet-$module $N_\bullet$ is equal to $\sum\limits_{d=1}^{2n-1} N_d$, where $N_d = \sum \bigwedge^p V^* x_q$ and the sum is taken over all pairs of nonnegative integers p and q with $p+q \le n-1$ and $p+2q = 2n-1-d$.
Caution: We use the symbol "$w_p x_q$" to represent an element of $T_\bullet$; no multiplication of w_p and x_q is involved. (Indeed, no multiplication of w_p and x_q is even defined.)

(d) One may combine (b) and (c) to give a clean description of the module action $S_t \times N_d \to N_{t+d}$. Let i and j be nonnegative integers with $i+j \le n-1$ and $i+2j = t$; and let p and q be nonnegative integers with $p+q \le n-1$ and $p+2q = 2n-1-d$. If $v_i \in \bigwedge^i V$ and $w_p \in \bigwedge^p V^*$, then

$$v_i h^{(j)} \cdot w_p x_q = \binom{q}{j} v_i(w_p) x_{q-j}, \tag{3.3}$$

where $v_i(w_p)$ is the element in $\bigwedge^{p-i} V^*$ which is given by the $\bigwedge^\bullet V-$action on $\bigwedge^\bullet V^*$.

Theorem 3.4. *Adopt the hypotheses of (3.1). If $T_\bullet$ is the algebra of Definition 3.2, then the graded $k-$algebras $\operatorname{Tor}_\bullet^R(A,k)$ and $T_\bullet$ are isomorphic.*

Proof. The $k-$algebra $\operatorname{Tor}_\bullet^R(A,k)$ is isomorphic to $\overline{\mathbb{M}}$, where $\mathbb{M}$ is given in Theorem 2.4 and "$-$" is the functor $\_\_ \otimes_R k$. Identify V with $\overline{F^*}$ and V^* with $\overline{F}$. Consider the map $\theta\colon T_\bullet \to \overline{\mathbb{M}}$ which is given by

$$\theta\left(v_i h^{(j)} + w_p x_q\right) = \operatorname{proj}\begin{bmatrix} v_i \overline{h}^{(j)} \\ (-1)^{\frac{p(p-1)}{2}} w_p \overline{\lambda}^{(n-1-q-p)} \end{bmatrix}$$

for $v_i \in \bigwedge^i V$ and $w_p \in \bigwedge^p V^*$. The natural projection $\operatorname{proj}\colon \mathbb{F} \to \mathbb{M}$ is defined in (2.7). It is clear that θ is an isomorphism of graded $k-$vector spaces. It remains to show that θ is a ring homomorphism. The only interesting calculation is

$$\theta(v_i h^{(j)}) \cdot \theta(w_p x_q) = \theta(v_i h^{(j)} \cdot w_p x_q). \tag{3.5}$$

Apply Lemma 3.6 to see that the left side of (3.5) is equal to

$$\begin{bmatrix} v_i \overline{h}^{(j)} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ (-1)^{\frac{p(p-1)}{2}} w_p \overline{\lambda}^{(n-1-q-p)} \end{bmatrix} = \varepsilon \binom{q}{j} \operatorname{proj}\begin{bmatrix} 0 \\ v_i(w_p) \overline{\lambda}^{(n-1-q-p+i+j)} \end{bmatrix}$$

for $\varepsilon = (-1)^{\frac{p(p-1)}{2}} (-1)^{\frac{i(i-1)}{2}} (-1)^{(i+2j)(2n-1-p-2q)}$. Apply (3.3) to see that the right side of (3.5) is equal to

$$\theta(\binom{q}{j} v_i(w_p) x_{q-j}) = \varepsilon' \binom{q}{j} \operatorname{proj}\begin{bmatrix} 0 \\ v_i(w_p) \overline{\lambda}^{(n-1-q-p+i+j)} \end{bmatrix}$$

for $\varepsilon' = (-1)^{\frac{(p-i)(p-i-1)}{2}}$. Observe that $\varepsilon = \varepsilon'$; and therefore, (3.5) is established and the proof is complete. $\square$

Lemma 3.6. *If $\overline{\mathbb{M}}$ is the graded $k-$algebra of Theorem 3.4, then the multiplication*

$$\overline{\mathbb{M}}_t \times \overline{\mathbb{M}}_{t'} \to \overline{\mathbb{M}}_{t+t'}$$

is given by

$$\begin{bmatrix} \overline{\alpha}_i \overline{h}^{(j)} \\ \overline{\beta}_p \overline{\lambda}^{(q)} \end{bmatrix} \begin{bmatrix} \overline{\alpha}_{i'} \overline{h}^{(j')} \\ \overline{\beta}_{p'} \overline{\lambda}^{(q')} \end{bmatrix} = \operatorname{proj}\left[\begin{array}{c} \binom{j+j'}{j} \overline{\alpha}_i \wedge \overline{\alpha}_{i'} \overline{h}^{(j+j')} \\ \hline (-1)^{\frac{i'(i'-1)}{2}} \binom{n-p-q-1}{j'} \overline{\alpha}_{i'}(\overline{\beta}_p) \overline{\lambda}^{(q+i'+j')} \\ +(-1)^{tt'} (-1)^{\frac{i(i-1)}{2}} \binom{n-p'-q'-1}{j} \overline{\alpha}_i(\overline{\beta}_{p'}) \overline{\lambda}^{(q'+i+j)} \end{array}\right],$$

where $\alpha_i \in \bigwedge^i F^*$, $\beta_p \in \bigwedge^p F$, *and the indices satisfy*

$$\begin{array}{llll} i+2j=t, & i'+2j'=t', & p+2q=t-1, & p'+2q'=t'-1, \\ i+j\le n-1, & i'+j'\le n-1, & p+q\le n-1, \text{ and} & p'+q'\le n-1. \end{array}$$

Proof. If m and m' are elements of $\mathbb{M}$, then $m \times_{\mathbb{M}} m' = \pi(m \times_{\mathbb{F}} m')$, where $\times_{\mathbb{M}}$ is multiplication in $\mathbb{M}$, $\times_{\mathbb{F}}$ is multiplication in $\mathbb{F}$, and π is given in (2.7). Write $m \equiv m'$ to mean $\overline{m} = \overline{m'}$. Keep in mind that $Y \equiv 0$, $g \equiv 0$, and

$$\varphi^{(j)} \equiv \begin{cases} 1 & \text{if } j=0, \text{ and} \\ 0 & \text{for any integer } j \text{ with } j \neq 0. \end{cases}$$

The interesting calculation involves

$$m = \begin{bmatrix} 0 \\ \beta_p \lambda^{(q)} \end{bmatrix} \quad \text{and} \quad m' = \begin{bmatrix} \alpha_{i'} h^{(j')} \\ 0 \end{bmatrix}.$$

Use Theorem 2.4 and (2.6) to see that

$$m \times_{\mathbb{F}} m' = \begin{bmatrix} 0 \\ \left((\beta_p\lambda^{(q)})\alpha_{i'}\right) h^{(j')} \end{bmatrix},$$

$$\left(\beta_p\lambda^{(q)}\right)\alpha_{i'} \equiv (-1)^{i'+\frac{i'(i'+1)}{2}} \alpha_{i'}(\beta_p)\lambda^{(q+i')}, \text{ and}$$

$$\left(\alpha_{i'}(\beta_p)\lambda^{(q+i')}\right) h^{(j')} \equiv \binom{n-q-p-1}{j'} \alpha_{i'}(\beta_p)\lambda^{(q+i'+j')}.$$

We complete the proof by showing that $\pi \equiv \text{proj}$. It is clear that $\pi|_{\mathbb{M}}$ is the identity map and that

$$\pi \begin{bmatrix} 0 \\ \beta_p\lambda^{(q)} \end{bmatrix} = 0 \quad \text{for } n \le p+q.$$

Finally, if $n \le i+j$, then

$$\pi_t \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} = \pm \text{proj}_t\, f_{t+1} \begin{bmatrix} 0 \\ \alpha_i(\eta)\lambda^{(i+j-n)} \end{bmatrix} \equiv \pm \text{proj}_t \begin{bmatrix} v_t\left(\alpha_i(\eta)\lambda^{(i+j-n)}\right) \\ 0 \end{bmatrix} \equiv \pm \text{proj}_t \begin{bmatrix} \alpha_i h^{(j)} \\ 0 \end{bmatrix} = 0,$$

where $t = i+2j$. The map v_t may be found in (2.5). □

4. The main Theorem.

In the present section

(4.1) $(R, \mathfrak{m}, k)$ is a regular local ring of embedding dimension e, $n \ge 3$ is an integer, $X^{\text{alt}}_{2n\times 2n}$ and $Y_{1\times 2n}$ are matrices with entries in $\mathfrak{m}$, $I = I_1(YX) + \text{Pf}(X)$ has grade $2n-1$, and $A = R/I$. The characteristic of k is denoted by $c \ge 0$.

Theorem 4.2. *Adopt the notation of (4.1). Let* $\text{Den}_A(z)$ *be the polynomial*

$$\text{Den}_A(z) = \begin{cases} (1+z)^{2n}[(1-z)^{2n} - z^2] & \text{if } c = 0 \text{ or } n \le c \\ (1+z)^{2n}[(1-z)^{2n}(1 - z^{2c+1} - z^{2c+2}) - z^2] & \text{if } (n+1)/2 \le c \le n-1. \end{cases}$$

If $0 = c$ *or* $(n+1)/2 \le c$, *then*

(a) *the Poincaré series* $P^k_A(z)$ *is given by*

$$P^k_A(z) = \frac{(1+z)^e(1+z^3)}{\text{Den}_A(z)}, \text{ and}$$

(b) $\text{Den}_A(z) P^M_A(z)$ *is a polynomial in* $\mathbb{Z}[z]$ *for every finitely generated* $A-$*module* M.

Remarks. (a) In the notation of (4.1), if n is taken to be 2, then the ideal I is generated by the maximal order pfaffians of an alternating 5×5 matrix. The Poincaré series of every module over R/I is known to be rational in this case; see [1, section 9] and [3].

(b) Let R be a regular local ring in which 2 is a unit, and let I be a grade five, seven-generated Gorenstein ideal in R. If I is in the linkage class of a complete intersection, then it is shown in [21], that either I is described in (4.1) with $n = 3$, or I is a double hypersurface section of the the ideal in Remark (a). In either event, the Poincaré series of every module over R/I is rational.

Proof. We saw in Theorem 2.4 that the minimal R−resolution of A is a DGΓ−algebra; therefore, we may apply the technique of [5] which is summarized in [20, section 4]. In Theorem 3.4 we proved that the graded k−algebra $\operatorname{Tor}^R_\bullet(R/I, k)$ is isomorphic to the algebra $T_\bullet$ of Definition 3.2. Avramov's Theorem [1, Corollary 3.3] gives

$$P^k_A(z) = P^k_R(z) P^k_{T_\bullet}(z) = (1+z)^e P^k_{T_\bullet}(z).$$

The DGΓ−algebra $\mathbb{B}$ of (4.6) is obtained from $T_\bullet$ by adjoining $2n + 1$ divided power variables of degree two and one divided power variable of degree three. It follows that

$$P^k_{T_\bullet}(z) = \frac{(1+z^3)}{(1-z^2)^{2n+1}} P^k_{\mathbb{B}}(z).$$

In Lemma 4.21 we prove that $\mathbb{B}$ is a Golod DGΓ−algebra. It follows from [2, Theorem 2.3] that

$$P^k_{\mathbb{B}}(z) = \frac{1}{1 - z \sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i};$$

and therefore,

$$P^k_A(z) = \frac{(1+z)^e (1+z^3)}{\operatorname{Den}_A(z)},$$

where

$$\operatorname{Den}_A(z) = (1-z^2)^{2n+1} \left(1 - z (\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i) \right).$$

The homology of $\mathbb{B}$ is calculated in Lemma 4.21. The proof is completed by appealing to [5, Corollary 1.6] or [20, Theorem 4.1]. □

Corollary 4.3. *Take A as in Theorem 4.2. Let M be a finitely generated A−module and let b_i be the i^{th} betti number of M; in other words, $b_i = b_i^A(M) = \dim_k \operatorname{Tor}_i^A(M, k)$. If the projective dimension of M is infinite, then*

- (a) *the betti numbers of M exhibit strong exponential growth; that is, there are real numbers M_0 and M_1, with $1 < M_0 \le M_1$, such that $M_0^i \le b_i \le M_1^i$ for all sufficiently large i, and*
- (b) *the betti numbers $\{b_i\}$ form an increasing sequence for all sufficiently large i.*

Proof. According to Theorem 4.2, the Poincaré series $P^M_A(z)$ is a rational function which does not have a pole at 1; consequently, we may apply the technique of [25]. Let $d(z) = \operatorname{Den}_A(z)/(1+z)^{2n}$. Fix a real root, r, of $d(z) = 0$, with $0 < r < 1$. It suffices to show that

$$r \text{ is a root of multiplicity 1, and} \tag{4.4}$$

$$\text{if } z \text{ is a complex number with } |z| = r, \text{ but } z \neq r, \text{ then } d(z) \neq 0. \tag{4.5}$$

If $c = 0$ or $n \le c$, then the analysis of $d(z) = (1-z)^{2n} - z^2$ is straightforward. It is clear that $d'(r) < 0$. Write $d(z) = h_1(z) - h_2(z)$, for $h_1(z) = (1-z)^{2n}$ and $h_2(z) = z^2$. It is easy to see that

$$\left|\frac{h_2(z)}{h_2(r)}\right| = 1 < \left|\frac{h_1(z)}{h_1(r)}\right| \quad \text{for all } z \text{ with } 0 < |z| = r < 1 \text{ and } z \ne r.$$

Conclusion (4.5) now follows readily.

The analysis of $d(z) = (1-z)^{2n}(1 - z^{2c+1} - z^{2c+2}) - z^2$ is slightly more complicated. Write $d'(r) = e(r) - f(r)$ with

$$\begin{aligned} e(r) &= (1-r)^{2n}(-(2c+1)r^{2c} - (2c+2)r^{2c+1}) \quad \text{and} \\ f(r) &= 2n(1-r)^{2n-1}(1 - r^{2c+1} - r^{2c+2}) + 2r. \end{aligned}$$

It is clear that $e(r) < 0$. We prove that $d'(r) < 0$ by showing that $0 < f(r)$. Since $r^{2c+2} < r^{2c+1} < r$, we see that $f_0(r) < f(r)$, where

$$f_0(r) = 2n(1-r)^{2n-1}(1-2r) + 2r.$$

If $0 < r < 1/2$, then $0 < 1 - 2r$ and $0 < f_0(r)$. If $1/2 \le r < 1$, then

$$0 < 2n(1-r)^{2n-2} < 1 \quad \text{and} \quad 0 < 2n(1-r)^{2n-2}[(1-r)(1-2r) + 2r] < f_0(r).$$

Thus, (4.4) holds. For (4.5), write $d(z) = u(z)[h_1(z) - h_2(z)]$, where $u(z) = (1-z)^{2n}$,

$$h_1(z) = 1 - z^{2c+1}, \quad \text{and} \quad h_2(z) = z^{2c+2}\left(1 + \frac{1}{z^{2c}(1-z)^{2n}}\right).$$

It is not difficult to see that

$$\left|\frac{h_2(z)}{h_2(r)}\right| < 1 \le \left|\frac{h_1(z)}{h_1(r)}\right| \quad \text{for all } z \text{ with } 0 < |z| = r < 1 \text{ and } z \ne r.$$

Once again, conclusion (4.5) follows readily. $\square$

Remark. The statement of the above result, and its proof, imitate the work of Li-Chuan Sun. Without Sun's techniques, only the weaker conclusion

$$M_0^i \le \sum_{j=0}^{i} b_j \le M_1^i$$

can be drawn. This weaker conclusion is established by observing that $P_A^M(z)$ is a rational function which does not have a pole at 1. See [3] and [4], or [20, Corollary 5.2] for more details.

Data 4.6. Let k be a field of characterisic $c \ge 0$, $n \ge 3$ be an integer, and $T_\bullet$ be the graded $k-$algebra of Definintion 3.2. Recall that, as a vector space, $T_\bullet$ is generated by elements of the form $v_i h^{(j)}$ and $w_p x_q$, where $v_i \in \bigwedge^i V$, $w_p \in \bigwedge^p V^*$, and V is a vector space of dimension $2n$. The element $v_i h^{(j)}$ is zero if $i < 0$, or $j < 0$, or $n \le i + j$; the element $w_p x_q$ is zero if $p < 0$, or $q < 0$, or $n \le p + q$. The multiplication in $T_\bullet$ is given by

$$\begin{aligned} &v_i h^{(j)} \cdot v_{i'} h^{(j')} = \tbinom{j+j'}{j} v_i \wedge v_{i'} h^{(j+j')}, \\ &v_i h^{(j)} \cdot w_p x_q = \tbinom{q}{j} v_i(w_p) x_{q-j}, \quad \text{and} \\ &w_p x_q \cdot w_{p'} x_{q'} = 0. \end{aligned}$$

The grading in $T_\bullet$ is given by

$$\deg v_i h^{(j)} = i + 2j \qquad \text{and} \qquad \deg w_p x_q = 2n - 1 - p - 2q.$$

Let $(\mathbb{B}, d)$ be the DG$\Gamma-$algebra

$$\mathbb{B} = T_\bullet{<}X_1, \dots, X_{2n}, Z, Y\ ;\ d(X_i) = e_i,\ d(Z) = 1x_{n-1},\ d(Y) = h{>},$$

where $e_1, \dots, e_{2n}$ is a basis for V over k, the divided power variables $X_1, \dots, X_{2n}, Z$ each have degree two, and the divided power variable Y has degree 3.

The rest of this section is devoted to calculating the homology of the complex $\mathbb{B}$. Our first step, in Proposition 4.8, is to decompose $\mathbb{B}$ into a direct sum of subcomplexes.

Definition 4.7. Adopt Data 4.6. For each integer ℓ, let $(X,Z)^{(\ell)}$ be the k-subspace of $\mathbb{B}$ which is generated by

$$\{X_1^{(a_1)} \cdots X_{2n}^{(a_{2n})} Z^{(b)} \mid a_1 + \cdots + a_{2n} + b = \ell\}.$$

For integers r and m, let $\mathbb{K}_{<m>}^{(r)}$ be the $k-$subspace

$$\left(\bigoplus_i \bigwedge^i V h^{(r-1)} Y (X,Z)^{(m-i)}\right) \oplus \left(\bigoplus_i \bigwedge^i V h^{(r)} (X,Z)^{(m-i)}\right)$$
$$\oplus \left(\bigoplus_p \bigwedge^p V^* x_{n-r} Y (X,Z)^{(m-1+p)}\right) \oplus \left(\bigoplus_p \bigwedge^p V^* x_{n-r-1} (X,Z)^{(m-1+p)}\right)$$

of $\mathbb{B}$.

Proposition 4.8. *Adopt the notation of Definition 4.7.*

(a) *If m and r are integers, then the restriction of d from $\mathbb{B}$ to $\mathbb{K}_{<m>}^{(r)}$ makes $\mathbb{K}_{<m>}^{(r)}$ a subcomplex of $\mathbb{B}$.*
(b) *The complex $\mathbb{B}$ is equal to the following direct sum of subcomplexes:*

$$\mathbb{B} = \mathbb{K}_{<0>}^{(0)} \oplus \left[\bigoplus_{r=0}^{n} \bigoplus_{m=1-r}^{\infty} \mathbb{K}_{<m>}^{(r)}\right].$$

Proof. Recall, from (4.6), that

$$\begin{aligned} &v_i h^{(j)} d(X_\ell) = v_i \wedge e_\ell h^{(j)}, && w_p x_q d(X_\ell) = (-1)^{p+1} e_\ell(w_p) x_q,\\ &v_i h^{(j)} d(Y) = (j+1) v_i h^{(j+1)}, && w_p x_q d(Y) = q w_p x_{q-1},\\ &v_i h^{(j)} d(Z) = \delta_{i0} \tbinom{n-1}{j} v_i x_{n-1-j}, \text{ and} && w_p x_q d(Z) = 0, \end{aligned} \tag{4.9}$$

for $v_i \in \bigwedge^i V$ and $w_p \in \bigwedge^p V^*$. Assertion (a) is now established. Assertion (b) is not difficult. □

We calculate the homology of the subcomplex $\mathbb{K}_{<m>}^{(r)}$ of $\mathbb{B}$ by concentrating on one "graded strand" at a time. For example, the graded strand

$$0 \to \bigwedge^0 V h^{(r)} (X,Z)^{(m)} \to \bigwedge^1 V h^{(r)} (X,Z)^{(m-1)} \to \cdots \to \bigwedge^{n-r-1} V h^{(r)} (X,Z)^{(m-n+r+1)} \to 0$$

of $\mathbb{K}_{<m>}^{(r)}$ is studied in Lemma 4.10. Every graded strand from $\mathbb{B}$ inherits the grading of $\mathbb{B}$; in particular, the right most non-zero module in the above graded strand sits in position $2m-n+3r+1$. The differential in the above graded strand is the "partial derivative with respect to X", which we denote by $\frac{\partial}{\partial X}$, and which is defined to be the graded $T_\bullet{<}Y, Z{>}-$divided power algebra derivation which sends X_i to e_i, for all i. The partial derivatives $\frac{\partial}{\partial Y}$ and $\frac{\partial}{\partial Z}$ are defined in a similar manner. It is clear that d is equal to the sum $\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}$.

Lemma 4.10. *Adopt Data 4.6.*

(a) *Let m and r be integers with $0 \le r \le n-2$ and $0 \le m$. If $\mathbb{G}$ is the following graded strand of $\mathbb{B}$:*

$$0 \to \bigwedge^0 Vh^{(r)}(X,Z)^{(m)} \to \bigwedge^1 Vh^{(r)}(X,Z)^{(m-1)} \to \dots \to \bigwedge^{n-r-1} Vh^{(r)}(X,Z)^{(m-n+r+1)} \to 0,$$

then

$$\dim_k H_i(\mathbb{G}) = \begin{cases} (-1)^{n-r} + \sum\limits_{\ell=0}^{n-1-r} (-1)^{n-1-r+\ell}\binom{2n}{\ell}\binom{2n+m-\ell}{2n}, & \text{if } i = 2m-n+3r+1 \\ 1, & \text{if } i = 2m+2r, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

(*Note: The modules $\bigwedge^0 Vh^{(r)}(X,Z)^{(m)}$ and $\bigwedge^{n-r-1} Vh^{(r)}(X,Z)^{(m-n+r+1)}$ sit in positions $2m+2r$ and $2m-n+3r+1$, respectively, in $\mathbb{G}$.*)

(b) *Let m and r be integers with $0 \le r \le n-1$ and $1-r \le m$. If $\widetilde{\mathbb{G}}$ is the following graded strand of $\mathbb{B}$:*

$$0 \to \bigwedge^r V^*x_{n-r-1}(X,Z)^{(m+r-1)} \to \bigwedge^{r-1} V^*x_{n-r-1}(X,Z)^{(m+r-2)} \to \dots$$
$$\dots \to \bigwedge^1 V^*x_{n-r-1}(X,Z)^{(m)} \to \bigwedge^0 V^*x_{n-r-1}(X,Z)^{(m-1)} \to 0,$$

then

$$\dim_k H_i(\widetilde{\mathbb{G}}) = \begin{cases} \sum\limits_{\ell=0}^{r} (-1)^{\ell}\binom{2n}{r-\ell}\binom{2n+m+r-1-\ell}{2n}, & \text{if } i = 2m+3r-1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

(*Note: The module $\bigwedge^r V^*x_{n-r-1}(X,Z)^{(m+r-1)}$ sits in position $2m+3r-1$ in $\widetilde{\mathbb{G}}$.*)

Remark 4.11. The hypothesis of (a) ensures that $2m-n+3r+1 < 2m+2r$. Also, the formula given in part (a) holds even when $i \le 0$ or $m-n+r+1 \le 0$, because

$$2m-n+3r+1 \le 0 \implies m-n+r+1 \le 0 \implies A = 1, \quad \text{where } A = \sum_{\ell=0}^{n-1-r} (-1)^{\ell}\binom{2n}{\ell}\binom{m+2n-\ell}{2n}.$$

Indeed, Observation 1.2 and Lemma 1.3 show that

$$A = \sum_{\ell=0}^{2n}(-1)^{\ell}\binom{2n}{\ell}\binom{m+2n-\ell}{2n} = \sum_{\ell=0}^{2n}(-1)^{\ell}\binom{2n}{\ell}\binom{m+2n-\ell}{m-\ell} = \sum_{\ell\in\mathbb{Z}}(-1)^{\ell}\binom{2n}{\ell}\binom{m+2n-\ell}{m-\ell} = 1.$$

Proof of Lemma 4.10. (a) Let $\mathbb{E}$ represent the DG$\Gamma-$algebra

$$\bigwedge_k^{\bullet} V<X_1,\dots,X_{2n}; dX_i = e_i>,$$

where $\bigwedge^{\bullet} V$ is the exterior algebra on the $2n-$dimensional vector space $V = \bigoplus_{i=1}^{2n} ke_i$, the differential on $\bigwedge^{\bullet} V$ is identically zero, and each of the divided power variables X_i has degree two. It is well known (see, for example, [16, Theorem 5.2]) that $\mathbb{E}$ is acyclic. It follows that the subcomplex

$$\mathbb{E}^{(\ell)}: \quad 0 \to \bigwedge^0 V(X)^{(\ell)} \longrightarrow \bigwedge^1 V(X)^{(\ell-1)} \longrightarrow \dots \longrightarrow \bigwedge^{2n-1} V(X)^{(\ell-2n+1)} \longrightarrow \bigwedge^{2n} V(X)^{(\ell-2n)} \longrightarrow 0$$

of $\mathbb{E}$ is exact for every integer ℓ, except $\ell=0$. If s is an integer, with $0\le s\le 2n$, then let $\mathbb{E}^{(\ell)}|_s$ represent the quotient

$$\frac{\mathbb{E}^{(\ell)}}{\sum\limits_{i=s+1}^{2n}\bigwedge^i V(X)^{(\ell-i)}}.$$

In other words, $\mathbb{E}^{(\ell)}|_s$ is the complex

$$\mathbb{E}^{(\ell)}|_s:\quad 0\to\bigwedge^0 V(X)^{(\ell)}\longrightarrow\bigwedge^1 V(X)^{(\ell-1)}\longrightarrow\dots\longrightarrow\bigwedge^{s-1} V(X)^{(\ell-s+1)}\longrightarrow\bigwedge^{s} V(X)^{(\ell-s)}\longrightarrow 0.$$

It is clear that

$$\dim_k H_i(\mathbb{E}^{(\ell)}|_s)=\begin{cases}\sum\limits_{j=0}^{s}(-1)^j\dim\bigwedge^{s-j}V(X)^{(\ell-s+j)}, & \text{if } \ell\ne 0 \text{ and } i=2\ell-s,\\ 1, & \text{if } i=\ell=0,\\ 0, & \text{otherwise.}\end{cases}$$

The complex $\mathbb{G}$ may be decomposed into the direct sum of complexes $\sum\limits_{\ell=0}^{m}\mathbb{G}^{(\ell)}$, where $\mathbb{G}^{(\ell)}$ is the complex

$$0\to\bigwedge^0 Vh^{(r)}(X)^{(\ell)}Z^{(m-\ell)}\to\bigwedge^1 Vh^{(r)}(X)^{(\ell-1)}Z^{(m-\ell)}\to .\to\bigwedge^{n-r-1} Vh^{(r)}(X)^{(\ell-n+r+1)}Z^{(m-\ell)}\to 0.$$

It is clear that $\mathbb{G}^{(\ell)}$ is isomorphic to $\mathbb{E}^{(\ell)}|_{n-r-1}[2\ell-2m-2r]$; and therefore,

$$\dim_k H_i(\mathbb{G}^{(\ell)})=\begin{cases}\sum\limits_{j=0}^{n-r-1}(-1)^j\binom{2n}{n-r-1-j}\binom{2n-1+\ell-n+r+1+j}{2n-1}, & \text{if } \ell\ne 0 \text{ and } i=2m-n+3r+1,\\ 1, & \text{if } \ell=0 \text{ and } i=2m+2r,\\ 0, & \text{otherwise.}\end{cases}$$

The calculation of $H_i(\mathbb{G})$ is complete for all i except $i=2m-n+3r+1$; furthermore,

$$\begin{aligned}\dim_k H_{2m-n+3r+1}(\mathbb{G})&=\sum_{\ell=1}^{m}\sum_{q=0}^{n-1-r}(-1)^{n-1-r+q}\binom{2n}{q}\binom{2n+\ell-q-1}{2n-1}\\ &=\sum_{q=0}^{n-1-r}(-1)^{n-1-r+q}\binom{2n}{q}\left[-\binom{2n-q-1}{2n-1}+\sum_{\ell=0}^{m}\binom{2n+\ell-q-1}{2n-1}\right]\\ &=(-1)^{n-r}+\sum_{q=0}^{n-1-r}(-1)^{n-1-r+q}\binom{2n}{q}\binom{2n+m-q}{2n}.\end{aligned}$$

(b) For each integer ℓ, consider the complex

$$\widetilde{\mathbb{E}}^{(\ell)}:\quad 0\to\bigwedge^{2n}V^*(X)^{(\ell)}\to\bigwedge^{2n-1}V^*(X)^{(\ell-1)}\to\dots\to\bigwedge^{0}V^*(X)^{(\ell-2n)}\to 0,$$

where $\bigwedge^a V^*X^{(b)}$ sits in position $2b+2n-a$, and the differential

$$\bigwedge^a V^*(X)^{(b)}\to\bigwedge^{a-1}V^*(X)^{(b-1)}$$

is given by

$$w_a X_1^{(b_1)} \cdots X_{2n}^{(b_{2n})} \mapsto \sum_{i=1}^{2n} e_i(w_a) X_1^{(b_1)} \cdots X_i^{(b_i-1)} \cdots X_{2n}^{(b_{2n})}.$$

Fix an orientation isomorphism $[\_\_]: \bigwedge^{2n} V \to k$. The module isomorphism $\bigwedge^{2n-a} V \to \bigwedge^a V^*$, given by $v \mapsto [\_\_ \wedge v]$, gives rise to an isomorphism of complexes $\mathbb{E}^{(\ell)} \to \widetilde{\mathbb{E}}^{(\ell)}$. (The signs are correct because $[\_\_ \wedge e \wedge v]$ and $e([\_\_ \wedge v])$ represent the same homomorphism $\bigwedge^{a-1} V \to k$, for all $e \in \bigwedge^1 V$.) It follows that $\widetilde{\mathbb{E}}^{(\ell)}$ is exact for all ℓ, except $\ell = 0$. For each fixed integer t, with $0 \le t \le 2n$, let $\widetilde{\mathbb{E}}^{(\ell)}|_t$ be the subcomplex

$$\widetilde{\mathbb{E}}^{(\ell)}|_t: \quad 0 \to \bigwedge^t V^*(X)^{(\ell+t-2n)} \to \bigwedge^{t-1} V^*(X)^{(\ell+t-2n-1)} \to \cdots \to \bigwedge^0 V^*(X)^{(\ell-2n)} \to 0$$

of $\widetilde{\mathbb{E}}^{(\ell)}$. We see that

$$\dim_k H_i(\widetilde{\mathbb{E}}^{(\ell)}|_t) = \begin{cases} \sum\limits_{q=0}^{t} (-1)^q \dim \bigwedge^{t-q} V^*(X)^{(\ell+t-2n-q)}, & \text{if } i = 2\ell + t - 2n, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

(The above formula is obvious when $0 < \ell$. It continues to hold when $\ell \le 0$.)

The complex $\widetilde{\mathbb{G}}$ may be decomposed into the direct sum of complexes $\sum\limits_{\ell=0}^{m+r-1} \widetilde{\mathbb{G}}^{(\ell)}$, where $\widetilde{\mathbb{G}}^{(\ell)}$ is the complex

$$0 \to \bigwedge^r V^* x_{n-r-1} X^{(\ell)} Z^{(m+r-\ell-1)} \to \cdots \to \bigwedge^0 V^* x_{n-r-1} X^{(\ell-r)} Z^{(m-\ell+r-1)} \to 0.$$

Use (3.3) to see that $\widetilde{\mathbb{G}}^{(\ell)}$ is isomorphic to $\widetilde{\mathbb{E}}^{(2n+\ell-r)}|_r[2\ell - 4r + 2n - 2m + 1]$; and therefore,

$$\dim_k H_i(\widetilde{\mathbb{G}}^{(\ell)}) = \begin{cases} \sum\limits_{q=0}^{r} (-1)^q \binom{2n}{r-q} \binom{2n-1+\ell-q}{2n-1}, & \text{if } i = 2m + 3r - 1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

The rest of the argument is now straightforward. $\square$

Let $B(\mathbb{B})$ and $Z(\mathbb{B})$ represent the boundaries and cycles of $\mathbb{B}$, respectively.

Lemma 4.12. *Retain the notation and hypotheses of Definition 4.7. Assume that, either,* $0 = c$ *or* $(n+1)/2 \le c$.

(1) *Let* $\mathbb{V}'$ *be the* $k-$*subspace*

$$\mathbb{V}' = \left(\sum_{i=0}^{n-1} \bigwedge^i V h^{(n-1-i)} + \sum_{p=0}^{n-1} \bigwedge^p V^* x_{n-p-1} + \bigwedge^1 V^* x_0 \right) k{<}X, Y, Z{>}$$

of $\mathbb{B}_+$. *If* $0 = c$ *or* $n \le c$, *then* $Z_+(\mathbb{B}) \subseteq \mathbb{V}' + B(\mathbb{B})$.

(2) *For each integer* q, *with* $0 \le q \le n$, *let* $K^{(q)}$ *be the following* $k-$*subspace of* $(\mathbb{B}, d)$:

$$K^{(q)} = \ker \begin{pmatrix} \bigwedge^{n-q-1} V^* x_q Y k{<}X,Z{>} & & \bigwedge^{n-q-2} V^* x_q Y k{<}X,Z{>} \\ \oplus & \xrightarrow{d} & \oplus \\ \bigwedge^{n-q} V^* x_{q-1} k{<}X,Z{>} & & \bigwedge^{n-q-1} V^* x_{q-1} k{<}X,Z{>} \end{pmatrix}.$$

If $(n+1)/2 \le c \le n-1$, *and* $\mathbb{V}''$ *is the* $k-$*subspace*

$$\begin{aligned}&\sum_{i=0}^{n-1}\bigwedge^{i} V h^{(n-1-i)} k<X,Y,Z> + h^{(c-1)} Y k<Z> + h^{(c)} k<Z> \\ &+\sum_{q=0}^{c-2}\sum_{p=0}^{n-1-q}\bigwedge^{p} V^* x_q k<X,Y,Z> + \sum_{p=0}^{n-c}\bigwedge^{p} V^* x_{c-1} Y k<X,Z> + \sum_{q=c}^{n} K^{(q)}\end{aligned}$$

of $\mathbb{B}$, *then* $Z_+(\mathbb{B}) \subseteq \mathbb{V}'' + B(\mathbb{B})$.

(3) *Let* i *and* m *be integers.*

(a) *If* $1 \le m$, *then*

$$\dim_k H_i(\mathbb{K}^{(0)}_{<m>}) = \begin{cases} (-1)^n + (-1)^{n+1}\sum\limits_{j=0}^{n-1}(-1)^j\binom{2n}{j}\binom{m+2n-j}{2n}, & \text{if } i = 2m-n+1, \\ \binom{m-1+2n}{2n} - 1, & \text{if } i = 2m-1, \text{ and} \\ 0, & \text{otherwise.}\end{cases}$$

(b) *If* $1 \le r \le n-1$ *and* $1-r \le m$, *then* $\sum\limits_{i\in\mathbb{Z}} \dim_k H_i(\mathbb{K}^{(r)}_{<m>}) z^i$ *is equal to*

$$\binom{2n}{n-r}\binom{m+n+r}{2n} z^{2m+3r-n+1} + \binom{2n}{r}\binom{2n+m+r-1}{2n} z^{2m+3r-1} + \varepsilon z^{2m+2c}(1+z),$$

$$\text{where}\quad \varepsilon = \begin{cases} 1, & \text{if } 0 \le m, \text{ and } (n+1)/2 \le r = c \le n-1, \\ 0, & \text{otherwise.}\end{cases}$$

(c) *If* $2-n \le m$, *then*

$$\dim_k H_i(\mathbb{K}^{(n)}_{<m>}) = \begin{cases} \sum\limits_{j=0}^{n-1}(-1)^j\binom{2n}{n-1-j}\binom{m+3n-2-j}{2n}, & \text{if } i = 2m+3n-1, \\ \binom{2n+m}{2n}, & \text{if } i = 2m+2n+1, \text{ and} \\ 0, & \text{otherwise.}\end{cases}$$

Remark. The formulas of part (3) give the correct dimension for H_i, even when $i \le 0$. Indeed, Remark 4.11 establishes this fact for (a). If $2m+3r-n+1 \le 0$ in (b), then $\binom{m+n+r}{2n} = 0$.

Proof of Lemma 4.12. The proof of (1) and (2) is incorporated in the proof of (3).

Fix an integer m with $1 \le m$. The complex $\mathbb{K}^{(0)}_{<m>}$ is the mapping cone of the following map of complexes:

$$\begin{array}{ccccccc} 0\to & \bigwedge^0 V h^{(0)}(X,Z)^{(m)} & \to & \bigwedge^1 V h^{(0)}(X,Z)^{(m-1)} & \to . \to & \bigwedge^{n-1} V h^{(0)}(X,Z)^{(m-n+1)} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \bigwedge^0 V^* x_{n-1}(X,Z)^{(m-1)} & \to & 0 & \to . \to & 0 & \to 0.\end{array}$$

(The horizontal maps are the derivative with respect to X. The vertical maps are the derivative with respect to Z.) According to Lemma 4.10, the top line of the above diagram has non-zero homology only at the far left and the far right. Furthermore, the homology at position $2m$ has dimension 1 and is generated by $1h^{(0)}Z^{(m)}$. The bottom line has homology of dimension $\binom{2n+m-1}{2n}$ at position $2m-1$. The vertical map in the above diagram sends

$$1h^{(0)}Z^{(m)} \mapsto 1x_{n-1}Z^{(m-1)}.$$

The long exact sequence of homology associated to a mapping cone establishes assertions (1), (2), and (3) for $\mathbb{K}^{(0)}_{<m>}$.

The complex $\mathbb{K}^{(n)}_{<m>}$ is the mapping cone of the following map of complexes:

$$\begin{array}{ccccccc}
0 \to & 0 & \to & 0 & \to . \to & \bigwedge^{0} Vh^{(n-1)}Y(X,Z)^{(m)} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigwedge^{n-1} V^* x_0 Y(X,Z)^{(m+n-2)} & \to & \bigwedge^{n-2} V^* x_0 Y(X,Z)^{(m+n-3)} & \to . \to & \bigwedge^{0} V^* x_0 Y(X,Z)^{(m-1)} & \to 0.
\end{array}$$

(The horizontal maps are the derivative with respect to X. The vertical maps are the derivative with respect to Z.) The top line of the above diagram has homology of dimension $\binom{2n+m}{2n}$ at position $2m+2n+1$. Lemma 4.10 shows that the homology of the bottom line is concentrated at the far left side. We conclude that assertions (1), (2), and (3) hold for the complex $\mathbb{K}^{(n)}_{<m>}$.

Henceforth, we assume that $1 \le r \le n-1$. The complex $\mathbb{K}^{(r)}_{<m>}$ consists of four graded strands. Two of the strands involve elements of the form $v_i h^{(j)}$, where $v_i \in \bigwedge^i V$. We refer to this part of $\mathbb{K}^{(r)}_{<m>}$ as "the right side of $\mathbb{K}^{(r)}_{<m>}$". The other two strands involve elements of the form $w_p x_q$, where $w_p \in \bigwedge^p V^*$. We refer to this part of $\mathbb{K}^{(r)}_{<m>}$ as "the left side of $\mathbb{K}^{(r)}_{<m>}$". In other words, the right side of $\mathbb{K}^{(r)}_{<m>}$ is the mapping cone of the following map of complexes. The horizontal maps are the derivative with respect to X. The vertical maps are the derivative with respect to Y.

$$\begin{array}{cccccc}
0 \to & \bigwedge^0 Vh^{(r-1)}Y(X,Z)^{(m)} & \to & \bigwedge^1 Vh^{(r-1)}Y(X,Z)^{(m-1)} & \to \dots \\
& \downarrow & & \downarrow & \\
0 \to & \bigwedge^0 Vh^{(r)}(X,Z)^{(m)} & \to & \bigwedge^1 Vh^{(r)}(X,Z)^{(m-1)} & \to \dots
\end{array}$$

$$\begin{array}{cccccc}
\dots \to & \bigwedge^{n-r-1} Vh^{(r-1)}Y(X,Z)^{(m-n+r+1)} & \to & \bigwedge^{n-r} Vh^{(r-1)}Y(X,Z)^{(m-n+r)} & \to 0 \\
& \downarrow & & \downarrow & \\
\dots \to & \bigwedge^{n-r-1} Vh^{(r)}(X,Z)^{(m-n+r+1)} & \to & 0 & \to 0.
\end{array}$$

The left side of $\mathbb{K}^{(r)}_{<m>}$ is the mapping cone of the following map of complexes. The horizontal maps are the derivative with respect to X. The vertical maps are the derivative with respect to Y.

$$\begin{array}{cccccc}
0 \to & 0 & \to & \bigwedge^{r-1} V^* x_{n-r} Y(X,Z)^{(m+r-2)} & \to \dots \\
& \downarrow & & \downarrow & \\
0 \to & \bigwedge^{r} V^* x_{n-r-1}(X,Z)^{(m+r-1)} & \to & \bigwedge^{r-1} V^* x_{n-r-1}(X,Z)^{(m+r-2)} & \to \dots
\end{array}$$

$$\begin{array}{cccccc}
\dots \to & \bigwedge^{1} V^* x_{n-r} Y(X,Z)^{(m)} & \to & \bigwedge^{0} V^* x_{n-r} Y(X,Z)^{(m-1)} & \to 0 \\
& \downarrow & & \downarrow & \\
\dots \to & \bigwedge^{1} V^* x_{n-r-1}(X,Z)^{(m)} & \to & \bigwedge^{0} V^* x_{n-r-1}(X,Z)^{(m-1)} & \to 0.
\end{array}$$

Finally, the complex $\mathbb{K}^{(r)}_{<m>}$ is the mapping cone of

$$\begin{array}{ccccccc}
0 & \to & \bigwedge^0 Vh^{(r-1)}Y(X,Z)^{(m)} & \to & \begin{array}{c}\bigwedge^1 Vh^{(r-1)}Y(X,Z)^{(m-1)} \\ \oplus \\ \bigwedge^0 Vh^{(r)}(X,Z)^{(m)}\end{array} & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \begin{array}{c}\bigwedge^0 V^* x_{n-r} Y(X,Z)^{(m-1)} \\ \oplus \\ \bigwedge^1 V^* x_{n-r-1}(X,Z)^{(m)}\end{array} & \to & \bigwedge^0 V^* x_{n-r-1}(X,Z)^{(m-1)} & \to & 0,
\end{array} \tag{4.13}$$

where the top line is the right side of $\mathbb{K}^{(r)}_{<m>}$, the bottom line is the left side of $\mathbb{K}^{(r)}_{<m>}$, and the vertical maps are the derivative with respect to Z.

The homology of each graded strand of $\mathbb{K}^{(r)}_{<m>}$ may be read from Lemma 4.10. (Keep in mind that, if $c \neq 0$, then

$$1 \le r \le n-1 < n+1 \le 2c;$$

and therefore, c divides r if and only if $c = r$.) Use the long exact sequence of homology associated to a mapping cone in order to draw the following conclusions.

(4.14) The homology of the left side of $\mathbb{K}^{(r)}_{<m>}$ is concentrated in position $2m+3r-1$ and has dimension $\binom{2n}{r}\binom{2n+m+r-1}{2n}$.

(4.15) If $c \neq r$, or, if $m < 0$, then homology of the right side of $\mathbb{K}^{(r)}_{<m>}$ is concentrated in position $2m-n+3r+1$ and has dimension $\binom{2n}{n-r}\binom{m+n+r}{2n}$.

(4.16) If $c = r$ and $0 \le m$, then

$$\sum_{i\in\mathbb{Z}} \dim_k H_i(\text{right side of } \mathbb{K}^{(r)}_{<m>})z^i = \binom{2n}{n-r}\binom{m+n+r}{2n} z^{2m-n+3r+1} + z^{2m+2c}(1+z).$$

A further comment about conclusions (4.15) and (4.16) is in order. Notice that $h^{(r-1)}YZ^{(m)}$ and $h^{(r)}Z^{(m)}$ are always cycles in the top strand, and the bottom strand, respectively, of the right side of $\mathbb{K}^{(r)}_{<m>}$. The vertical map on the right side of $\mathbb{K}^{(r)}_{<m>}$ carries

$$h^{(r-1)}YZ^{(m)} \mapsto rh^{(r)}Z^{(m)}. \tag{4.17}$$

In (4.15), the map (4.17) is an injection; but in (4.16), (4.17) is the zero map.

Now that we know the homology of each side of $\mathbb{K}^{(r)}_{<m>}$, we compute the homology of the entire complex $\mathbb{K}^{(r)}_{<m>}$ by using the long exact sequence of homology which is associated to the mapping cone of (4.13). Notice, in the case $(n+1)/2 \le c = r \le n-1$, that the cycles $h^{(c-1)}YZ^{(m)}$ and

$$h^{(c)}Z^{(m)} - \binom{n-1}{c} e_1^* x_{n-c-1} X_1^{(1)} Z^{(m-1)}$$

are both elements of $\mathbb{V}''$, where $e_1^*, \dots, e_{2n}^*$ is the basis for V^* which is dual to the basis $e_1, \dots, e_{2n}$ for V.

The subspaces $\mathbb{V}'$ and $\mathbb{V}''$ contain many elements of $\mathbb{B}$ which are not cycles. We have chosen them to be extra large so that they may be described quickly; however our ultimate use for them occurs in Lemma 4.21, where we prove that $\mathbb{B}$ is Golod. For example, we have included all of

$$\bigwedge^1 V^* x_{n-2}(X,Z)^{(m)} + \bigwedge^0 V^* x_{n-1} Y (X,Z)^{(m-1)} \tag{4.18}$$

in $\mathbb{V}'$ even though a quick examination of $\mathbb{K}^{(1)}_{<m>}$ shows that

$$\left[\bigwedge^1 V^* x_{n-2}(X,Z)^{(m)} + \bigwedge^0 V^* x_{n-1} Y (X,Z)^{(m-1)}\right] \cap Z(\mathbb{B}) = K^{(n-1)}_{2m+2}.$$

If $c = 0$ or $n \le c$, then we are able to put all of line (4.18) into $\mathbb{V}'$ in our proof that $\mathbb{B}$ is Golod; however, the more careful description $K^{(n-1)}_{2m+2} \subseteq \mathbb{V}''$ is needed in our proof that $\mathbb{B}$ is Golod when $(n+1)/2 \le c \le n-1$. $\square$

The next calculation is used in our proof that $\mathbb{B}$ is Golod when $(n+1)/2 \le c \le n-1$.

Lemma 4.19. *Retain the notation and hypotheses of Definition 4.7. Let $\mathbb{V}''$ be the $k-$subspace of $\mathbb{B}$ which is described in Lemma 4.12. For integers ℓ and q, let $u = u[\ell, q]$ be the integer $n+2\ell+2c+1-q$, and let $L[\ell, q]$ and $M[\ell, q]$ be the $k-$subspaces*

$$L[\ell,q] = \ker\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_u \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u-1}\right) \quad \textit{and}$$
$$M[\ell,q] = \ker\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+1} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u}\right)$$

of $\mathbb{B}$, where $\mathbb{L}^{(r)}_{<m>}$ represents the left side of $\mathbb{K}^{(r)}_{<m>}$. If $0 \le \ell$, $(n+1)/2 \le c \le n-1$, and $c \le q \le n$, then

$$L[\ell,q] + M[\ell,q] \subseteq d\mathbb{V}''.$$

Proof. Consider the subcomplex

$$\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+2} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+1} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u} \xrightarrow{d} \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u-1} \tag{4.20}$$

of $\mathbb{B}$; in other words, complex (4.20) is the same as

$$\begin{array}{c} \bigwedge^{n-q+1} V^* x_{q-c} Y(X,Z)^{(\ell+1)} \\ \oplus \\ \bigwedge^{n-q+2} V^* x_{q-c-1}(X,Z)^{(\ell+2)} \end{array} \xrightarrow{d} \begin{array}{c} \bigwedge^{n-q} V^* x_{q-c} Y(X,Z)^{(\ell)} \\ \oplus \\ \bigwedge^{n-q+1} V^* x_{q-c-1}(X,Z)^{(\ell+1)} \end{array} \xrightarrow{d}$$
$$\begin{array}{c} \bigwedge^{n-q-1} V^* x_{q-c} Y(X,Z)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-c-1}(X,Z)^{(\ell)} \end{array} \xrightarrow{d} \begin{array}{c} \bigwedge^{n-q-2} V^* x_{q-c} Y(X,Z)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-c-1}(X,Z)^{(\ell-1)}. \end{array}$$

We saw in the proof of Lemma 4.12 (see, in particular, (4.14)) that the homology of $\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}$ is concentrated in degree

$$i = 2(\ell+q-n+1) + 3(n+c-q) - 1.$$

Observe that $u < u+1 < i$. We conclude that (4.20) is exact. It follows that

$$M[\ell,q] \subseteq d\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+2}\right) \quad \text{and} \quad L[\ell,q] \subseteq d\left(\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+1}\right).$$

On the other hand, the hypothesis $(n+1)/2 \le c \le q \le n$ ensures that

$$q-c-1 < q-c \le c-1;$$

thus, $\left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+2} + \left(\mathbb{L}^{(n+c-q)}_{<\ell+q-n+1>}\right)_{u+1}$ is contained in

$$\sum_{q'=0}^{c-2} \sum_{p=0}^{n-1-q'} \bigwedge^p V^* x_{q'} k<X,Y,Z> + \sum_{p=0}^{n-c} \bigwedge^p V^* x_{c-1} Y k<X,Z> \subseteq \mathbb{V}'',$$

and $L[\ell,q] + M[\ell,q] \subseteq d\mathbb{V}''$. $\square$

Lemma 4.21. *Adopt the data of (4.6). If $c = 0$ or $(n+1)/2 \le c$, then $\mathbb{B}$ is a Golod algebra, and*

$$\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = \begin{cases} \dfrac{z}{(1-z)^{2n+1}(1+z)} - \dfrac{z}{1-z^2} & \textit{if } 0 = c \textit{ or } n \le c \\ \dfrac{z}{(1-z)^{2n+1}(1+z)} - \dfrac{z}{1-z^2} + \dfrac{z^{2c}}{1-z} & \textit{if } (n+1)/2 \le c \le n-1. \end{cases}$$

Proof. Define the integer δ by

$$\delta = \begin{cases} 1, & \text{if } (n+1)/2 \le c \le n-1, \\ 0, & \text{if } c = 0, \text{ or } n \le c. \end{cases}$$

Proposition 4.8 shows that

$$\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = S_1 + S_2 + S_3, \tag{4.22}$$

$$\text{where } \ S_1 = \sum_{i\in\mathbb{Z}} \sum_{m=1}^{\infty} \dim_k H_i(\mathbb{K}^{(0)}_{<m>}) z^i, \quad S_2 = \sum_{i\in\mathbb{Z}} \sum_{r=1}^{n-1} \sum_{m=1-r}^{\infty} \dim_k H_i(\mathbb{K}^{(r)}_{<m>}) z^i, \text{ and}$$

$$S_3 = \sum_{i\in\mathbb{Z}} \sum_{m=2-n}^{\infty} \dim_k H_i(\mathbb{K}^{(n)}_{<m>}) z^i.$$

(Notice that if $i \le 0$, then the coefficient of z^i is zero in each S_j.) The homology of each complex $\mathbb{K}^{(r)}_{<m>}$ has been calculated in Lemma 4.12. The identity

$$\sum_{m=a-b}^{\infty} \binom{m+b}{a} z^{2m} = \frac{z^{2(a-b)}}{(1-z^2)^{a+1}},$$

which holds for all integers a and b provided $0 \le a$, is the key to simplifying the S_i. The calculation

$$S_2 = \frac{1}{(1-z^2)^{2n+1}} \sum_{r=1}^{n-1} \left[\binom{2n}{n-r} z^{n+r+1} + \binom{2n}{r} z^{1+r} \right] + \delta \frac{z^{2c}}{1-z}$$

requires the observation that if $1 - r \le m < n - r$, then $\binom{m+n+r}{2n} = 0$. The calculation

$$S_1 = (-1)^n z^{1-n} + \frac{(-1)^n z^{3-n} - z}{1-z^2} + \frac{1}{(1-z^2)^{2n+1}} \left[z + \sum_{j=0}^{n-1} (-1)^{n+1+j} \binom{2n}{j} z^{2j+1-n} \right]$$

requires the observation that if $1 \le m < j$, then $\binom{m+2n-j}{2n} = 0$. The calculation

$$S_3 = \frac{1}{(1-z^2)^{2n+1}} \left[z^{2n+1} + \sum_{j=0}^{n-1} (-1)^j \binom{2n}{n-1-j} z^{n+2j+3} \right]$$

requires the observations that

$$2-n \le m < 0 \implies \binom{2n+m}{2n} = 0 \qquad \text{and} \qquad 2-n \le m < j+2-n \implies \binom{m+3n-j-2}{2n} = 0.$$

It follows, from (4.22), that $\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = A + B + C$, where

$$A = \frac{1}{(1-z^2)^{2n+1}} \left[z + z^{2n+1} + \sum_{r=1}^{n-1} \binom{2n}{n-r} z^{n+r+1} + \sum_{r=1}^{n-1} \binom{2n}{r} z^{1+r} \right],$$

$$B = \frac{1}{(1-z^2)^{2n+1}} \left[\sum_{j=0}^{n-1} (-1)^{n+1+j} \binom{2n}{j} z^{2j+1-n} + \sum_{j=0}^{n-1} (-1)^j \binom{2n}{n-1-j} z^{n+2j+3} \right],$$

$$\text{and } C = (-1)^n z^{1-n} + \frac{(-1)^n z^{3-n} - z}{1-z^2} + \delta \frac{z^{2c}}{1-z}.$$

Straightforward calculations yield

$$A = \frac{1}{(1-z^2)^{2n+1}} \left[(1+z)^{2n} z - \binom{2n}{n} z^{1+n} \right],$$

$$B = \frac{1}{(1-z^2)^{2n+1}} \left[(-1)^{n+1}(1-z^2)^{2n} z^{1-n} + \binom{2n}{n} z^{1+n} \right], \text{ and}$$

$$C = \frac{(-1)^n z^{1-n} - z}{1-z^2} + \delta \frac{z^{2c}}{1-z}; \text{ thus,}$$

$$\sum_{i=1}^{\infty} \dim_k H_i(\mathbb{B}) z^i = \frac{z}{(1-z)^{2n+1}(1+z)} - \frac{z}{1-z^2} + \delta \frac{z^{2c}}{1-z}.$$

To show that $\mathbb{B}$ is a Golod algebra we exhibit a $k-$subspace $\mathbb{V}$ of $\mathbb{B}_+$ such that

$$Z_+(\mathbb{B}) \subseteq \mathbb{V} + B(\mathbb{B}) \quad \text{and} \tag{4.23}$$

$$\mathbb{V}^2 \subseteq d\mathbb{V}; \tag{4.24}$$

and then we apply [6, Lemma 5.7] or [20, Lemma 2.6].

We first assume that $c = 0$ or $n \le c$. Let $\mathbb{V}$ be the subspace $\mathbb{V}'$ of Lemma 4.12. We know that condition (4.23) holds. It is apparent that

$$\mathbb{V}^2 \subseteq \left(\bigwedge^0 V^* x_0 \right) k{<}X, Y, Z{>}.$$

If

$$E = 1 x_0 X_1^{(a_1)} \cdots X_{2n}^{(a_{2n})} Y^{(b)} Z^{(b')}$$

is an element of $(\bigwedge^0 V^* x_0) k{<}X, Y, Z{>}$, then

$$d\left(e_1^* x_0 X_1^{(a_1+1)} X_2^{(a_2)} \cdots X_{2n}^{(a_{2n})} Y^{(b)} Z^{(b')} \right) = E; \tag{4.25}$$

and condition (4.24) also holds. (The element e_1^* of V^* is defined between (4.17) and (4.18).)

Now we assume that $(n+1)/2 \le c \le n-1$. Let $\mathbb{V}$ be the vector space $\mathbb{V}''$ from part (2) of Lemma 4.12. Lemma 4.12 shows that condition (4.23) holds. The hypothesis $(n+1)/2 \le c$ ensures that $h^{(c-1)} \cdot h^{(c)} = 0$. This hypothesis also ensures that $h^{(c-1)} w_p x_q$ is equal to zero, whenever

$c \le q \le n-1$. Recall, also, that Y has degree 3; thus $Y^2 = 0$. Furthermore, $dZ \cdot K^{(q)} = 0$; and therefore, $Z^{(m)} \cdot K^{(q)} \subseteq K^{(q)}$. It now follows that

$$\mathbb{V}^2 \subseteq \bigwedge^0 V^* x_0 k<X,Y,Z> + h^{(c-1)} Y K^{(c)} + h^{(c)} \cdot \sum_{q=c}^{n} K^{(q)}.$$

The argument of (4.25) shows that $\bigwedge^0 V^* x_0 k<X,Y,Z> \subseteq d\mathbb{V}$. Fix an integer q, with $c \le q \le n$. We next prove that $h^{(c)}K^{(q)} \subseteq d\mathbb{V}$. Let $\ell \ge 0$ be an integer, and let $u = u[\ell, q]$ and $L = L[\ell, q]$ be the integer and vector space, respectively, of Lemma 4.19. The element $h^{(c)}$ of $\mathbb{B}$ is a cycle; and therefore, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^{(q)}_{u-2c} & \longrightarrow & \begin{array}{c}\bigwedge^{n-q-1} V^* x_q Y(X,Z)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-1}(X,Z)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-q-2} V^* x_q Y(X,Z)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-1}(X,Z)^{(\ell-1)}\end{array} \\
 & & \downarrow{\scriptstyle h^{(c)}} & & \downarrow{\scriptstyle h^{(c)}} & & \downarrow{\scriptstyle h^{(c)}} \\
0 & \longrightarrow & L & \longrightarrow & \begin{array}{c}\bigwedge^{n-q-1} V^* x_{q-c} Y(X,Z)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-q} V^* x_{q-c-1}(X,Z)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-q-2} V^* x_{q-c} Y(X,Z)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-q-1} V^* x_{q-c-1}(X,Z)^{(\ell-1)}\end{array}
\end{array}$$

commutes and has exact rows, where all of the vertical maps are multiplication by $h^{(c)}$. It follows that $(h^{(c)}K^{(q)})_u \subseteq L$. Lemma 4.19 guarantees that $L \subseteq d\mathbb{V}$. Since ℓ is an arbitrary non-negative integer, we conclude that $h^{(c)}K^{(q)} \subseteq d\mathbb{V}$. The proof that $h^{(c-1)}YK^{(c)}$ is contained in $d\mathbb{V}$ is very similar. This time, we let $u = u[\ell, c]$ and $M = M[\ell, c]$ for some $\ell \ge 0$. The element $h^{(c-1)}Y$ is a cycle of $\mathbb{B}$; and therefore, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^{(c)}_{u-2c} & \longrightarrow & \begin{array}{c}\bigwedge^{n-c-1} V^* x_c Y(X,Z)^{(\ell-1)} \\ \oplus \\ \bigwedge^{n-c} V^* x_{c-1}(X,Z)^{(\ell)}\end{array} & \xrightarrow{d} & \begin{array}{c}\bigwedge^{n-c-2} V^* x_c Y(X,Z)^{(\ell-2)} \\ \oplus \\ \bigwedge^{n-c-1} V^* x_{c-1}(X,Z)^{(\ell-1)}\end{array} \\
 & & \downarrow{\scriptstyle h^{(c-1)}Y} & & \downarrow{\scriptstyle h^{(c-1)}Y} & & \downarrow{\scriptstyle h^{(c-1)}Y} \\
0 & \longrightarrow & M & \longrightarrow & \bigwedge^{n-c} V^* x_0 Y(X,Z)^{(\ell)} & \xrightarrow{d} & \bigwedge^{n-c-1} V^* x_0 Y(X,Z)^{(\ell-1)}
\end{array}$$

also commutes and has exact rows. Thus, $(h^{(c-1)}YK^{(c)})_{u+1} \subseteq M$. Once again, Lemma 4.19 ensures that $M \subseteq d\mathbb{V}$ and we let $\ell \ge 0$ vary in order to see that $h^{(c-1)}K^{(c)}$ is contained in $d\mathbb{V}$. Condition (4.24) has been established and the proof is complete. □

Remark. The above proof fails when $2 \le c \le n/2$, because, in this case, $h^{(c-1)}Y \cdot h^{(c)}$, which is equal to $\binom{2c-1}{c} h^{(2c-1)}Y$, is not a boundary in $\mathbb{B}$; and therefore, it is not in $d\mathbb{V}$ for any choice of $\mathbb{V}$. This observation makes it very likely that $\mathbb{B}$ is not Golod. We do not know what form Theorem 4.2 takes under the present hypothesis on c.

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