THE COTANGENT COMPLEX, QUASI-COMPLETE INTERSECTIONS IN THE SENSE OF AVRAMOV, AND EXACT ZERO DIVISORS.

The talk has two parts.
1. qcis and ezds about joint work with Liana Şega and Adela Vraciu
2. André-Quillen cohomology and the cotangent complex.
I am happy to e-mail a copy of this talk.

1. Qcis and ezds.
Throughout this part of the talk $(R,\mathfrak{m})$ is a local ring.

Definition. Let I be an ideal of the local ring R. Let $\underline{f} = f_1,\dots,f_n$ be a minimal generating set of I, E be the Koszul complex on $\underline{f}$, and $S = R/I$. The ideal I is a *quasi-complete intersection* (q.c.i.) if $\mathrm{H}_1(E)$ is free over S and the canonical homomorphism

$$\textstyle\bigwedge_S^{\bullet}(\mathrm{H}_1(E)) \longrightarrow \mathrm{H}_{\bullet}(E)$$

is an isomorphism.

Example. Every complete intersection is a q.c.i..

Example. Principal q.c.i. ideals are well understood. If $x \neq 0$ is an element of $\mathfrak{m}$, then the ideal (x) is q.c.i. if and only if x is either regular or else $\operatorname{ann}(x) \cong R/(x)$; in the last case we say that x is an *exact zero-divisor*. The name EZD was introduced by Henriques and Şega a couple of years ago.

Remark. R. Wiegand and his collaborators use EZD's to create totally reflexive modules and totally acyclic resolutions. I noticed while preparing this talk that our paper sends the reader to a completely different set of applications for EZD.

Example. Let I be an ideal in a local ring R. If there exists a faithfully flat extension $R \to R'$, a local ring Q and complete intersection ideals $\mathfrak{a} \subseteq \mathfrak{b}$ of Q such that $R' = Q/\mathfrak{a}$ and $R'/IR' = Q/\mathfrak{b}$, then I is a qci. Such q.c.i. ideals are *embedded.*

• Qci ideals were introduced in recent paper by of Avramov, Henriques, and Şega because

Fact. Let I be an ideal in a local ring R. Then

I is a qci
$\iff$ the homomorphism $\phi : R \to R/I$ satisfies the conclusion of the Quillen conjecture
$\iff$ $D^n(\phi,-) = 0$ for all n with $3 \leq n$, where $D^n(\phi,-)$ is André-Quillen cohomology.

Conjecture. [Quillen] Let $\phi : A \to B$ be a homomorphism of commutative Noetherian rings. Then the following statements are equivalent:

(1) $D^n(\phi,-) = 0$ for all large n
(2) $D^n(\phi,-) = 0$ for all n with $3 \leq n$

Goal. The ultimate goal is to understand qci ideals in order to prove or disprove the Quillen conjecture for local surjections and then use Avramov's machinery in particular his 1999 Annals paper which deals with locally complete intersections to decide the fate of the Quillen conjecture in the general case.

Immediate Question. Have I already described all qcis? In particular, how will I know if a given two-generated grade zero ideal is a qci?

Lemma. *Let I be a grade zero ideal of the local ring $(R,\mathfrak{m})$ which is minimally generated by (f_1, f_2) Then the following statements are equivalent:*
(a) *I is a q.c.i. and*
(b) *there exist elements a,b,c,d in $\mathfrak{m}$ with*

$$R^4 \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \xrightarrow{d_0} R \xrightarrow{d_1^{\mathrm{T}}} R^2,$$

an exact sequence, with $d_0 = [ad-bc]$, $d_1 = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$,

$$d_2 = \begin{bmatrix} -f_2 & a & b \\ f_1 & c & d \end{bmatrix}, \quad \textit{and} \quad d_3 = \begin{bmatrix} -c & -d & a & b \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix}.$$

• The next result is a step toward proving that we already know all qcis or a step toward finding where the unknown ones live.

Theorem. *Let $(R,\mathfrak{m})$ be an Artinian local ring which is not a complete intersection. Assume that one of the following holds:*
(a) $\mathfrak{m}^3 = 0$*;*
(b) $\mathfrak{m}^4 = 0$ *and R is Gorenstein.*

Then every q.c.i. ideal of R is generated by an exact zero divisor.

Our most important result is:

Example. Fix a field k. Let

$$B = \frac{k[X_1, X_2, \ldots, X_5]}{(X_1^2 - X_2X_3,\ X_2^2 - X_3X_5,\ X_3^2 - X_1X_4,\ X_4^2,\ X_5^2,\ X_3X_4,\ X_2X_5,\ X_4X_5)},$$

We denote the image of the variable X_i in B by x_i. Let f_1 and f_2 be the elements

$$f_1 = x_1 + x_2 + x_4 \quad \text{and} \quad f_2 = x_2 + x_3 + x_5.$$

of B and I be the ideal $(f_1, f_2)B$. Then the following statements hold:

(a) B is an Artinian local ring with Hilbert series $H_B(z) = 1+5z+7z^2+3z^3$,
(b) the algebra B is Koszul,
(c) $\zeta^2(B)$ is a 1-dimensional vector space, and
(d) the ideal I of B is a (grade zero) non-embedded q.c.i. which does not contain any exact zero divisors ideal.

•The degree two component of the center of the homotopy Lie algebra of B is denoted by $\zeta^2(B)$. Assertion (c) is the key to showing that I is not an embedded qci.

•We use the fact that B is a Koszul algebra in the calculation of $\zeta^2(B)$.

• My favorite result from the paper.

Theorem. *Let P be a standard graded polynomial ring over the field k, I be a homogeneous ideal of P which is primary to the maximal homogeneous ideal of P. Assume that I is generated by forms of the same degree. Let Θ_1 and Θ_2 be homogeneous elements of P whose images in $S = P/I$ form an* **E***xact* **P***air of* **H***omogeneous* **Z***ero* **D***ivisors. Then $\Theta_1 \cdot \Theta_2$ is a minimal generator of I.*

• An application:

Theorem. *Let k be an algebraically closed field of characteristic different from 2 and let P denote the polynomial ring $k[x_1,\dots,x_n]$. If $5 \le n$ and I is an ideal in P generated by n general quadratic forms. Then $P/(I)$ does not contain any exact zero divisors.*

2. André-Quillen cohomology.

Let $\phi : A \to B$ be a homomorphism of commutative Noetherian rings and M be a B-module.

$$\{D^n(-|A,-)\} \text{ are the derived functors of } (B,M) \mapsto \mathrm{Der}_A(B,M)$$
$$\{D_n(-|A,-)\} \text{ are the derived functors of } (B,M) \mapsto \Omega_{B|A} \otimes_B M$$

Let P be a simplicial resolution of B over A. So, P is an A-algebra and $P = \{P_n\}$ is a family of A-algebras with degeneracy maps and face maps. The object

$$\Omega_{P|A} \otimes_P B$$

is a simplicial B-module; the associated complex of B-modules is called the *cotangent complex* $\mathcal{L}_\phi$ of ϕ. Define the André-Quillen cohomology module $D^n(\phi,M)$ and homology modules $D_n(\phi,M)$ by

$$D^n(\phi,M) = \mathrm{H}^n(\mathrm{Hom}_B(\mathcal{L}_\phi,M))$$
$$D_n(\phi,M) = \mathrm{H}_n(\mathcal{L}_\phi \otimes_B M)$$

A little history:

If $A \to B \to C$ are ring homomorphisms and M is a C-module, then there are well-known exact sequences:

$$0 \to \mathrm{Der}_B(C,M) \to \mathrm{Der}_A(C,M) \to \mathrm{Der}_A(B,M) \quad \text{and} \tag{1}$$

$$\Omega_{B|A} \otimes_A C \to \Omega_{C|A} \to \Omega_{C|B} \to 0 \tag{2}$$

Matsumura (Commutative Algebra) calls (2) "The first fundamental exact sequence". Some authors (For example, Avramov, Iyengar) refer to (2) as the Jacobi-Zarsiki exact sequence.

Grothendieck (1964) was the first to extend (1):

$$\begin{aligned} &0 \to \mathrm{Der}_B(C,M) \to \mathrm{Der}_A(C,M) \to \mathrm{Der}_A(B,M) \\ &\to \mathrm{Exalcomm}_B(C,M) \to \mathrm{Exalcomm}_A(C,M) \to \mathrm{Exalcomm}_A(B,M), \end{aligned} \tag{3}$$

where "Exalcomm" stands for Commutative Algebra Extensions; so $\mathrm{Exalcomm}_B(C,M)$ parameterizes the set of isomorphism classes of B-algebra surjections $E \twoheadrightarrow C$ with kernel equal to M and $M^2 = 0$.

Lichtenbaum and Schlessinger (1967) extend (3) a little further

$$\begin{aligned} 0 &\to \mathrm{Der}_B(C,M) \to \mathrm{Der}_A(C,M) \to \mathrm{Der}_A(B,M) \\ &\to T^1(C|B,M) \to T^1(C|A,M) \to T^1(B|A,M) \\ &\to T^2(C|B,M) \to T^2(C|A,M) \to T^2(B|A,M) \end{aligned}$$

with $\mathrm{Der}_B(C,M) = T^0(C|B,M)$ and $\mathrm{Exalcomm}_B(C,M) = T^1(C|B,M)$.

There are three advantages to the Lichtenbaum and Schlessinger approach.

(a) There is a 3-term complex $\mathbb{L}_{B|A}$ so that $T^i(B|A,M) = \mathrm{H}^i(\mathrm{Hom}_B(\mathbb{L}_{B|A},M))$
(b) It makes sense to think of homology modules $T_i(B|A,M) = (\mathbb{L}_{B|A} \otimes_B M)$.
(c) The T^i actually mean something. Take A to be a field.
  - $T^1(B|A,B)$ parameterizes the set of first order deformations of B over A.
  - $T^2(B|A,B)$ measures the obstructions to lifting the first order deformations to second order deformations.

- That is, T^1 parameterizes the set of isomorphism classes of flat maps $A' \to B'$ with $B' \otimes_{A'} A = B$, where $A' = A[\varepsilon]/(\varepsilon^2)$.

- Each isomorphism class of first order deformations B' corresponds to an element $\delta_{B'}$ of T^2 and B' can be lifted to a second order deformation B'' of B if and only if $\delta_{B'} = 0$. Of course, this B'' would have a flat map $A'' \to B''$ with $B'' \otimes_{A''} A' = B'$ for $A'' = k[\varepsilon]/(\varepsilon^3)$.

- The situation is screaming to have a complete cohomology theory.

• The definitive criterion for regularity in terms of André-Quillen homology is due to André.

Theorem. *Let $\phi : A \to B$ be a homomorphism of Noetherian rings. The following conditions are equivalent*:

(a) ϕ *is regular,* (that is, ϕ is flat and has geometrically regular fibers (i.e., $B \otimes_A \underline{\ell}$ is regular whenever $A \to \underline{\ell}$ is a homomorphism essentially of finite type and $\underline{\ell}$ is a field)),
(b) $D^n(B|A;-) = 0$ for each $1 \leq n$,
(c) $D^1(B|A;-) = 0$.

• Vanishing of André-Quillen homology is linked to the locally complete intersection property by following result, which was proved by Lichtenbaum and Schlessinger (TAMS 1967), André (1974), and Quillen (1970) in the case when is essentially of finite type, and by Avramov (1999 Annals) in the general case.

Theorem. *Let $\phi : A \to B$ be a homomorphism of Noetherian rings. The following conditions are equivalent*:

(a) *$\phi : A \to B$ is locally complete intersection.*
(b) *$D^n(B|A;-) = 0$ for $n \geq 2$.*
(c) *$D^2(B|A;-) = 0$.*

• The Quillen conjecture.

Conjecture. Let $\phi : A \to B$ be a homomorphism of Noetherian rings. The following conditions are equivalent:

(a) $D^n(B|A,-) = 0$ for all $n \geq 3$.
(b) $D^3(B|A,-) = 0$.
(c) $D^n(B|Q,-) = 0$ for all large n

If (a) occurs, then ϕ is called a quasi-complete intersection.