RESOLUTIONS WHICH ARE DIFFERENTIAL GRADED ALGEBRAS
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I put a copy of this talk on my website.

1. The statement of the main result.

Theorem. *Let P be a commutative Noetherian ring and F be a resolution by finitely generated free P-modules. Assume that $F_0 = P$, F has length four, and F is self-dual, that is, $F \cong \mathrm{Hom}_P(F,P)$. Then F is a Differential Graded Algebra with Divided Powers (denoted $\mathrm{DG}\Gamma$-algebra) and F exhibits Poincaré duality.*

This result is already known if P is a local Gorenstein ring and F is a minimal resolution. The purpose of the present project is to remove the unnecessary hypotheses that P is local, P is Gorenstein, and F is minimal.

Example. The Koszul complex is an example of a $\mathrm{DG}\Gamma$-algebra with Poincaré duality.

The assertion of the Theorem is that an arbitrary F (as described in the Theorem) has a multiplication which has the same properties as exterior multiplication on the Koszul complex has. That is, there is a multiplication on F which respects the grading of F, is graded-commutative, associative, satisfies the graded product rule, $\theta_2^{(2)}$ is meaningful (and behaves like $\frac{1}{2}\theta_2^2$ would behave if 2 were a unit) for each $\theta_2 \in F_2$, and multiplication $F_i \otimes F_{4-i} \to F_4$ is a perfect pairing for all i.

2. Which change to the old theorem is the important change?

I promised to remove the hypotheses P is local, P is Gorenstein, and F is minimal from the old theorem.

I suspect that P is Gorenstein is not used in the original result.

I know that F is minimal is NOT needed in the old result. An arbitrary resolution F (as described in the hypotheses of the Theorem) over a local ring is isomorphic to a minimal resolution plus

$$0 \to \underset{\text{spot } 3}{E_1^*} \xrightarrow{\begin{bmatrix} 0 \\ \cong^* \end{bmatrix}} \underset{\text{spot } 2}{\begin{matrix} E_2 \\ \oplus \\ E_2^* \end{matrix}} \xrightarrow{\begin{bmatrix} \cong & 0 \end{bmatrix}} \underset{\text{spot } 1}{E_1} \to 0$$

One can easily extend the multiplication on the minimal resolution to a multiplication on the direct sum.

(This observation is the starting point of the present project.)

Removing the hypothesis "local" is much sneakier. The hypothesis local is used in two main spots in the original proof.

3. AN OUTLINE OF THE PROOF OF THE THEOREM, WITH A SPECIAL EMPHASIS ON WHERE THE ORIGINAL HYPOTHESIS "LOCAL" HAD BEEN USED.

An outline of the proof follows.

Step 1. Find maps $\psi_3 : F_1 \otimes F_3 \to F_4$ and $\psi_4 : D_2F_2 \to F_4$ such that

(a) ψ_3 and ψ_4 satisfy the product rule for $0 = \theta_1 \cdot \theta_4$ and $0 = \theta_2 \cdot \theta_3$, and
(b) ψ_3 is a perfect pairing and $\theta_2 \mapsto \psi_4(\theta_2 \cdot -)$ is an isomorphism $F_2 \to \mathrm{Hom}(F_2, F_4)$.

Step 2. Take ψ_3 and ψ_4 from Step 1 to be the multiplication. Make $\psi_1 : F_1 \otimes F_1 \to F_2$ do all the work. That is, figure out what ψ_1 must do in order for ψ_1, ψ_2, ψ_3, ψ_4 to be the multiplication, where $\psi_2 : F_1 \otimes F_2 \to F_3$ is defined to make $\theta_1'(\theta_1\theta_2) = (\theta_1'\theta_1)\theta_2$:

$$\psi_3(\theta_1' \otimes \psi_2(\theta_1 \otimes \theta_2)) = \psi_4(\psi_1(\theta_1' \otimes \theta_1) \cdot \theta_2).$$

Of course, this definition makes sense because ψ_3 is a perfect pairing.

It turns out that ψ_1 must satisfy 3 hypotheses.

(a) one differential condition for $F_1 \otimes F_1 \to F_2$,
(b) one differential condition for $F_1 \otimes F_2 \to F_3$, and
(c) ψ_1 factors through $\bigwedge^2 F_1$.

Step 3. Prove that there exists a ψ_1 which satisfies all of the conditions of Step 2.

Here is how Step 3 turns out.

Step 1.(a) is easy. I will show it to you.

Step 1.(b) is obvious if P is local. (Again, I will show it to you.) It is not obvious in general.

It is not hard to satisfy to find a ψ_1 which satisfies conditions 2.(a) and 2.(b). One then modifies ψ_1 (numerous times) to make the ultimate ψ_1 satisfy condition 2.(c). The proof in the local case is spread over two papers. The first paper proves the result when 2 is a unit (by dividing by 2). The second paper proves the result when 3 is a unit (by dividing by 3). Of course, in a local ring, either 2 is a unit or 3 is a unit. The present argument multiplies instead of dividing. It solves 2^n times 2.(a), 2.(b), and 2.(c), for some large n, and it solves 3 times 2.(a), 2.(b), and 2.(c) and then it solves the problem by taking the appropriate integral linear combination of the two solutions.

4. How to get started.

We learned the technique that is used in the proof the Buchsbaum-Eisenbud Am. J. paper. The technique is similar to the Tate method of killing cycles. One kills cycles of even degree with exterior variables and one kills cycles of odd degree with divided power variables.

The maps on the top from a complex. The maps on the bottom are a resolution. The comparison theorem yields a map of complexes from the top to the bottom. Focus on ψ_3 and ψ_4. The fact that the left most square commutes ensures that

$$0 = f_1(\theta_1)\cdot\theta_4 - \psi_3(\theta_1 \otimes f_3(\theta_3)) \quad \text{and}$$

$$0 = \psi_3(f_2(\theta_2)\otimes\theta_3) + \psi_4(\theta_2 \cdot f_3(\theta_3)).$$

$$\begin{tikzcd}[ampersand replacement=\&]
{\begin{matrix} F_2\otimes F_3 \\ \oplus \\ F_1\otimes F_4 \end{matrix}} \arrow[r, "{\begin{bmatrix} 1\otimes f_3 & 0 \\ f_2\otimes 1 & -1\otimes f_4 \\ 0 & f_1\otimes 1 \end{bmatrix}}"] \arrow[d] \& {\begin{matrix} D_2F_2 \\ \oplus \\ F_1\otimes F_3 \\ \oplus \\ F_4 \end{matrix}} \arrow[r, "{\begin{bmatrix} f_2 & -1\otimes f_3 & 0 \\ 0 & f_1\otimes 1 & f_4 \end{bmatrix}}"] \arrow[d, "{c_4=\begin{bmatrix}\psi_4 & \psi_3 & \mathrm{id}_{F_4}\end{bmatrix}}"] \& {\begin{matrix} F_1\otimes F_2 \\ \oplus \\ F_3 \end{matrix}} \arrow[r, "{\begin{bmatrix} -1\otimes f_2 & 0 \\ f_1\otimes 1 & f_3 \end{bmatrix}}"] \arrow[d, "{c_3=\begin{bmatrix}\psi_2 & \mathrm{id}_{F_3}\end{bmatrix}}"] \& {\begin{matrix} \bigwedge^2 F_1 \\ \oplus \\ F_2 \end{matrix}} \arrow[r, "{\begin{bmatrix} f_1 & f_2 \end{bmatrix}}"] \arrow[d, "{c_2=\begin{bmatrix}\psi_1 & \mathrm{id}_{F_2}\end{bmatrix}}"] \& F_1 \arrow[r, "f_1"] \arrow[d, "="] \& F_0 \arrow[d, "="] \\
0 \arrow[r] \& F_4 \arrow[r, "f_4"] \& F_3 \arrow[r, "f_3"] \& F_2 \arrow[r, "f_2"] \& F_1 \arrow[r, "f_1"] \& F_0.
\end{tikzcd}$$

We modify ψ_3 and ψ_4 in order to make them induce the appropriate isomorphisms. (No modification is needed in the local case.) In the 2^n part of the argument, we keep ψ_3 and ψ_4, ignore the given ψ_1 and ψ_2, and build a new ψ_1 (and ψ_2) from scratch! In the 3 part of the argument we modify the ψ_1 and ψ_2 that come from the Buchsbaum-Eisenbud-Tate technique.

5. Here is why no modification of ψ_3 and ψ_4 is needed in the local case.

Let $(-)^\vee$ denote the functor $\mathrm{Hom}_P(-,F_4)$. Define

$$\begin{tikzcd}
F \arrow[d, "\Phi"] & 0 \arrow[r] & F_4 \arrow[r, "f_4"] \arrow[d, "\Phi_4"] & F_3 \arrow[r, "f_3"] \arrow[d, "\Phi_3"] & F_2 \arrow[r, "f_2"] \arrow[d, "-\Phi_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\Phi_1"] & F_0 \arrow[d, "\Phi_0"] \\
F^\vee & 0 \arrow[r] & F_0^\vee \arrow[r, "f_1^\vee"] & F_1^\vee \arrow[r, "f_2^\vee"] & F_2^\vee \arrow[r, "f_3^\vee"] & F_3^\vee \arrow[r, "f_4^\vee"] & F_4^\vee
\end{tikzcd}$$

by

$$\begin{aligned}
\Phi_0(\theta_0) &= \theta_0 \cdot -,\\
\Phi_1(\theta_1) &= \psi_3(\theta_1 \otimes -),\\
\Phi_2(\theta_2) &= \psi_4(\theta_2 \cdot -),\\
\Phi_3(\theta_3) &= \psi_3(- \otimes \theta_3), \text{ and}\\
\Phi_4(\theta_4) &= - \cdot \theta_4.
\end{aligned}$$

It is easy to see that Φ is a map of complexes.

If P is a local ring and F is a minimal resolution, then Φ is a map of complexes from one minimal resolution of $P/\operatorname{im} f_1$ to another minimal resolution of $P/\operatorname{im} f_1$ and Φ_0 is an isomorphism. It follows immediately that Φ is an isomorphism of complexes.

6. Application to matrix factorization

Project. Let P be a commutative Noetherian ring, $\mathfrak{K}$ be an ideal of P generated by a regular sequence, g be a regular element of P, and $\overline{P} = P/(g)$. The goal is to resolve $\overline{P}/\mathfrak{K}\overline{P}$ by free $\overline{P}$- modules. In particular, the goal is to find the matrix factorization of g which is the infinite tail of this resolution.

The motivation is to study invariants of Frobenius powers $(\mathfrak{K}\overline{P})^{[q]}$. (Many people do this; including, for example, Bernd.)

Here is an answer to this project when $\mathfrak{K}$ has grade 4. Let F be a resolution of $P/(\mathfrak{K} : g)$ by free P-modules. I insist that F be a DGΓ-algebra with Poincaré duality. Let $X : F_1 \to F_2$ be a homomorphism which satisfies a few properties. Let K be a Koszul complex which resolves $P/\mathfrak{K}$, $\alpha : K \to F$ be a map of DGΓ-algebras which extends the natural quotient map $P/\mathfrak{K} \to P/(\mathfrak{K} : g)$. Define $\beta : F \to K$ to be "an adjoint" of α:

$$[\beta_i(\theta_i) \wedge \phi_{4-i}]_K = [\theta_i \cdot \alpha_{4-i}(\phi_{4-i})]_F.$$

Let ω be a basis element of K_4. The matrix factorization of $\beta_0(1) = [\alpha_4(\omega)]_K$ "is"

$$\begin{bmatrix} X & \alpha_2 & f_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_2 \\ \beta_2 \\ X^{\mathrm{adj}} \end{bmatrix}.$$

The map X is chosen so that

$$\begin{tikzcd}
0 \arrow[r] & F_4 \arrow[r, "f_4"] \arrow[d, "w_4=0"'] & F_3 \arrow[r, "f_3"] \arrow[d, "w_3"] \arrow[dl, "0"'] & F_2 \arrow[r, "f_2"] \arrow[d, "w_2"] \arrow[dl, "X^{\mathrm{adj}}"'] & F_1 \arrow[r, "f_1"] \arrow[d, "w_1"] \arrow[dl, "X"'] & F_0 \arrow[d, "w_0=0"] \arrow[dl, "0"'] \\
0 \arrow[r] & F_4 \arrow[r, "f_4"] & F_3 \arrow[r, "f_3"] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0,
\end{tikzcd}$$

is a homotopy, where $w_i : F_i \to F_i$ is given by

$$w_i(\theta_i) = \beta_0(1)\theta_i - (\alpha_i \circ \beta_i)\theta_i.$$

The map $X^{\mathrm{adj}} : F_2 \to F_3$ is defined by

$$X^{\mathrm{adj}}(\theta_2)\cdot\theta_1 = \theta_2 \cdot X(\theta_1).$$

(The ideals $(\mathfrak{K}, g)$ and $(\mathfrak{K}, \beta_0(1))$ are equal. Once one knows a matrix factorization of $\beta_0(1)$, then it is not difficult to record the corresponding matrix factorization of g. The answer will not be as pretty.)

7. Here is an alternate description of $X : F_1 \to F_2$.

If one takes a basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ for K_1, then

$$w_1(\theta_1) = \begin{cases} \quad [\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)]_F \cdot \theta_1 - [\theta_1\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_1) \\ +[\theta_1\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_2) - [\theta_1\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_3) \\ +[\theta_1\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)]_F \cdot \alpha_1(\varepsilon_4) \end{cases}$$

and

$$w_2(\theta_2) = \begin{cases} -[\theta_2\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2) + [\theta_2\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3) \\ -[\theta_2\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)]_F \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_4) - [\theta_2\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)]_F \cdot \alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4) \\ +[\theta_2\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3)]_F \cdot \alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4) - [\theta_2\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_4)]_F \cdot \alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3) \\ +[\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)]_F \cdot \theta_2. \end{cases}$$

My student Susan Palmer Slattery proved that in a local ring P in which 2 is a unit there always exists a map $X : F_1 \to F_2$ with $f_2 \circ X = w_1$ and $X \circ f_2 + f_3 \circ X^{\mathrm{adj}} = w_2$. She and I used this map to prove that if I is a grade 4 almost complete intersection in P, then the minimal resolution of P/I by free P-modules is a DGΓ-algebra. The present work proves that X exists without assuming that P is local and without assuming that 2 is a unit.