THE MINIMAL RESOLUTION OF A CODIMENSION FOUR ALMOST COMPLETE INTERSECTION IS A DG–ALGEBRA

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Abstract. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring in which two is a unit. We prove that if J is a five generated grade four perfect ideal in R, then the minimal resolution of R/J by free $R-$modules is an associative, differential, graded$-$commutative algebra. This result extends and completes the work in [16] and [17], where the conclusion is shown to hold provided certain technical conditions on Tor are satisfied. The multiplication on the resolution of R/J is constructed using appropriate higher order multiplication on the resolution of R/I, where I is a Gorenstein ideal which is linked to J.

For the time being, let A be a quotient of a regular local ring $(R, \mathfrak{m}, k)$, and let $\mathbb{F}$ be the minimal resolution of A by free $R-$modules. If $\mathbb{F}$ has the structure of a differential graded algebra (DG$-$algebra), then many interesting and difficult questions about A can be translated into questions about the algebra $T_\bullet = \operatorname{Tor}_\bullet^R(A, k)$. The algebra $T_\bullet$, although graded-commutative instead of commutative, is in many ways simpler than the original ring A. This philosophy has lead to some striking theorems in the case that A has small codimension or small linking number. If any one of the following conditions hold:

(a) $\operatorname{codim} A \le 3$, or
(b) $\operatorname{codim} A = 4$ and A is Gorenstein, or
(c) A is one link from a complete intersection, or
(d) A is two links from a complete intersection and A is Gorenstein,

then it is shown in [2] and [3] that all of the following conclusions hold:

(1) The Poincaré series $P_A^M(t) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(k, M)\, t^i$ is a rational function for all finitely generated $A-$modules M.
(2) If R contains the field of rational numbers, then the Herzog Conjecture [7] holds for the ring A. That is, the cotangent cohomology $T_i(A/R)$ vanishes for all large i if and only if A is a complete intersection.
(3) The Eisenbud Conjecture [5] holds for the ring A. That is, if M is a finitely generated $A-$module whose Betti numbers are bounded, then the minimal resolution of M eventually becomes periodic of period at most two.

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In each case, (a) – (d), there are three steps to the process:

(i) one proves that the resolution $\mathbb{F}$ is a DG−algebra;
(ii) one classifies the Tor−algebras $\mathrm{Tor}_\bullet^R(A,k)$; and
(iii) one completes the proof of (1) – (3).

Eventually, we hope to extend the list (a) – (d) to include the hypothesis

(e) A is an almost complete intersection of codimension four in which two is a unit.

Indeed, step (ii) is carried out in [9]; step (i) was begun in [16] and [17], and is completed in the present paper; and we anticipate that step (iii) will be contained in a future paper.

Roughly speaking, there are two ways to put a DG−structure on $\mathbb{F}$. One approach is to record an explicit multiplication table for $\mathbb{F}$ and show that it satisfies all of the relevant axioms. This approach works if A is:

- a complete intersection, (in this case, the resolution $\mathbb{F}$ is an exterior algebra);
- one link from a complete intersection [3];
- two links from a complete intersection and is Gorenstein [12];
- a codimension four Gorenstein ring defined by the $(n-1)\times(n-1)$ minors of an $n\times n$ matrix [6];
- a determinantal ring defined by the maximal minors of a matrix [18]; or
- a Gorenstein ring defined by a Huneke−Ulrich deviation two ideal [19].

The other approach is to observe that $\mathbb{F}$ always has a multiplication which satisfies all of the DG axioms except it is associative only up to homotopy. If $\mathbb{F}$ is sufficiently short, then this multiplication might be modified in order to become associative "on the nose." This is the approach of:

- [4] for $\operatorname{codim} A \le 3$;
- [10] for $\operatorname{codim} A = 4$, $\operatorname{char} k \neq 2$, and A Gorenstein;
- [8] for $\operatorname{codim} A = 4$, $\operatorname{char} k = 2$, and A Gorenstein; and
- [16, 17] for a codimension four almost complete intersection A in which two is a unit, provided hypothesis (W) holds for the defining ideal of A, see (3.3).

The main theorem in this paper is Theorem 3.13, which states that if J is a grade four almost complete intersection ideal in a local ring R and two is a unit in R, then the minimal resolution of R/J is a DG−algebra. The outline of our proof is quite simple: a DG−resolution $\mathbb{M}$ of R/J has been introduced in [16] and [17], we find a DG−ideal $\mathbb{I}$ of $\mathbb{M}$ for which $\mathbb{M}/\mathbb{I}$ is the minimal resolution of R/J; however, a significant amount of effort is involved in finding the ideal $\mathbb{I}$. The resolution $\mathbb{M}$ is built using higher order multiplication on the minimal resolution $\mathbb{L}$ of R/I, where I is a Gorenstein ideal which is linked to J. In order to find the DG−ideal $\mathbb{I}$ we must modify the multiplicative structure of $\mathbb{L}$. We do this by building a non-minimal resolution $\mathbb{F}$ of R/I which exhibits the proper multiplicative structure. Our final step is to carry the multiplicative structure of $\mathbb{F}$ to $\mathbb{L}$.

In section 1 we explain what is meant by higher order multiplication. Section 2 is a review of the DG−resolution $\mathbb{M}$. Section 3 consists of two parts. First, we use the classification of $\mathrm{Tor}_\bullet^R(R/J,k)$ from [9] to catalogue those almost complete intersection ideals J which are not covered in [17]. Then, for the ideals J which are not covered in [17], we identify our candidate for the DG−ideal $\mathbb{I}$. The candidate for $\mathbb{I}$ is introduced in Lemma 3.10, which also shows how we would like the multiplication on $\mathbb{L}$ to behave. Theorem 3.11 states that there is a multiplication on $\mathbb{L}$ which exhibits the correct behavior. Theorem 3.11 is probably an interesting result in its own right. It shows that there is a multiplication on the minimal resolution of R/I which satisfies some of the same equations as the multiplication in $\operatorname{Tor}^R_\bullet(R/I,k)$, where I is a grade four Gorenstein ideal. We suspect that a much stronger result along these lines is true and we hope that this potential stronger result, if it exists, has as nice an application as the use of Theorem 3.11 in the proof of Theorem 3.13.

All of sections 4, 5, and 6 are devoted to the proof of Theorem 3.11. The aforementioned resolution $\mathbb{F}$ of R/I is built using the "big from small construction" of [11]. This construction and the related notion of tight double linkage are reviewed in section 4. In section 5 we pass higher order multiplications across the big from small construction in order to endow $\mathbb{F}$ with the multiplication that we wish $\mathbb{L}$ to have. The easiest part of the argument occurs in section 6, where we carry the multiplicative structure on $\mathbb{F}$ down to the minimal resolution $\mathbb{L}$.

In this paper "ring" means commutative noetherian ring with one. If M is a module over a ring R, then we write "$\mathbb{F}$ is an R−resolution of M" to mean that $\mathbb{F}$ is an acyclic complex of **finitely generated**, **free** R−modules with $H_0(\mathbb{F})=M$. If I is an ideal in the local ring $(R,\mathfrak{m},k)$, then we will often consider the map $\psi\colon I\to \operatorname{Tor}^R_1(R/I,k)$ which is the following composition of natural homomorphisms:

$$I\to I/\mathfrak{m}I \xrightarrow{\cong} \operatorname{Tor}^R_1(R/I,k). \tag{0.1}$$

In other words, if $(\mathbb{F},f)$ is an R−resolution of R/I and x is an element of I, then $\psi(x)$ is equal to the class of e in $H_1(\mathbb{F}\otimes_R k)$ for any $e\in F_1$ with $f_1(e)=x\in F_0=R$.

The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. The ideal I of R is called *perfect* if the grade of I is equal to the projective dimension of the R−module R/I. A grade g ideal I is called a *complete intersection* if it can be generated by g generators. Complete intersection ideals are necessarily perfect. The grade g ideal I is called an *almost complete intersection* if it is a **perfect** ideal which is **not** a complete intersection and which can be generated by $g+1$ generators. The grade g ideal I is called *Gorenstein* if it is perfect and $\operatorname{Ext}^g_R(R/I,R)\cong R/I$.

We always take "DG−algebra" to mean **associative** DG−algebra. Elementary results about DG−algebras and linkage may be found in [10] and [4]. In particular, we use the symmetry property of linkage quite often. Let $K\subsetneqq I$ be grade g perfect ideals in a commutative noetherian ring R. If K is a complete intersection and $J=K:I$, then J is a grade g perfect ideal and $I=K:J$.

Section 1. Higher order multiplication in Poincaré algebras.

In this section R is a fixed commutative noetherian ring.

Definition 1.1. If $\mathbb{F}: 0\to F_g \xrightarrow{f_g} \dots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$ is a DG−algebra over R, then $\mathbb{F}$ is a *length g Poincaré DG−algebra over R* if

(a) $F_0=R$,

(b) there is an isomorphism $[\ ]\colon F_g \to R$,
(c) the map $F_i \otimes F_{g-i} \to R$, which is given by $u_i \otimes u_{g-i} \mapsto [u_i u_{g-i}]$, is a perfect pairing for all i, and
(d) if $\operatorname{char} R = 2$ and $g = 2n$, then $x^2 = 0$ for every $x \in F_n$.

NOTE: Condition (d) is irrelevant to this paper. It has been included in order to keep our definition consistent with [1, Definition 9.3].

There are many well known examples of Poincaré DG−algebras. If $\mathbb{F}$ is a Koszul complex, then exterior multiplication gives $\mathbb{F}$ the structure of a Poincaré DG−algebra. It is shown in [10] and [8] that if I is a grade four Gorenstein ideal in a local ring R, then the minimal R−resolution of R/I is a Poincaré DG−algebra over R. If I is a grade g Gorenstein ideal in a local ring $(R, \mathfrak{m}, k)$, then Avramov [1, Example 9.4] has shown that $\operatorname{Tor}^R_\bullet(R/I, k)$ is a Poincaré DG−algebra over k of length g.

Notation 1.2. For each length four Poincaré DG−algebra $\mathbb{F}$ over R, we define maps

$$\Gamma\colon F_3 \otimes \bigwedge^5 F_1 \to R \quad \text{and} \quad \Phi\colon S_2F_2 \otimes \bigwedge^4 F_1 \to R \quad \text{by}$$

$$\begin{aligned}
\Gamma(v_3 \otimes v_1 \wedge v_1' \wedge v_1'' \wedge v_1''' \wedge v_1'''') = &+ [v_3v_1][v_1'v_1''v_1'''v_1''''] - [v_3v_1'][v_1v_1''v_1'''v_1''''] + [v_3v_1''][v_1v_1'v_1'''v_1''''] \\
&- [v_3v_1'''][v_1v_1'v_1''v_1''''] + [v_3v_1''''][v_1v_1'v_1''v_1'''], \quad \text{and} \\
\Phi(v_2 \otimes v_2' \otimes v_1 \wedge v_1' \wedge v_1'' \wedge v_1''') = &- [v_2v_1v_1'][v_2'v_1''v_1'''] + [v_2v_1v_1''][v_2'v_1'v_1'''] - [v_2v_1v_1'''][v_2'v_1'v_1''] \\
&- [v_2v_1''v_1'''][v_2'v_1v_1'] + [v_2v_1'v_1'''][v_2'v_1v_1''] - [v_2v_1'v_1''][v_2'v_1v_1'''] \\
&+ [v_2v_2'][v_1v_1'v_1''v_1''']
\end{aligned}$$

for all $v_i \in F_i$.

Definition 1.3. Let $(\mathbb{F}, f)$ be a Poincaré DG−algebra of length four over the ring R, and let Γ and Φ be the maps which are defined in (1.2).

(a) The map $\varphi\colon F_2 \otimes \bigwedge^5 F_1 \to R$ is a *complete higher order multiplication* on $\mathbb{F}$ if it satisfies
  (i) $\varphi(f_3(v_3) \otimes V^{(5)}) = \Gamma(v_3 \otimes V^{(5)})$, and
  (ii) $\varphi(v_2 \otimes f_2(v_2') \wedge V^{(4)}) + \varphi(v_2' \otimes f_2(v_2) \wedge V^{(4)}) = \Phi(v_2 \otimes v_2' \otimes V^{(4)})$
  for all $v_i \in F_i$ and $V^{(i)} \in \bigwedge^i F_1$.
(b) Let $\mathfrak{A}$ be a four−generated submodule of F_1. The map $\varphi'\colon F_2 \otimes F_1 \to R$ is a *partial higher order multiplication* on $\mathbb{F}$, with respect to $\mathfrak{A}$, if it satisfies
  (i) $\varphi'(f_3(v_3) \otimes v_1) = \Gamma(v_3 \otimes v_1 \wedge A)$,
  (ii) $\varphi'(v_2 \otimes f_2(v_2')) + \varphi'(v_2' \otimes f_2(v_2)) = \Phi(v_2 \otimes v_2' \otimes A)$, and
  (iii) $\varphi'(v_2 \otimes a) = 0$,
  for all $v_i \in F_i$ and $a \in \mathfrak{A}$, where $A \in \bigwedge^4 F_1$ is a fixed generator of the image of $\bigwedge^4 \mathfrak{A} \to \bigwedge^4 F_1$.

NOTE: In the above definition we say that φ' is associated to A. If rA is another generator of $\operatorname{im}(\bigwedge^4 \mathfrak{A} \to \bigwedge^4 F_1)$, for some element $r \in R$, then $r\varphi'$ is a partial higher order multiplication which is associated to rA.

Example 1.4. If $\mathbb{F}$ is a Koszul complex of length four, then the zero map from $F_2\otimes\bigwedge^5 F_1$ to R is a complete higher order multiplication on $\mathbb{F}$ because Γ and Φ are both identically zero.

Observation 1.5. *Adopt the notation of Definition 1.3. If* $\varphi\colon F_2\otimes\bigwedge^5 F_1\to R$ *is a complete higher order multiplication on* $\mathbb{F}$ *and* $\mathfrak{A}=(a_1,a_1',a_1'',a_1''')$ *is a submodule of* F_1*, then the map* $\varphi'\colon F_2\otimes F_1\to R$*, which is defined by*

$$\varphi'(v_2\otimes v_1)=\varphi(v_2\otimes v_1\wedge a_1\wedge a_1'\wedge a_1''\wedge a_1''')\quad \textit{for all } v_i\in F_i,$$

is a partial higher order multiplication on $\mathbb{F}$ *with respect to* M*.* $\square$

The following result, which establishes the existence of Poincaré algebras with higher order multiplication, plays a crucial role in the present paper. The only proof [16, 17] which is known at present is quite brutal.

Theorem 1.6. (Palmer) *Let* I *be a grade four Gorenstein ideal in a local ring* R*. If two is a unit in* R*, then the minimal* $R-$*resolution of* R/I *admits a complete higher order multiplication.* $\square$

Section 2. The algebra resolution $\mathbb{M}$ of R/J.

Throughout this section J is a grade four almost complete intersection ideal in the local ring $(R,\mathfrak{m},k)$. A DG$-$algebra resolution $\mathbb{M}$ of R/J was introduced in [16]. The resolution $\mathbb{M}$ is, in general, not the minimal resolution of R/J; however, we prove in section 3 that it is possible to choose $\mathbb{M}$ in such a way that $\mathbb{M}/\mathbb{I}$ is the minimal $R-$resolution of R/J for some DG$-$ideal $\mathbb{I}$ in $\mathbb{M}$.

Our description of $\mathbb{M}$ is taken from [9]. Let K be a grade four complete intersection ideal with $K\subseteq J$ and $\mu(J/K)=1$. (We use $\mu(M)$ to mean the minimal number of generators of the $R-$module M.) The ideal $I=K:J$ is known to be a grade four Gorenstein ideal. It is shown in [10] and [8] (the results in these references hold for Gorenstein ideals in arbitrary local rings) that the minimal resolution $\mathbb{L}$ of R/I is a Poincaré DG$-$algebra. Let $\mathbb{K}$ be a Koszul complex which is the minimal resolution of R/K and let $\alpha_\bullet\colon\mathbb{K}\to\mathbb{L}$ be a map of DG$-$algebras which extends the identity map $\alpha_0\colon R\to R$. A routine mapping cone argument establishes the following result.

Proposition 2.1. *Let* J *be a grade four almost complete intersection in the local ring* $(R,\mathfrak{m},k)$ *and let* K *be a grade four complete intersection ideal with* $K\subseteq J$ *and* $\mu(J/K)=1$*. Let* $I=K:J$*,* $(\mathbb{L},\ell)$ *be the minimal resolution of* R/I*,* $(\mathbb{K},k)$ *be the minimal resolution of* R/K*, and* $\alpha_\bullet\colon\mathbb{K}\to\mathbb{L}$ *be a map of DG$-$algebras. If* $\beta_i\colon L_i\to K_i$ *is the map defined by*

$$[\beta_i(v_i)u_{4-i}]=[v_i\alpha_{4-i}(u_{4-i})]$$

for all $u_j\in K_j$ *and all* $v_i\in L_i$*, then*

$$\mathbb{M}:\qquad 0\to M_4\xrightarrow{m_4}M_3\xrightarrow{m_3}M_2\xrightarrow{m_2}M_1\xrightarrow{m_1}M_0$$

is a resolution of R/J*, where* $M_0 = R$*,* $M_1 = K_1 \oplus L_0$*,* $M_2 = K_2 \oplus L_1$*,* $M_3 = K_3 \oplus L_2$*,* $M_4 = L_3$*,* $m_1 = [\, k_1 \quad \beta_0 \,]$*,*

$$m_2 = \begin{bmatrix} k_2 & -\beta_1 \\ 0 & \ell_1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} k_3 & \beta_2 \\ 0 & \ell_2 \end{bmatrix}, \quad \textit{and} \quad m_4 = \begin{bmatrix} -\beta_3 \\ \ell_3 \end{bmatrix}. \quad \square$$

The following result is Proposition 2.5 in both [16] and [17]. The main result in these two papers (that $\mathbb{M}$ is a DG$-$algebra provided two is a unit in R) is proved by combining Theorem 1.6 and Lemma 2.2.

Lemma 2.2. *Adopt the notation and hypotheses of Proposition 2.1. If two is a unit in* R *and* $\mathbb{L}$ *admits a complete higher order multiplication, then* $\mathbb{M}$ *is a DG$-$algebra.* $\square$

At a crucial step (Lemma 3.10) we will be forced to use a partial higher order multiplication on $\mathbb{L}$ (rather than a complete higher order multiplication); consequently, we will appeal to Lemma 2.3 in place of Lemma 2.2. It is clear from Observation 1.5 that Lemma 2.3 implies Lemma 2.2; unfortunately, it is necessary prove Lemma 2.3 from scratch. The proof is long and tedious; however it is straightforward and very similar to the proof of Lemma 2.2. We will sketch its outline and omit most details.

Lemma 2.3. *Adopt the notation and hypotheses of Proposition 2.1. If two is a unit in* R *and* $\mathbb{L}$ *admits a partial higher order multiplication with respect to the submodule* $\alpha_1(K_1)$ *of* L_1*, then* $\mathbb{M}$ *is a DG$-$algebra.*

Sketch of proof. Let $\mathfrak{A} = \alpha_1(K_1)$, $h \in L_4$ and $\eta \in \bigwedge^4 K_1 = K_4$ satisfy $[h] = [\eta] = 1$, A be the generator $\left(\bigwedge^4 \alpha_1\right)(\eta)$ of the image of $\bigwedge^4 \mathfrak{A} \to \bigwedge^4 F_1$, and $\varphi'\colon L_2 \otimes L_1 \to R$ be a partial higher order multiplication with respect to $\mathfrak{A}$ which is associated to A. If maps $p\colon L_1 \to L_2$ and $q\colon L_2 \to L_3$ are defined by

$$[v_2 p(v_1)] = \varphi'(v_2 \otimes v_1) = [v_1 q(v_2)],$$

then the following multiplication gives $\mathbb{M}$ the structure of a DG$-$algebra:

$$M_1 \otimes M_1 \to M_2 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_1' \\ v_0' \end{bmatrix} = \begin{bmatrix} u_1 u_1' \\ v_0' \alpha_1(u_1) - v_0 \alpha_1(u_1') \end{bmatrix}$$

$$M_1 \otimes M_2 \to M_3 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_1 u_2 \\ v_0 \alpha_2(u_2) + \alpha_1(u_1) v_1 + v_0 p(v_1) \end{bmatrix}$$

$$M_1 \otimes M_3 \to M_4 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_3 \\ v_2 \end{bmatrix} = -[u_1 u_3]\ell_4(h) - v_0 \alpha_3(u_3) + \alpha_1(u_1) v_2 - v_0 q(v_2)$$

$$M_2 \otimes M_2 \to M_4 : \begin{bmatrix} u_2 \\ v_1 \end{bmatrix} \begin{bmatrix} u_2' \\ v_1' \end{bmatrix} = -[u_2 u_2']\ell_4(h) + \alpha_2(u_2) v_1' + v_1 \alpha_2(u_2') + v_1 p(v_1') + v_1' p(v_1)$$

for all $u_i \in K_i$ and $v_i \in L_i$.

One must verify associativity and the differential property

$$m_{i+j}(x_i x_j) = m_i(x_i) x_j + (-1)^i x_i m_j(x_j) \tag{2.4}$$

for all $x_k \in M_k$. The following formulas are used in these verifications:

(1) $\beta_{i+j}(v_i\alpha_j(u_j)) = (\beta_i v_i)u_j$
(2) $a_1 p(v_1) = q(a_1 v_1)$
(3) $q(a_1 a_1') = 0$
(4) $p(v_1)p(v_1') = 0$
(5) $\beta_0(1)v_1 - \alpha_1\beta_1(v_1) = \ell_2 p(v_1)$
(6) $\beta_0(1)v_2 - \alpha_2\beta_2(v_2) = p\ell_2(v_2) - \ell_3 q(v_2)$
(7) $\beta_0(1)v_3 - \alpha_3\beta_3(v_3) + q\ell_3(v_3) = 0$
(8) $v_2\alpha_1\beta_1(v_1) - v_1\alpha_2\beta_2(v_2) = v_1 p\ell_2(v_2) + \ell_2(v_2)p(v_1) - \ell_1(v_1)q(v_2)$
(9) $[u_4]\beta_3\ell_4(h) = k_4(u_4)$

for all $a_1 \in \mathfrak{A}$, $u_i \in K_i$, and $v_i \in L_i$.

Formula (1) is an immediate consequence of the definition of β_i. To prove (2), first observe that axioms (ii) and (iii) in Definition 1.3 yield

$$0 = 2[a_1 v_1 p(\ell_2(a_1 v_1))] = 2(\ell_1 a_1)[a_1 v_1 p(v_1)]$$

for all $a_1 \in \mathfrak{A}$ and $v_1 \in L_1$. The ideal $\ell_1(\mathfrak{A}) = K$ is generated by regular elements of R; consequently, we divide by two and conclude that

$$a_1 v_1 p(v_1) = 0 \quad \text{for all } a_1 \in \mathfrak{A} \text{ and } v_1 \in L_1. \tag{2.5}$$

Apply (2.5) to the element $v_1 + v_1'$ of L_1 in order to see that $a_1(v_1' p(v_1) + v_1 p(v_1')) = 0$ for all $a_1 \in \mathfrak{A}$ and $v_1, v_1' \in L_1$. Formula (2) now holds, and (3) follows from (2). Since two is a unit, we prove (4) by showing that $p(v_1)p(v_1) = 0$ for all $v_1 \in L_1$. Let a_1 be an element of $\mathfrak{A}$ with $\ell_1(a_1)$ a regular element of R. The differential property of $\mathbb{L}$ yields

$$0 = \ell_5(a_1 p(v_1)p(v_1)) = \ell_1(a_1)p(v_1)p(v_1) + 2a_1\ell_2(p(v_1))p(v_1).$$

Axiom (i) shows that $\ell_2(p(v_1)) \in (\mathfrak{A}, v_1)$. Use formulas (3) and (2.5) to see that the final term in the most recent equation is zero.

Recall that η is the element of K_4 with $[\eta] = 1$. Let $\varepsilon_1, \dots, \varepsilon_4$ be a basis for K_1 with $\varepsilon_1 \wedge \dots \wedge \varepsilon_4 = \eta$. The definition of β_i yields

$$\begin{aligned}
\beta_0(1) &= + [\alpha_4(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)] \\
\beta_1(v_1) &= + [v_1\alpha_3(\varepsilon_2\varepsilon_3\varepsilon_4)]\varepsilon_1 - [v_1\alpha_3(\varepsilon_1\varepsilon_3\varepsilon_4)]\varepsilon_2 + [v_1\alpha_3(\varepsilon_1\varepsilon_2\varepsilon_4)]\varepsilon_3 - [v_1\alpha_3(\varepsilon_1\varepsilon_2\varepsilon_3)]\varepsilon_4 \\
\beta_2(v_2) &= + [v_2\alpha_2(\varepsilon_3\varepsilon_4)]\varepsilon_1\varepsilon_2 - [v_2\alpha_2(\varepsilon_2\varepsilon_4)]\varepsilon_1\varepsilon_3 + [v_2\alpha_2(\varepsilon_2\varepsilon_3)]\varepsilon_1\varepsilon_4 \\
&\quad + [v_2\alpha_2(\varepsilon_1\varepsilon_2)]\varepsilon_3\varepsilon_4 - [v_2\alpha_2(\varepsilon_1\varepsilon_3)]\varepsilon_2\varepsilon_4 + [v_2\alpha_2(\varepsilon_1\varepsilon_4)]\varepsilon_2\varepsilon_3 \\
\beta_3(v_3) &= + [v_3\alpha_1(\varepsilon_4)]\varepsilon_1\varepsilon_2\varepsilon_3 - [v_3\alpha_1(\varepsilon_3)]\varepsilon_1\varepsilon_2\varepsilon_4 + [v_3\alpha_1(\varepsilon_2)]\varepsilon_1\varepsilon_3\varepsilon_4 - [v_3\alpha_1(\varepsilon_1)]\varepsilon_2\varepsilon_3\varepsilon_4 \\
\beta_4(v_4) &= + [v_4]\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4
\end{aligned}$$

for all $v_i \in L_i$. Use the differential property in $\mathbb{L}$, together with axiom (i) of Definition 1.3 to see that

$$\begin{aligned}
[v_3\ell_2 p(v_1)] = [\ell_3(v_3)p(v_1)] &= \Gamma\left(v_3 \otimes v_1 \wedge \alpha_1(\varepsilon_1) \wedge \alpha_1(\varepsilon_2) \wedge \alpha_1(\varepsilon_3) \wedge \alpha_1(\varepsilon_4)\right) \\
&= \left[v_3\left(\beta_0(1)v_1 - \alpha_1\beta_1(v_1)\right)\right];
\end{aligned}$$

hence, (5) is established. In a similar manner, we see that

$$\begin{aligned}\left[v_2'\left(p\ell_2(v_2)-\ell_3q(v_2)\right)\right]&=\left[v_2'p\ell_2(v_2)+v_2p\ell_2(v_2')\right]=\Phi\left(v_2\otimes v_2'\otimes\alpha_1(\varepsilon_1)\wedge\alpha_1(\varepsilon_2)\wedge\alpha_1(\varepsilon_3)\wedge\alpha_1(\varepsilon_4)\right)\\&=\left[v_2'\left(\beta_0(1)v_2-\alpha_2\beta_2v_2\right)\right];\end{aligned}$$

thus, (6) holds. Formula (7) follows from (5) because

$$[v_1\left(\beta_0(1)v_3-\alpha_3\beta_3(v_3)+q\ell_3(v_3)\right)]=[\left(\beta_0(1)v_1-\alpha_1\beta_1(v_1)-\ell_2p(v_1)\right)v_3]=0.$$

Apply (6) to see that the right side of (8) is equal to

$$\beta_0(1)v_1v_2-v_1\alpha_2\beta_2(v_2)+B,$$

where

$$\begin{aligned}B=&\ v_1\ell_3q(v_2)+\ell_2(v_2)p(v_1)-\ell_1(v_1)q(v_2)=-\ell_4(v_1q(v_2))+\ell_2(v_2)p(v_1)\\=&-\ell_4(v_2p(v_1))+\ell_2(v_2)p(v_1)=-v_2\ell_2p(v_1);\end{aligned}$$

hence, (8) follows from (5). Finally, (9) is a consequence of the differential properties on $\mathbb{L}$ and $\mathbb{K}$ because

$$[([u_4]\beta_3\ell_4(h))\,u_1]=[u_4][\ell_4(h)\alpha_1(u_1)]=-[u_4]k_1(u_1)=[(k_4u_4)u_1]\,.\quad\square$$

Section 3. The main theorem.

The main result in this paper is Theorem 3.13, where we convert the DG−algebra resolution $\mathbb{M}$ of Proposition 2.1 into a minimal resolution which is still a DG−algebra. Suppose $\mathbb{F}$ is a DG−resolution of a cyclic module M over the local ring $(R,\mathfrak{m},k)$. It is always true that the minimal resolution $\mathbb{F}'$ of M is a summand of $\mathbb{F}$; however, most attempts to use the multiplication on $\mathbb{F}$ to induce a multiplication on $\mathbb{F}'$ yield something which does not associate. However, if the kernel of $\mathbb{F}\to\mathbb{F}'$ is a DG−ideal of $\mathbb{F}$, then $\mathbb{F}'$ is a DG−resolution of M. For example, if $\mathbb{F}$ and $\mathbb{F}'$ differ only at the back end, then $\mathbb{F}'$ is a DG−resolution.

Observation 3.1. *If $(\mathbb{F},f)$ is a length g, DG−resolution of a nonzero cyclic module M over the local ring $(R,\mathfrak{m},k)$, then there exists a quotient $(\mathbb{F}',f')$ of $\mathbb{F}$ such that $\mathbb{F}'$ is a DG−resolution of M, and $f_g'\otimes k$ is identically zero.*

Proof. The ring R is local, so it is possible to find a submodule B of F_g such that $f_g(B)$ is a summand of F_{g-1} and $\operatorname{im}(f_g)\subseteq(f_g(B)+\mathfrak{m}F_{g-1})$. Let $\mathbb{I}$ be the complex

$$0\to B\xrightarrow{f_g}f_g(B)\to 0$$

(with B in position g). It is clear that B is a summand of F_g and that $\mathbb{F}/\mathbb{I}$ is a resolution of M by free R−modules. Furthermore, $\mathbb{I}$ is an ideal of $\mathbb{F}$ because

$$x_1f_g(b)=f_1(x_1)b \tag{3.2}$$

for all $x_1\in F_1$ and $b\in B$. $\square$

Sometimes Observation 3.1 is all that is needed in order to convert $\mathbb{M}$ into a minimal DG−algebra resolution.

Definition 3.3. Let J be a grade four almost complete intersection in the local ring $(R,\mathfrak{m},k)$. For each grade four complete intersection ideal with $K \subseteq J$ and $\mu(J/K)=1$, let

$$W(K) = \operatorname{im}\left(\operatorname{Tor}_1^R\left(\tfrac{R}{K}\,,\,k\right) \xrightarrow{\pi_*} \operatorname{Tor}_1^R\left(\tfrac{R}{K:J}\,,\,k\right)\right),$$

where $\pi\colon R/K \to R/(K:J)$ is the natural map. If it is possible to choose K so that $(W(K))^2=0$ in $\operatorname{Tor}_\bullet^R(R/(K:J),k)$, then we say that *hypothesis* (W) *holds for* J.

Corollary 3.4. ([17]) *Let J be a grade four almost complete intersection in the local ring $(R,\mathfrak{m},k)$. If two is a unit in R and hypothesis (W) holds for J, then the minimal $R-$resolution of R/J is a DG$-$algebra.*

Proof. Fix a complete intersection K which exhibits hypothesis (W) for J. Adopt the notation of Proposition 2.1. Combine Theorem 1.6 and Lemma 2.2 in order to see that the resolution $\mathbb{M}$ of R/J is a DG$-$algebra. The hypothesis $(W(K))^2=0$ is equivalent to the statement $\alpha_2 \otimes k = 0$. It follows that $\alpha_i \otimes k = 0$ for $2 \le i \le 4$. Since

$$\operatorname{rank}\beta_j \otimes k = \operatorname{rank}\alpha_{4-j} \otimes k, \tag{3.5}$$

we conclude that $\beta_j \otimes k = 0$ for $0 \le j \le 2$ and $m_i \otimes k = 0$ for $1 \le i \le 3$. An application of Observation 3.1 completes the proof. □

Now that the Tor$-$algebras of grade four almost complete intersections have been classified, it is possible to reformulate hypothesis (W) in an intrinsic manner. It is not necessary for us to recapitulate the entire classification from [9]; but it is worth while to let $\mathbf{C}[0]$ represent the graded$-$commutative $k-$algebra

$$\mathbf{C}[0] = \bigwedge^{\bullet} k(-1)^2 \bigotimes_k \frac{\bigwedge^{\bullet} k(-1)^3}{\bigwedge^2 k(-1)^3} = \frac{k[e_1,e_2,e_3,e_4,e_5]}{(e_3e_4,e_3e_5,e_4e_5)}\,,$$

where each variable e_i has degree one.

Proposition 3.6. *Let J be a grade four almost complete intersection in the local ring $(R,\mathfrak{m},k)$, and let $T_\bullet$ be the algebra $\operatorname{Tor}_\bullet^R(R/J,k)$. If two is a unit in R, then the following statements are equivalent.*

(a) *Hypothesis (W) holds for J.*
(b) *There is a four$-$dimensional subspace S of T_1 with $\dim_k S^2 = 6$.*
(c) *The subalgebra $k[T_1]$ of $T_\bullet$ is not isomorphic to $\mathbf{C}[0]$.*

Furthermore, if hypothesis (W) does not hold for J, then there exists a complete intersection ideal K with $K \subseteq J$ and $\mu(J/K)=1$ such that

$$\operatorname{rank}\left(\pi_*\colon\ \operatorname{Tor}_i^R(R/K,k) \to \operatorname{Tor}_i^R(R/I,k)\right) = \begin{cases} 1, & \text{if } i=2,\\ 2, & \text{if } i=1,\end{cases} \tag{3.7}$$

where $I = K:J$ and $\pi\colon R/K \to R/I$ is the natural map.

Proof. Lemma 3.9 shows that conditions (a) and (b) are equivalent. Conditions (b) and (c) are shown to be equivalent in Lemma 1.2 of [9]. The proof of (b) $\Rightarrow$ (c) is a straightforward calculation. The proof of (c) $\Rightarrow$ (b) uses the main theorem in [9], which is proved under the additional hypothesis

$$k \text{ is closed under the square root operation.} \tag{3.8}$$

However, it is not difficult to see that condition (3.8) is only used in the description of the multiplication $T_1 \otimes T_3 \to T_4$. All of the products in T_1^4 are zero (because J is not a complete intersection); consequently, the classification of the subalgebras $k[T_1] \subseteq T_\bullet$ in Corollary 1.6 of [9] is valid even in the absence of (3.8); and therefore, (c) $\Rightarrow$ (b) even in the absence of (3.8).

The final statement is also a consequence of Lemma 3.9. If hypothesis (W) does not hold for J, then $k[T_1] = \mathbf{C}[0]$ and the subspace $S = (e_1, e_2, e_3, e_4)$ of $\mathbf{C}[0]$ satisfies $\dim_k S^2 = 5$ and $\dim_k S^3 = 2$. $\square$

In the notation of Proposition 3.6, let $\psi\colon J \to T_1$ be the map of (0.1). If S is any four−dimensional subspace of T_1, then a routine general position argument will produce a complete intersection ideal K with $K \subseteq J$, $\mu(J/K) = 1$, and $\psi(K) = S$.

Lemma 3.9. *Adopt the notation of Proposition 3.6. If S is a four−dimensional subspace of T_1, and K is a complete intersection ideal with $K \subseteq J$, $\mu(J/K) = 1$, and $\psi(K) = S$, then*

$$\begin{aligned} \dim_k S^2 &= 6 - \operatorname{rank}\left(\pi_*\colon\ \operatorname{Tor}_2^R(R/K,k) \to \operatorname{Tor}_2^R(R/I,k)\right), \text{ and} \\ \dim_k S^3 &= 4 - \operatorname{rank}\left(\pi_*\colon\ \operatorname{Tor}_1^R(R/K,k) \to \operatorname{Tor}_1^R(R/I,k)\right) \end{aligned}$$

where $I = K\colon J$ and $\pi\colon R/K \to R/I$ is the natural map.

Proof. The idea for this proof is taken from (4.1) and (4.2) in [9]. Apply Proposition 2.1 to the ideals $K \subseteq I$ and compute the powers of S in $H_0(\overline{\mathbb{M}}) = T_\bullet$, where we write $\overline{}$ to mean $\_\_ \otimes k$. We see that

$$T_1 = \overline{K}_1 \oplus \overline{L}_0, \quad T_2 = \frac{\overline{K}_2}{\operatorname{im}\overline{\beta}_2} \oplus \overline{L}_1, \quad \text{and} \quad T_3 = \frac{\overline{K}_3}{\operatorname{im}\overline{\beta}_3} \oplus \ker\overline{\beta}_2.$$

It is clear that S is the subspace $\overline{K}_1$ of T_1. Use Lemma 2.3 to read the multiplication in $H_0(\overline{\mathbb{M}})$:

$$S^2 = \frac{\overline{K}_2}{\operatorname{im}\overline{\beta}_2} \subseteq T_2, \quad \text{and} \quad S^3 = \frac{\overline{K}_3}{\operatorname{im}\overline{\beta}_3} \subseteq T_3.$$

The proof is complete because of (3.5). $\square$

Proposition 3.6 serves two purposes. On the one hand, hypothesis (W) from [17] can now be replaced with the less awkward hypothesis $k[T_1] \not\cong \mathbf{C}[0]$. But even more importantly, we now have a good handle on precisely which almost complete intersection ideals J are not covered in Corollary 3.4. Lemma 3.10 is a straightforward calculation in which we demonstrate how we plan to put a DG−structure on the minimal resolution of R/J for such ideals.

Lemma 3.10. *Let $K \subseteq I$ be grade four ideals in the local ring $(R, \mathfrak{m}, k)$, with K a complete intersection and I Gorenstein. Assume that (3.7) holds for these ideals and that two is a unit of R. Let $(\mathbb{L}, \ell)$ be a Poincaré DG−algebra which is the minimal resolution of R/I. If there exists a four−generated submodule $\mathfrak{A}$ of L_1, elements $e_1, e_2 \in \mathfrak{A}$, $f_1' \in L_2$, and a partial higher order multiplication φ' on $\mathbb{L}$, with respect to $\mathfrak{A}$, such that*

(a) $\ell_1(\mathfrak{A}) = K$,
(b) $[e_1 e_2 f_1'] = 1$,
(c) $\mathfrak{A} f_1' \subseteq (e_1, e_2) f_1'$, *and*
(d) $\varphi'(f_1' \otimes v_1) = 0$ *for all* $v_1 \in L_1$,

then the minimal R-resolution of $R/(K : I)$ is a DG−algebra.

Proof. Use (b) and (c) to find a_3 and a_4 with $\mathfrak{A} = (e_1, e_2, a_3, a_4)$ and $a_3 f_1' = a_4 f_1' = 0$. Let $\mathbb{K}$ be a minimal resolution of R/K. Choose a basis $\varepsilon_1, \dots, \varepsilon_4$ for K_1 and orient $\mathbb{K}$ by insisting that $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4] = 1$. Let $\alpha_\bullet : \mathbb{K} \to \mathbb{L}$ be the DG−algebra map with $\alpha_1(\varepsilon_i) = e_i$ for $1 \le i \le 2$ and $\alpha_1(\varepsilon_i) = a_i$ for $3 \le i \le 4$. Construct the resolution $\mathbb{M}$ of Proposition 2.1. We see from (3.7) that $\operatorname{rank} \alpha_1 \otimes k = 2$ and $\operatorname{rank} \alpha_2 \otimes k = 1$. It follows that $\operatorname{rank} \beta_3 \otimes k = 2$ and $\operatorname{rank} \beta_2 \otimes k = 1$. The definition of β_i yields $\beta_2(f_1') = \varepsilon_3 \varepsilon_4$. Formula (1), from the proof of Lemma 2.3, shows that $\beta_3(e_i f_1') = \varepsilon_i \varepsilon_3 \varepsilon_4$, for $i = 1$ and 2. Let x_3 represent the element

$$\begin{bmatrix} 0 \\ f_1' \end{bmatrix}$$

of M_3, and let $\mathbb{I}$ be the subcomplex

$$\mathbb{I}: \quad 0 \to (e_1 f_1',\, e_2 f_1') \xrightarrow{m_4} (m_4(e_1 f_1'),\, m_4(e_2 f_1'),\, x_3) \xrightarrow{m_3} (m_3(x_3)) \to 0 \to 0$$

of $\mathbb{M}$. It is clear that $\mathbb{M}/\mathbb{I}$ is the minimal R−resolution of $R/(K : I)$. On the other hand, $\mathbb{M}$ is a DG−algebra by Lemma 2.3; and hypotheses (c) and (d) have been chosen in order to ensure that $\mathbb{I}$ is an ideal of $\mathbb{M}$. Indeed, if

$$x_1 = \begin{bmatrix} u_1 \\ v_0 \end{bmatrix}$$

is an arbitrary element of M_1, then $x_1 x_3$ is equal to $\alpha_1(u_1) f_1' - v_0 q(f_1')$, which is in $\mathbb{I}$ by hypothesis. The rest of the products $\mathbb{M} \cdot \mathbb{I}$ are in $\mathbb{I}$ for formal reasons:

$$x_1 \cdot m_3(x_3) = -m_4(x_1 x_3) + m_1(x_1) x_3 \quad \text{and} \quad x_2 \cdot m_3(x_3) = -(m_2(x_2)) x_3,$$

together with (3.2). $\square$

Hypotheses (a) and (b) in Lemma 3.10 pose no difficulty. Indeed, the existence of elements e_1, e_2, and f_1' which satisfy (b) is a consequence of $i = 2$ in (3.7). Furthermore, a routine calculation, using the classification of $\operatorname{Tor}_\bullet^R(R/I, k)$ in [9] or [15], shows that e_1, e_2, and f_1' can be chosen so that (a), (b) and

(c′) $L_1 f_1' \subseteq (e_1, e_2) f_1' + \mathfrak{m} L_3$.

hold. However, a significant amount of work (sections 4, 5, and 6) is needed in order to guarantee that hypothesis (c) is met on the nose. Hypothesis (d) is also non-trivial; but, our method of attacking (c) leads to a natural proof of (d). At any rate, Theorem 3.11 is all that is needed in order to complete the proof of Theorem 3.13, which is the main result in this paper. The proof of Theorem 3.11 appears in section 6.

Theorem 3.11. *Let I be a grade four Gorenstein ideal in the local ring $(R, \mathfrak{m}, k)$, $(\mathbb{L}, \ell)$ be the minimal $R-$resolution of R/I, and S be a subspace of T_1, where $T_\bullet = \operatorname{Tor}^R_\bullet(R/I, k)$. If two is a unit in R and the subspace S^2 of T_2 is not zero, then there exists a Poincaré DG$-$algebra structure on $\mathbb{L}$ and there exist elements $e_1, e_2 \in L_1$ and $f_1' \in L_2$ such that*

(a) *the class of e_i in T_1 is an element of S, for $i = 1$ and 2,*
(b) $[e_1 e_2 f_1'] = 1$, *and*
(c) $L_1 f_1' \subseteq (e_1, e_2) f_1'$.

Furthermore, if $\mathfrak{A}$ is any four generated submodule of L_1 which contains (e_1, e_2), then there exists a partial higher order multiplication φ' on $\mathbb{L}$, with respect to $\mathfrak{A}$, which satisfies

(d) $\varphi'(f_1' \otimes v_1) = 0$ *for all* $v_1 \in L_1$.

Example 3.12. Theorem 3.11 obviously holds when I is a complete intersection. Indeed, if e_1, e_2, e_3, e_4 is any basis for L_1 for which (a) holds, then let $f_1' = e_3 e_4$. Recall, from Example 1.4, that the zero map is a partial higher order multiplication on $\mathbb{L}$.

Remark. Consider the resolution $\mathbb{L}$ as given in the hypotheses of Theorem 3.11. We know, from [10] and [16], that $\mathbb{L}$ already comes equipped with a Poincaré DG$-$structure and a complete higher order multiplication. Unfortunately, we do not know if these products satisfy conclusions (c) and (d). Consequently, in our proof of the result we ignore the existing multiplicative structures and create new ones from scratch.

Theorem 3.13. *Let J be a grade four almost complete intersection ideal in a local ring $(R, \mathfrak{m}, k)$. If two is a unit in R, then the minimal resolution of R/J by free $R-$modules is a DG$-$algebra.*

Proof. Corollary 3.4 takes care of the case when hypothesis (W) holds for J. Henceforth, we assume that hypothesis (W) does not hold for J. Use Proposition 3.6 to select ideals $K \subseteq I$ for which (3.7) holds. Let $(\mathbb{L}, \ell)$ be the minimal resolution of R/I. Apply Theorem 3.11 with S equal to the image of $\operatorname{Tor}^R_1(R/K, k)$ in T_1. Now that $e_1, e_2 \in L_1$ have been chosen, pick a_3 and a_4 with $\ell_1(\mathfrak{A}) = K$, for $\mathfrak{A} = (e_1, e_2, a_3, a_4)$, and apply part (d) of Theorem 3.11 in order to verify that the rest of the hypotheses of Lemma 3.10. The proof is complete because $K : I = J$. $\square$

Section 4. Tight double linkage and the big from small construction.

Theorem 3.11 can be thought of as a lifting theorem. Let I be a grade four Gorenstein ideal in the local ring $(R, \mathfrak{m}, k)$, and let $T_\bullet$ be the algebra $\operatorname{Tor}^R_\bullet(R/I, k)$.

Suppose that that we are given elements $\bar{e}_1, \bar{e}_2 \in T_1$ and $\overline{f'_1} \in T_2$ which satisfy

$$(*) \qquad \left[\bar{e}_1\bar{e}_2\overline{f'_1}\right] = 1, \quad \text{and} \quad T_1\overline{f'_1} \subseteq (\bar{e}_1, \bar{e}_2)\,\overline{f'_1}.$$

If $\mathbb{L}$ is a minimal DG$-$resolution of R/I, then $(*)$ takes place in $\mathbb{L} \otimes_R k = T_\bullet$. In the course of proving Theorem 3.11, we must produce representatives e_1, e_2, and f'_1 in $\mathbb{L}$ (for $\bar{e}_1$, $\bar{e}_2$, and $\overline{f'_1}$, respectively), such that $(*)$ holds in $\mathbb{L}$, that is, without bars and with $T_1 = \overline{L}_1$ replaced by L_1. Our plan is to ignore the original multiplication on $\mathbb{L}$ and to create a brand new multiplication which exhibits the desired properties. The process has two steps. In section 5 we use the "big from small construction" of [11] and [12] in order to produce a DG$-$resolution $\mathbb{F}$ whose multiplication has the desired properties. The resolution $\mathbb{F}$ is not usually minimal; we convert it into the appropriate minimal resolution in section 6. In the present section we review the big from small construction and the related notion of tight double linkage. The majority of the section is consumed by the statement and the proof of Proposition 4.6. This result identifies the elements e_1, e_2, and f'_1 which are used in the proof of Theorem 3.11 in section 6.

Definition 4.1. Let R be a ring, r be an element of R, and $\alpha_\bullet\colon \mathbb{B} \to \mathbb{A}$ be a map of DG$-$algebras, where $(\mathbb{B}, b)$ is a Koszul complex of length three over R and $(\mathbb{A}, a)$ is a Poincaré DG$-$algebra of length four over R. Define maps $\beta_i\colon A_i \to B_{i-1}$ for $1 \le i \le 4$ by

$$[(\beta_i x_i) z_{4-i}] = (-1)^{i+1}[x_i \alpha_{4-i} z_{4-i}]$$

for all $x_i \in A_i$ and $z_{4-i} \in B_{4-i}$. When the *big from small construction* is applied to the data $(\alpha_\bullet, r)$, the resulting complex is

$$\mathbb{F} = \mathbb{F}(\alpha_\bullet, r): \quad 0 \to F_4 \xrightarrow{f_4} F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0,$$

where $F_0 = R$, $F_1 = B_1 \oplus A_1$, $F_2 = B_2 \oplus A_2 \oplus B_1$, $F_3 = A_3 \oplus B_2$, $F_4 = A_4$, $f_1 = [\, b_1 \quad \beta_1 + r a_1\,]$,

$$f_2 = \begin{bmatrix} b_2 & \beta_2 & r \\ 0 & -a_2 & -\alpha_1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} \beta_3 & -r \\ -a_3 & -\alpha_2 \\ 0 & b_2 \end{bmatrix}, \quad \text{and} \quad f_4 = \begin{bmatrix} \alpha_3\beta_4 - r a_4 \\ -b_3\beta_4 \end{bmatrix}.$$

The relationship between $H_0(\mathbb{A})$ and $H_0(\mathbb{F})$ is explained in Proposition 4.3.

Definition 4.2. ([14]) If I and I' are grade g Gorenstein ideals in the commutative noetherian ring R and K is a grade $g-1$ complete intersection in R, then there is a *tight double link between I and I' over* K, if there exists an almost complete intersection ideal $J = (K, y, y')$ with (K, y) and (K, y') both grade g complete intersection ideals, and

$$(K, y)\colon I = J = (K, y')\colon I'.$$

Caution: The ideal J in the above definition must be proper and is not permitted to be a complete intersection.

Proposition 4.3. *Let $K \subseteq I$ be ideals in the commutative noetherian ring R, where K is a grade three complete intersection and I is a grade four Gorenstein ideal. Let $\mathbb{B}$ be a Koszul complex of length three which resolves R/K, $\mathbb{A}$ be a Poincaré DG−algebra of length four which resolves R/I, and $\alpha_\bullet \colon \mathbb{B} \to \mathbb{A}$ be a map of DG−algebras over R. If there is a tight double link between I and a grade four Gorenstein ideal I' over K, then there is an element $r \in R$ so that $\mathbb{F}(\alpha_\bullet, r)$ resolves R/I'.*

Sketch of proof. The proof of Theorem 2.1 in [13] shows that I' is equal to the image of $[\, b_1 \quad \beta_1 + ra_1\,]$ for some element r in R. The maps β_i have been defined in order to make

$$\begin{tikzcd}
 & 0 \arrow[r] & B_3 \arrow[r, "b_3"] \arrow[d, "\alpha_3"'] & B_2 \arrow[r, "b_2"] \arrow[d, "\alpha_2"'] & B_1 \arrow[r, "b_1"] \arrow[d, "\alpha_1"'] & B_0 \arrow[d, "\alpha_0"'] \\
0 \arrow[r] & A_4 \arrow[r, "a_4"] \arrow[d, "\beta_4"'] & A_3 \arrow[r, "a_3"] \arrow[d, "\beta_3"'] & A_2 \arrow[r, "a_2"] \arrow[d, "\beta_2"'] & A_1 \arrow[r, "a_1"] \arrow[d, "\beta_1"'] & A_0 \\
0 \arrow[r] & B_3 \arrow[r, "b_3"] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 &
\end{tikzcd}$$

be a commutative diagram. An iterated mapping cone argument produces the the complex $\mathbb{F} = \mathbb{F}(\alpha_\bullet, r)$ and shows that $H_i(\mathbb{F}) = 0$ for $2 \le i$. One uses the fact that $H_0(\mathbb{F})$ is a perfect R−module in the course of proving $H_1(\mathbb{F}) = 0$. While performing the calculations, it is useful to notice that

$$[((\alpha_3\beta_4 - ra_4)x_4) \cdot x_1] = [x_4](\beta_1 + ra_1)x_1 \quad \text{and} \quad -[(b_3\beta_4 x_4) \cdot z_1] = [x_4]b_1(z_1)$$

for all $x_i \in A_i$ and $z_i \in B_i$, and that $\beta_i\alpha_i = 0$ for $1 \le i \le 3$. See [11, Theorem 1.3] or [15, Lemma 1.5] for details. $\square$

The name "big from small construction" refers to the fact that when the construction was introduced in [11], it was used (with $\alpha_1 \otimes k = 0$) to produce a Gorenstein ideal I' which required more generators than the original Gorenstein ideal I. In the meantime, the process has been found to be even more useful when used "in reverse". Indeed, the key induction tool in [15] is the big from small construction applied with $\operatorname{rank} \alpha_1 \otimes k = 3$. The next result is essentially the same as Theorem 1.6 in [15].

Proposition 4.4. *Let I be a grade four Gorenstein ideal in the local ring $(R, \mathfrak{m}, k)$, and $K \subseteq I$ be a grade three complete intersection with $\mu(I/K) = \mu(I) - 3$. If I' is a Gorenstein ideal which is one tight double link over K away from I, then*

$$\mu(I') = \mu(I) - \operatorname{rank}\left(\pi_* \colon \operatorname{Tor}_2^R(R/K, k) \to \operatorname{Tor}_2^R(R/I, k)\right),$$

where $\pi \colon R/K \to R/I$ is the natural surjection.

Note. If the ideal I, in the above result, is a complete intersection, then there do not exist any ideals I' which are one tight double link over K away from I, because any candidate for the intermediate almost complete intersection $J = (K, y) \colon I$ would be a complete intersection instead.

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be the minimal resolutions of R/I and R/K, respectively. Apply Proposition 4.3 in order to find a resolution $\mathbb{F}(\alpha_\bullet, r)$ of R/I'. (It makes no difference whether r is a unit or r is in $\mathfrak{m}$.) The hypothesis guarantees that $\operatorname{rank}\alpha_1 \otimes k = 3$; thus,

$$\mu(I') = \mu(I) + 3 - \operatorname{rank} f_2 \otimes k = \mu(I) - \operatorname{rank}\beta_2 \otimes k.$$

The definition of β_i shows that

$$\operatorname{rank}\beta_i \otimes k = \operatorname{rank}\alpha_{4-i} \otimes k; \tag{4.5}$$

and it is clear that $\operatorname{rank}\alpha_2 \otimes k = \operatorname{rank}\big(\pi_*\colon \operatorname{Tor}_2^R(R/K,k) \to \operatorname{Tor}_2^R(R/I,k)\big)$. $\square$

Proposition 4.6. *Let I be a grade four Gorenstein ideal in the local ring $(R, \mathfrak{m}, k)$, $T_\bullet$ be the algebra $\operatorname{Tor}_\bullet^R(R/I,k)$, and S be a two dimensional subspace of T_1 with S^2 not equal to zero in T_2. If I is not a complete intersection, then there exists an element $r \in R$, a DG$-$algebra map $\alpha_\bullet\colon \mathbb{B} \to \mathbb{A}$, and a basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for B_1 such that $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] = 1$ and the following statements hold.*

(a) *The complex $\mathbb{F} = \mathbb{F}(\alpha_\bullet, r)$ is an $R-$resolution of R/I.*

(b) *The subspace S of T_1 is generated by the classes of $\overline{e}_1$ and $\overline{e}_2$ in $H_1(\overline{\mathbb{F}})$, where $^-$ means $\_\_ \otimes_R k$ and*

$$e_i = \begin{bmatrix} \varepsilon_i \\ 0 \end{bmatrix} \in F_1, \quad \textit{for } i = 1 \textit{ and } 2.$$

(c) *If $\mathbb{F}$ is not the minimal resolution of R/I, then the minimal resolution of R/I has the form $\mathbb{F}/\mathbb{J}$, where $\mathbb{J}$ is described in either* (i) *or* (ii) *below.*

(i) *There is an element $g_1 \in A_3$ such that $[g_1\alpha_1(\varepsilon_3)] = 0$, $[g_1\alpha_1(\varepsilon_1)] = 1$, and $\mathbb{J}$ is the subcomplex*

$$\mathbb{J}: \quad 0 \longrightarrow 0 \xrightarrow{f_4} (y_3) \xrightarrow{f_3} (f_3(y_3),\, y_2) \xrightarrow{f_2} (f_2(y_2)) \xrightarrow{f_1} 0$$

of $\mathbb{F}$, for

$$y_3 = \begin{bmatrix} g_1 \\ 0 \end{bmatrix} \in F_3 \quad \textit{and} \quad y_2 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_1 \end{bmatrix} \in F_2.$$

(ii) *There are elements $g_1, g_2 \in A_3$ such that*

$$\begin{aligned} [g_1\alpha_1(\varepsilon_3)] &= [g_2\alpha_1(\varepsilon_3)] = 0, \\ [g_1\alpha_1(\varepsilon_2)] &= [g_2\alpha_1(\varepsilon_1)] = 0, \\ [g_1\alpha_1(\varepsilon_1)] &= [g_2\alpha_1(\varepsilon_2)] = 1, \end{aligned}$$

and $\mathbb{J}$ is the subcomplex

$$\mathbb{J}: \quad 0 \longrightarrow 0 \xrightarrow{f_4} (y_3,\, y_3') \xrightarrow{f_3} (f_3(y_3),\, f_3(y_3'),\, y_2,\, y_2') \xrightarrow{f_2} (f_2(y_2'),\, f_2(y_2)) \xrightarrow{f_1} 0$$

of $\mathbb{F}$, for

$$y_3 = \begin{bmatrix} g_1 \\ 0 \end{bmatrix},\ y_3' = \begin{bmatrix} g_2 \\ 0 \end{bmatrix} \in F_3 \quad \textit{and} \quad y_2 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_1 \end{bmatrix},\ y_2' = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_2 \end{bmatrix} \in F_2.$$

Remark. When we apply Proposition 4.6 in our proof of Theorem 3.11 (see section 6), we will take f_1' to be the element

$$\begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \end{bmatrix}$$

of F_2. It is very important that this element not be in $\mathbb{J}$. Most of the maneuvers which occur in the proof of Proposition 4.6 are designed with this goal in mind.

Proof. Select an element $\theta_3 \in T_1 \backslash S$ with the property that $\dim_k(\theta_3 S)$ is maximized. Choose a basis θ_1, θ_2 for S with $\theta_2\theta_3 = 0$ if $\dim_k(\theta_3 S) \leq 1$. Let d be the dimension of the subspace $(\theta_3, S)^2$ of T_2. The hypothesis $S^2 \neq 0$ guarantees that $\theta_1\theta_2 \neq 0$; consequently, $1 \leq d \leq 3$. Recall the map $\psi\colon I \to T_1$ from (0.1). A general position argument produces a regular sequence w_1, w_2, w_3 in I such that w_1, w_2, w_3 begins a minimal generating set for I and $\psi(w_i) = \theta_i$ for $1 \leq i \leq 3$. Let K be the ideal (w_1, w_2, w_3) and let I' be an ideal which is one tight double link over K away from I. (The ideal I' exists because I is not a complete intersection. Indeed, if y is any element of I for which w_1, w_2, w_3, y is a regular sequence, then the ideal $J = (K, y)\colon I$ is necessarily an almost complete intersection. Select y' in J so that $J = (K, y, y')$ and (K, y') is a grade four complete intersection. Let $I' = (K, y')\colon J$.)

We use the fact that tight double linkage is a symmetric relation. On the one hand, Proposition 4.4 shows that

$$\mu(I') = \mu(I) - d. \tag{4.7}$$

(In the proof of Proposition 4.4 a resolution for R/I' was built out of a resolution for R/I; we have no further need for this resolution.) On the other hand, we may use Proposition 4.3 to produce a resolution for R/I out of a resolution for R/I'. More precisely, let $\alpha_\bullet\colon \mathbb{B} \to \mathbb{A}$ be a map of DG$-$algebras over R, where $(\mathbb{A}, a)$ is the minimal resolution of R/I' and $(\mathbb{B}, b)$ is the minimal resolution of R/K. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3'$ be a basis for B_1 with $b_1(\varepsilon_i) = w_i$, for $1 \leq i \leq 2$, and $b_1(\varepsilon_3') = w_3$. (Eventually, we will replace ε_3' with $\varepsilon_3 = \varepsilon_3' - r_1\varepsilon_1 - r_2\varepsilon_2$ for some $r_i \in R$.) Orient $\mathbb{B}$ by insisting that $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3'] = 1$. Proposition 4.3 guarantees that $\mathbb{F}(\alpha_\bullet, r) = \mathbb{F}$ resolves R/I for some element $r \in R$. We see that conclusions (a) and (b) hold.

We must determine how far $\mathbb{F}$ is from the minimal resolution of R/I. Use the presentation

$$F_2 \xrightarrow{f_2} F_1 \to I$$

to see that $\mu(I) = \mu(I') + 3 - \operatorname{rank} \overline{f}_2$; thus, (4.7) shows that $\operatorname{rank} \overline{f}_2 = 3 - d \leq 2$. It quickly follows that $\overline{r} = 0$. Furthermore, we also conclude that $\overline{\beta}_2 = 0$. Indeed, if $\overline{\beta}_2 \neq 0$, then (4.5) shows that $\overline{\alpha}_2 \neq 0$; hence, $\operatorname{rank} \overline{\alpha}_1 \geq 2$ and the bound on $\operatorname{rank} \overline{f}_2$ has been contradicted. When the above information is put together, we see that $\overline{f}_1 = 0$, $\overline{f}_4 = 0$,

$$\overline{f}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{\alpha}_1 \end{bmatrix}, \quad \text{and} \quad \overline{f}_3 = \begin{bmatrix} \overline{\beta}_3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $0 \le \operatorname{rank} \overline{\alpha}_1 = \operatorname{rank} \overline{\beta}_3 = 3 - d$. If $d = 3$, then $\mathbb{F}$ is the minimal resolution of R/I.

We next consider the case $d = 2$. Let z_1 be an element of B_1 with $\overline{\alpha}_1(\overline{z}_1)$ a basis for $\operatorname{im} \overline{\alpha}_1$. The minimal resolution $\mathbb{A}$ is a Poincaré algebra so there exists $g_1 \in A_3$ with $[g_1 \alpha_1(z_1)] = 1$. The definition of β_3 shows that $\overline{\beta}_3(\overline{g}_1)$ is a basis for $\operatorname{im} \overline{\beta}_3$. Let

$$y_2 = \begin{bmatrix} 0 \\ 0 \\ z_1 \end{bmatrix} \in F_2 \quad \text{and} \quad y_3 = \begin{bmatrix} g_1 \\ 0 \end{bmatrix} \in F_3. \tag{4.8}$$

It follows that $\mathbb{F}/\mathbb{J}$ is a minimal R–resolution of R/I, where $\mathbb{J}$ is the subcomplex

$$0 \longrightarrow 0 \xrightarrow{f_4} (y_3) \xrightarrow{f_3} (f_3(y_3),\, y_2) \xrightarrow{f_2} (f_2(y_2)) \xrightarrow{f_1} 0$$

of $\mathbb{F}$.

The case $d = 1$ proceeds in an identical manner. Select $z_1, z_1' \in B_1$ and $g_1, g_2 \in A_3$ with $\overline{\alpha}_1(\overline{z}_1), \overline{\alpha}_1(\overline{z'}_1)$ a basis for $\operatorname{im} \overline{\alpha}_1$ and

$$[g_1 \alpha_1(z_1)] = [g_2 \alpha_1(z_1')] = 1, \quad [g_1 \alpha_1(z_1')] = [g_2 \alpha_1(z_1)] = 0.$$

Let y_2 and y_3 be as in (4.8) and define

$$y_2' = \begin{bmatrix} 0 \\ 0 \\ z_1' \end{bmatrix} \in F_2 \quad \text{and} \quad y_3' = \begin{bmatrix} g_2 \\ 0 \end{bmatrix} \in F_3.$$

It follows that $\mathbb{F}/\mathbb{J}$ is a minimal resolution of R/I where $\mathbb{J}$ is the subcomplex

$$0 \longrightarrow 0 \xrightarrow{f_4} (y_3,\, y_3') \xrightarrow{f_3} (f_3(y_3),\, f_3(y_3'),\, y_2,\, y_2') \xrightarrow{f_2} (f_2(y_2),\, f_2(y_2')) \xrightarrow{f_1} 0$$

of $\mathbb{F}$.

We next show that if $d \le 2$, then $\overline{\alpha}_1(\overline{\varepsilon}_1) \ne 0$. Recall that the elements $\theta_i \in T_1$ have been defined so that $\theta_2\theta_3 = 0$. The multiplication in $T_\bullet$ is induced by the multiplication in $\mathbb{F}$, so

$$0 = \theta_2\theta_3 = \text{ the class of } \begin{bmatrix} \varepsilon_2\varepsilon_3' \\ 0 \\ 0 \end{bmatrix} \text{ in } (\mathbb{F}/\mathbb{J})_2 \otimes_R k.$$

One consequence of this is that there exists an element $g \in A_3$ with $\varepsilon_2\varepsilon_3 - \beta_3(g) \in \mathfrak{m}B_2$. Multiply the last expression by ε_1 in order to draw the desired conclusion.

If $d = 1$, then we may take the element z_1 of (4.8) to be ε_1. Define $\varepsilon_3 = \varepsilon_3' - [g_1\alpha_1(\varepsilon_3')]\varepsilon_1$. Notice that $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3']$ and $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3]$ have the same value. We conclude that the minimal resolution of R/I is described in (i).

A similar calculation shows that if $d = 1$, then $\overline{a}_1(\overline{\varepsilon}_1), \overline{a}_1(\overline{\varepsilon}_2)$ is a basis for $\operatorname{im} \overline{\alpha}_1$. Indeed, the fact that $\theta_1\theta_3 = 0$ in T_2 implies that $\varepsilon_1\varepsilon_3 \in (\operatorname{im} \beta_3 + \mathfrak{m}B_2)$. It follows that $\alpha_1(\varepsilon_2) - r_0\alpha_1(\varepsilon_1) \notin \mathfrak{m}B_1$ for any $r_0 \in R$. If we take $z_1 = \varepsilon_1$, $z_1' = \varepsilon_2$, and

$$\varepsilon_3 = \varepsilon_3' - [g_1\alpha_1(\varepsilon_3')]\varepsilon_1 - [g_2\alpha_1(\varepsilon_3')]\varepsilon_2,$$

then the minimal resolution of R/I is given in (ii). $\square$

Section 5. Passing multiplication across the big from small construction.

The big from small construction of Definition 4.1 builds the complex $\mathbb{F}$ from complexes $\mathbb{A}$ and $\mathbb{B}$. The DG$-$multiplication on $\mathbb{F}$ in Theorem 5.1 is built from something akin to partial higher order multiplication on $\mathbb{A}$. (In fact, we will use complete higher order multiplication on $\mathbb{A}$ in our description of multiplication on $\mathbb{F}$; but, some lesser (not yet named) structure would work.) The idea of using the big from small construction to convert higher order multiplication on $\mathbb{A}$ into ordinary DG$-$multiplication on $\mathbb{F}$ is not new; it is the key idea in [12]. However, the $\mathbb{A}$ in [12] is a Koszul complex; and therefore, all higher order multiplication on $\mathbb{A}$ is zero; consequently, Theorem 5.1 is the first successful and interesting use of the idea. In Theorem 5.1 we also endow $\mathbb{F}$ with partial higher order multiplication (as required by Theorem 3.11). This **partial** higher order multiplication on $\mathbb{F}$ requires **complete** higher order multiplication on $\mathbb{A}$. At present it is not possible to use multiplicative structures on $\mathbb{A}$ in order to endow $\mathbb{F}$ with a complete higher order multiplication. (Indeed, such a result would require an even higher order level of multiplication on $\mathbb{A}$. Actually, we believe that an entire hierarchy of higher order multiplications live on $\mathbb{A}$; however, at present, the best existence theorem is Theorem 1.6 and the best applications are Lemma 2.3 and Theorem 3.11; and we have not pursued the issue any further.) At any rate, the fact that it is impossible to pass complete higher order multiplication across the big from small construction explains why Lemma 3.10 uses Lemma 2.3 rather than Lemma 2.2.

Theorem 5.1. *Let R be a ring in which two is a unit, r be an element of R, and $\alpha_\bullet\colon \mathbb{B}\to\mathbb{A}$ be a map of DG$-$algebras, where $(\mathbb{B},b)$ is a Koszul complex of length three over R and $(\mathbb{A},a)$ is a Poincaré DG$-$algebra of length four over R. Construct the complex $\mathbb{F}=\mathbb{F}(\alpha_\bullet,r)$ of Definition 4.1. Assume that $\varphi\colon A_2\otimes\bigwedge^5 A_1\to R$ is a complete higher order multiplication on $\mathbb{A}$. Let $h\in A_4$ and $\eta\in B_3=\bigwedge^3 B_1$ satisfy $[h]=[\eta]=1$; and define maps*

$$p\colon \bigwedge^2 A_1\to A_2 \quad \textit{and} \quad q\colon A_2\otimes A_1\to A_3 \quad \textit{by}$$

$$[x_2p(x_1\wedge x_1')]=\varphi\left(x_2\otimes x_1\wedge x_1'\wedge(\bigwedge^3\alpha_1)(\eta)\right)=[x_1q(x_2\otimes x_1')]$$

(a) *The following multiplication gives $\mathbb{F}$ the structure of a Poincaré DG$-$algebra:*

$$F_1 \otimes F_1 \to F_2 : \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \begin{bmatrix} z_1' \\ x_1' \end{bmatrix} = \begin{bmatrix} z_1 z_1' \\ -(\alpha_1 z_1)x_1' - x_1(\alpha_1 z_1') - r x_1 x_1' + p(x_1 \wedge x_1') \\ (a_1 x_1) z_1' - (a_1 x_1') z_1 + \beta_2(x_1 x_1') \end{bmatrix}$$

$$F_1 \otimes F_2 \to F_3 : \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \begin{bmatrix} z_2 \\ x_2 \\ z_1' \end{bmatrix} = \begin{bmatrix} x_1(\alpha_2 z_2) - [z_1 z_2] a_4(h) - (\alpha_1 z_1) x_2 - r x_1 x_2 + q(x_2 \otimes x_1) \\ z_1 z_1' - (a_1 x_1) z_2 - \beta_3(x_1 x_2) \end{bmatrix}$$

$$F_1 \otimes F_3 \to F_4 : \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \begin{bmatrix} x_3 \\ z_2 \end{bmatrix} = [z_1 z_2] h - x_1 x_3$$

$$F_2 \otimes F_2 \to F_4 : \begin{bmatrix} z_2 \\ x_2 \\ z_1 \end{bmatrix} \begin{bmatrix} z_2' \\ x_2' \\ z_1' \end{bmatrix} = [z_2 z_1'] h + [z_1 z_2'] h - x_2 x_2'.$$

(b) *Let $\mathfrak{A}$ be the submodule of F_1 which is generated by the fixed elements*

$$\begin{bmatrix} \varepsilon_1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z_1' \\ x_1' \end{bmatrix}, \quad \textit{and} \quad \begin{bmatrix} z_1'' \\ x_1'' \end{bmatrix},$$

with $\varepsilon_1, \varepsilon_2, z_1', z_1'' \in B_1$, and $x_1', x_1'' \in A_1$. If the map $\varphi' : F_2 \otimes F_1 \to R$ is defined by

$$\varphi'\left(\begin{bmatrix} z_2 \\ x_2 \\ \check{z}_1 \end{bmatrix} \otimes \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \right) = \begin{array}{l} -[(\alpha_1\varepsilon_2)x_1x_1'x_1''][\varepsilon_1 z_2] + [(\alpha_1\varepsilon_1)x_1x_1'x_1''][\varepsilon_2 z_2] \\ -\varphi\,(x_2 \otimes \alpha_1\varepsilon_1 \wedge \alpha_1\varepsilon_2 \wedge x_1 \wedge x_1' \wedge x_1'') \\ +[\varepsilon_1\varepsilon_2 z_1'][x_2 x_1 x_1''] - [\varepsilon_1\varepsilon_2 z_1][x_2 x_1' x_1''] - [\varepsilon_1\varepsilon_2 z_1''][x_2 x_1 x_1'], \end{array}$$

for all $z_i, \check{z}_i \in B_i$ and $x_i \in A_i$, then φ' is a partial higher order multiplication on $\mathbb{F}$, with respect to $\mathfrak{A}$.

Remark. If ε_1, ε_2, ε_3 are elements of B_1 with $\varepsilon_1\varepsilon_2\varepsilon_3 = \eta$, then conclusions (b), (c), and (d) of Theorem 3.11 hold for the elements

$$e_1 = \begin{bmatrix} \varepsilon_1 \\ 0 \end{bmatrix} \in F_1, \quad e_2 = \begin{bmatrix} \varepsilon_2 \\ 0 \end{bmatrix} \in F_1, \quad \text{and} \quad f_1' = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \end{bmatrix} \in F_2.$$

Recall that the elements e_1 and e_2 also appear in Proposition 4.6.

Proof. In this proof we denote elements of A_i, B_i, and $\bigwedge^i A_1$ by x_i, z_i, and $X^{(i)}$, respectively.

(a) The proof consists of a long but straightforward verification that the proposed multiplication on $\mathbb{F}$ is associative and satisfies the differential property (2.4). As in the proof of Lemma 2.3, we record facts that one needs in order to complete the verification, but we omit the verification itself. One of the crucial properties of a complete higher order multiplication appears as (3) in the proof of Proposition 2.5 in [16] and [17]:

$$\varphi(x_1 x_1' \otimes x_1'' \wedge x_1 \wedge X^{(3)}) + \varphi(x_1 x_1'' \otimes x_1' \wedge x_1 \wedge X^{(3)}) = 0. \tag{5.2}$$

It immediately follows that

(1) $\beta_2 p(x_1 \wedge x_1') = 0$,
(2) $\beta_3 q(x_2 \otimes x_1) = 0$,
(3) $q(\alpha_2(z_2) \otimes x_1) = 0$, and
(4) $\alpha_1(z_1)x_1''p(x_1 \wedge x_1') + \alpha_1(z_1)x_1 p(x_1'' \wedge x_1') = 0$.

It is also shown in [16, 17] that if the map $P\colon \bigwedge^5 A_1 \to A_2$ is defined by

$$[x_2 P(X^{(5)})] = \varphi(x_2 \otimes X^{(5)}), \quad \text{then} \quad P(x_1 \wedge X^{(4)})P(x_1' \wedge X^{(4)}) = 0. \tag{5.3}$$

The proof also uses the following formulas:

(5) $\beta_{i+j}((\alpha_i z_i)x_j) = z_i \beta_j(x_j)$
(6) $a_2 p(x_1 \wedge x_1') = (\beta_1 x_1')x_1 - (\beta_1 x_1)x_1' - \alpha_1\beta_2(x_1 x_1')$
(7) $q(a_3(x_3) \otimes x_1) = x_1\alpha_2\beta_3(x_3) - (\beta_1 x_1)x_3 - \alpha_3\beta_4(x_1 x_3)$
(8) $p(x_1 \wedge a_2(x_2)) + a_3 q(x_2 \otimes x_1) = \alpha_2\beta_3(x_1 x_2) - x_1\alpha_1\beta_2(x_2) - (\beta_1 x_1)x_2$
(9) $q(x_2' \otimes a_2(x_2)) + q(x_2 \otimes a_2(x_2')) = \alpha_3\beta_4(x_2 x_2') - (\alpha_1\beta_2 x_2)x_2' - (\alpha_1\beta_2 x_2')x_2$.

Formula (5) is an immediate consequence of the definition of β_i in Definition 4.1. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis for B_1 with $\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \eta$. The definition of β_i yields

$$\begin{aligned}
\beta_1(x_1) &= +\,[x_1\alpha_3(\varepsilon_1\varepsilon_2\varepsilon_3)]\\
\beta_2(x_2) &= -\,[x_2\alpha_2(\varepsilon_2\varepsilon_3)]\varepsilon_1 + [x_2\alpha_2(\varepsilon_1\varepsilon_3)]\varepsilon_2 - [x_2\alpha_2(\varepsilon_1\varepsilon_2)]\varepsilon_3\\
\beta_3(x_3) &= +\,[x_3\alpha_1(\varepsilon_3)]\varepsilon_1\varepsilon_2 - [x_3\alpha_1(\varepsilon_2)]\varepsilon_1\varepsilon_3 + [x_3\alpha_1(\varepsilon_1)]\varepsilon_2\varepsilon_3\\
\beta_4(x_4) &= -\,[x_4]\varepsilon_1\varepsilon_2\varepsilon_3.
\end{aligned}$$

Formulas (6) and (7) are reformulations of axiom (i) from Definition 1.3; and (8) and (9) are reformulations of axiom (ii).

(b) This proof is also a long straightforward verification. The identity

$$[z_1z_1'z_1'']z_1''' - [z_1z_1'z_1''']z_1'' + [z_1z_1''z_1''']z_1' - [z_1'z_1''z_1''']z_1 = 0$$

is used quite often; it holds because $\bigwedge^4 B_1 = 0$. The multiplication of part (a) yields the following values for $\bigwedge^4 F_1 \to F_4 \to R$:

$$\begin{aligned}
&[\begin{bmatrix} z_1\\0\end{bmatrix}\begin{bmatrix} z_1'\\0\end{bmatrix}\begin{bmatrix} z_1''\\0\end{bmatrix}\begin{bmatrix} z_1'''\\0\end{bmatrix}] = 0\\
&[\begin{bmatrix} z_1\\0\end{bmatrix}\begin{bmatrix} z_1'\\0\end{bmatrix}\begin{bmatrix} z_1''\\0\end{bmatrix}\begin{bmatrix} 0\\x_1'''\end{bmatrix}] = -a_1(x_1''')[z_1z_1'z_1'']\\
&[\begin{bmatrix} z_1\\0\end{bmatrix}\begin{bmatrix} z_1'\\0\end{bmatrix}\begin{bmatrix} 0\\x_1''\end{bmatrix}\begin{bmatrix} 0\\x_1'''\end{bmatrix}] = -[\alpha_1(z_1)\alpha_1(z_1')x_1''x_1''']\\
&[\begin{bmatrix} z_1\\0\end{bmatrix}\begin{bmatrix} 0\\x_1'\end{bmatrix}\begin{bmatrix} 0\\x_1''\end{bmatrix}\begin{bmatrix} 0\\x_1'''\end{bmatrix}] = -r[\alpha_1(z_1)x_1'x_1''x_1'''] + [\alpha_1(z_1)x_1'p(x_1'' \wedge x_1''')]. \quad \square
\end{aligned}$$

Section 6. The proof of Theorem 3.11.

Lemma 6.1 is the final piece in the puzzle. When we began this paper, we had a multiplicative structure on the resolution $\mathbb{M}$ of the almost complete intersection R/J. Our job was to push the multiplication down to the minimal resolution of R/J. In Lemma 3.10 we exchanged the original job for the problem of finding appropriate multiplicative structures on the minimal resolution of R/I, for some Gorenstein ideal I. In section 5 we found the appropriate multiplicative structures on the (non-minimal) resolution $\mathbb{F}$ of R/I. We still must drag the multiplicative information from $\mathbb{F}$ to the minimal resolution of R/I. This last step is accomplished in Lemma 6.1. The trick of section 3 will not work on $\mathbb{F}$. Indeed, it is impossible to view the minimal resolution of R/I as $\mathbb{F}/\mathbb{I}$ for some DG$-$ideal $\mathbb{I}$ of $\mathbb{F}$, because $\mathbb{F}$ is a Poincaré algebra and therefore F_4 is contained in every nonzero DG$-$ideal of $\mathbb{F}$. Nonetheless, the proof of Lemma 6.1 is remarkably simple. If $\mathbb{F}$ is a length four Poincaré algebra and M_i is a submodule of F_i, then

$$M_i^{\perp} = \{x_{4-i} \in F_{4-i} \mid x_{4-i}M_i = 0\}.$$

Lemma 6.1. *Let $(\mathbb{F}, f)$ be a length four Poincaré DG$-$algebra over R. Suppose that $y_2 \in F_2$ and $y_3 \in F_3$ are elements of $\mathbb{F}$ which satisfy $y_2y_2 = 0$ and $[y_2 \cdot f_3(y_3)] = 1$. Let $\mathbb{J}$ be the subcomplex*

$$0 \longrightarrow 0 \xrightarrow{f_4} (y_3) \xrightarrow{f_3} (f_3(y_3),\, y_2) \xrightarrow{f_2} (f_2(y_2)) \xrightarrow{f_1} 0$$

of $\mathbb{F}$, and let $\mathfrak{A}$ be a four generated submodule of F_1 which has the property that $y_3 \cdot \mathfrak{A} = 0$. The following statements hold.

- (a) *The complex $\mathbb{F}/\mathbb{J}$ is isomorphic to the subcomplex $(\mathbb{L}, \ell)$ of $\mathbb{F}$, where $L_4 = F_4$, $L_3 = (f_2(y_2))^{\perp}$, $L_2 = (f_3(y_3),\, y_2)^{\perp}$, $L_1 = (y_3)^{\perp}$, and $L_0 = F_0$.*
- (b) *If multiplication $*\colon \mathbb{L} \otimes \mathbb{L} \to \mathbb{L}$ is given by*

$$\begin{aligned} x_1 * x_1' &= x_1x_1' - [x_1x_1'y_2]f_3(y_3) & \qquad x_1 * x_3 &= x_1x_3 \\ x_1 * x_2 &= x_1x_2 - [x_1x_2f_2(y_2)]y_3 & \qquad x_2 * x_2' &= x_2x_2', \end{aligned}$$

  for all $x_i \in L_i$, where x_ix_j represents multiplication in $\mathbb{F}$, then $\mathbb{L}$ is a Poincaré DG$-$algebra.
- (c) *If $\varphi'\colon F_2 \otimes F_1 \to R$ is a partial higher order multiplication on $\mathbb{F}$, with respect to $\mathfrak{A}$, then the restriction of φ' to $L_2 \otimes L_1$ is a partial higher order multiplication on $\mathbb{L}$, with respect to the submodule $\mathfrak{A}$ of L_1.*

Proof. (a) A straightforward calculation shows that $f_i(L_i) \subseteq L_{i-1}$. Define $\pi_i\colon F_i \to L_i$ by $\pi_0 = \mathrm{id}$, $\pi_4 = \mathrm{id}$,

$$\pi_1(x_1) = x_1 + [x_1y_3]f_2(y_2), \quad \pi_2(x_2) = x_2 - [x_2y_2]f_3(y_3) - [x_2f_3(y_3)]y_2,$$

and $\pi_3(x_3) = x_3 - [x_3f_2(y_2)]y_3$. It is clear that $\pi_\bullet\colon \mathbb{F} \to \mathbb{L}$ is a map of complexes with kernel equal to $\mathbb{J}$.

(b) An alternate description of multiplication in $\mathbb{L}$ is

$$x_i * x_j = \pi_{i+j}(x_ix_j),$$

for $x_i \in L_i$. The multiplication $*$ satisfies differential property (2.4), because the projection $\pi_\bullet$ of $\mathbb{F}$ onto $\mathbb{L}$ and the inclusion of $\mathbb{L}$ into $\mathbb{F}$ are both maps of complexes. A small calculation is needed to see that $*$ is associative:

$$\begin{aligned}(x_1 * x_1') * x_1'' &= \pi_3\,(x_1x_1'x_1'')\\ x_1 * (x_1' * x_2) &= x_1x_1'x_2 = (x_1 * x_1') * x_2.\end{aligned}$$

For example,

$$(x_1 * x_1') * x_1'' = \pi_3(x_1x_1'x_1'') - [x_1x_1'y_2]X,$$

where

$$X = f_3(y_3)x_1'' - [f_3(y_3)x_1''f_2(y_2)]y_3 = f_3(y_3)x_1'' - f_1(x_1'')y_3 = f_4(y_3x_1'') = 0.$$

It is also clear that the multiplication on $\mathbb{L}$ exhibits Poincaré duality.

(c) The hypothesis guarantees that $\mathfrak{A}$ is a submodule of L_1. Multiplication in $\mathbb{L}$ satisfies

$$x_1 * x_1' * x_1'' * x_1''' = x_1x_1'x_1''x_1'''.$$

It follows that the restriction of the map $\Gamma\colon F_3 \otimes \bigwedge^5 F_1 \to R$ of (1.2) to $\mathbb{L}$ is exactly the same as the map Γ on $\mathbb{L}$ which is defined using the multiplication $*$. The same statement holds for $\Phi\colon S_2L_2 \otimes \bigwedge^4 L_1 \to R$. $\square$

We record the next result only for the convenience of being able to refer to it. One proves it by iterating Lemma 6.1. For this iteration to work it is necessary for $y_3' \in (f_2(y_2))^\perp$ and $y_2' \in (f_3(y_3), y_2)^\perp$; hence, these conditions are included in the hypotheses.

Corollary 6.2. *Let $(\mathbb{F}, f)$ be a length four Poincaré DG$-$algebra over R. Suppose that $y_2, y_2' \in F_2$ and $y_3, y_3' \in F_3$ are elements of $\mathbb{F}$ which satisfy $(y_2, y_2')^2 = 0$, and*

$$[y_2 \cdot f_3(y_3)] = [y_2' \cdot f_3(y_3')] = 1, \quad [y_2 \cdot f_3(y_3')] = [y_2' \cdot f_3(y_3)] = 0.$$

Let $\mathbb{J}$ be the subcomplex

$$0 \to 0 \xrightarrow{f_4} (y_3,\, y_3') \xrightarrow{f_3} (f_3(y_3),\, f_3(y_3'),\, y_2,\, y_2') \xrightarrow{f_2} (f_2(y_2),\, f_2(y_2')) \xrightarrow{f_1} 0$$

of $\mathbb{F}$, and let $\mathfrak{A}$ be a four generated submodule of F_1 with $(y_3, y_3') \cdot \mathfrak{A} = 0$. The following statements hold.

(a) *The complex $\mathbb{F}/\mathbb{J}$ is isomorphic to the subcomplex $(\mathbb{L}, \ell)$ of $\mathbb{F}$, where $L_4 = F_4$,*

$$L_3 = (f_2(y_2),\, f_2(y_2'))^\perp,\ L_2 = (f_3(y_3),\, f_3(y_3'),\, y_2,\, y_2')^\perp,\ L_1 = (y_3,\, y_3')^\perp,$$

and $L_0 = F_0$.

(b) *If multiplication $*\colon \mathbb{L} \otimes \mathbb{L} \to \mathbb{L}$ is given by*

$$\begin{aligned}x_1 * x_1' &= x_1x_1' - [x_1x_1'y_2]f_3(y_3) - [x_1x_1'y_2']f_3(y_3') &\quad x_1 * x_3 &= x_1x_3\\ x_1 * x_2 &= x_1x_2 - [x_1x_2f_2(y_2)]y_3 - [x_1x_2f_2(y_2')]y_3' &\quad x_2 * x_2' &= x_2x_2',\end{aligned}$$

for all $x_i \in L_i$, where x_ix_j represents multiplication in $\mathbb{F}$, then $\mathbb{L}$ is a Poincaré DG$-$algebra.

(c) *If $\varphi'\colon F_2 \otimes F_1 \to R$ is a partial higher order multiplication on $\mathbb{F}$, with respect to $\mathfrak{A}$, then the restriction of φ' to $L_2 \otimes L_1$ is a partial higher order multiplication on $\mathbb{L}$, with respect to the submodule $\mathfrak{A}$ of L_1.* $\square$

Proof of Theorem 3.11. We are given a grade four Gorenstein ideal I in a local ring $(R, \mathfrak{m}, k)$ and a subspace S of T_1 with $S^2 \neq 0$, where $T_\bullet = \operatorname{Tor}_\bullet^R(R/I, k)$. There is no harm in assuming that $\dim_k S = 2$. Example 3.12 shows that the theorem holds if I is a complete intersection; henceforth, we assume that I is not a complete intersection. Proposition 4.6 furnishes the resolution $\mathbb{F}$ of R/I and elements e_1 and e_2 of F_1 which fulfill conclusion (a). The resolution $\mathbb{F}$ is endowed with a Poincaré DG$-$algebra structure in Theorem 5.1. (We use the hypothesis that two is a unit in R when we apply Theorem 1.6 in order to know that there exists a complete higher order multiplication on $\mathbb{A}$.) The remark following Theorem 5.1 exhibits an element $f_1' \in F_2$ so that conditions (b), (c), and (d) also hold. The proof is complete in the case that $\mathbb{F}$ is the minimal resolution of R/I.

If $\mathbb{F}$ is not the minimal resolution, then we apply either Lemma 6.1 or Corollary 6.2 to the complex $\mathbb{J}$ of Proposition 4.6. In either case, we see that $e_1, e_2 \in L_1$ and $f_1' \in L_2$. Observe that (b) and (c) hold in the algebra $\mathbb{L}$ (with multiplication $*$). In particular, if

$$v_1 = \begin{bmatrix} z_1 \\ x_1 \end{bmatrix}$$

is an element if L_1 for some $z_1 \in B_1$ and $x_1 \in A_1$, then

$$v_1 * f_1' = \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \end{bmatrix} = \pi_3 \begin{bmatrix} 0 \\ z_1\varepsilon_3 \end{bmatrix} \in \pi_3(e_1 f_1',\, e_2 f_1') = (e_1 * f_1',\, e_2' * f_1').$$

Finally, if $\mathfrak{A}$ is a four generated submodule of L_1 with $(e_1, e_2) \subset L_1$, then $\mathfrak{A}$ is a submodule of F_1. Use part (b) of Theorem 5.1 to create a partial higher order multiplication φ' on $\mathbb{F}$, with respect to $\mathfrak{A}$. It is clear that φ' satisfies (d). Proposition 6.1 and Corollary 6.2 show that the restriction of φ' to $\mathbb{L}$ is a partial higher order multiplication which also satisfies (d). $\square$

We conclude with a comment about the hypothesis “two is a unit”. This hypothesis is used extensively in the proof of Theorem 1.6. We presume that a comparable result is valid in characteristic two; although we note that the proof in [8] is substantially different than the proof in [10]. Division by two also appears in the proof of Lemma 2.3, (5.2), and (5.3). These divisions will probably disappear when a characteristic two formulation of Theorem 1.6 is established.

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