THE WEAK LEFSCHETZ PROPERTY FOR EQUICHARACTERISTIC, STANDARD GRADED, ARTINIAN GORENSTEIN ALGEBRAS OF EMBEDDING DIMENSION FOUR AND SOCLE DEGREE THREE

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ABSTRACT. Let $\boldsymbol{k}$ be an arbitrary field and A be a standard graded Artinian Gorenstein $\boldsymbol{k}$-algebra of embedding dimension four and socle degree three. Then, except for exactly one exception, A has the weak Lefschetz property. Furthermore, the exception occurs only in characteristic two.

CONTENTS

1. INTRODUCTION.

The Lefschetz property is a ring-theoretic abstraction of the Hard Lefschetz Theorem for compact Kähler manifolds. Let $\boldsymbol{k}$ be a field. A graded $\boldsymbol{k}$-algebra A, equal to $\bigoplus_{i=0}^{s} A_i$, has the weak Lefschetz property if there is a linear form $\ell \in A_1$ so that multiplication by ℓ from A_i to A_{i+1} has maximal rank for each i. (Similarly, A has the strong Lefschetz property if multiplication by ℓ^s has maximal rank in each degree for every positive integer s for some $\ell \in A_1$.) Stanley introduced the concept

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We made extensive use of Macaulay2 [15]. We are very appreciative of this computer algebra system.

in [27] where he proved that if the characteristic of $\boldsymbol{k}$ is zero, $P = \boldsymbol{k}[x_1, \dots, x_n]$, and $A = P/(x_1^{a_1}, \dots, x_n^{a_n})$, then A has the strong Lefschetz property. Other early proofs of Stanley's theorem were also given by [25] and [28].

Our interest in the weak Lefschetz property for fields of arbitrary characteristic is a consequence of our desire to record the minimal resolution of A by free P-modules in terms of the coefficients of the Macaulay inverse system which determines A. See [8], [9], and especially [10], where this project is carried out for quotient rings A with Castelnuovo regularity two as a P-module, and [11] for quotient rings A of regularity three in three variables. (We also have results along these lines pertaining to ideals of regularity three in four and five variables.) We are able to carry out this project provided A has the weak Lefschetz property, independent of the characteristic of $\boldsymbol{k}$. The absence of the weak Lefschetz property is an obstruction to this project; but positive characteristic, in and of itself, does not cause any problem.

The weak Lefschetz property is very sensitive to change in characteristic. In positive characteristic p, Brenner and Kaid [4] gave an explicit description of those d and p for which $P/(x_1^d, x_2^d, \dots, x_n^d)$ has the weak Lefschetz property when P equals $\boldsymbol{k}[x_1, \dots, x_n]$, $\boldsymbol{k}$ is a field of characteristic p, and $n = 3$. The paper [21] is devoted to the analogous project for $4 \le n$.

Artinian rings with socle degree three are somewhat mysterious. Bøgvad's [3] examples of Artinian Gorenstein rings with transcendental Poincaré series have socle degree three. Rossi and Şega [26] prove that if R is a compressed Artinian Gorenstein local ring with socle degree not equal to three, then the Poincaré series of all finitely generated R-modules are rational, sharing a common denominator. Similarly, it is shown in [20] that if R is a compressed local Artinian ring with odd top socle degree at least five, then the Poincaré series of all finitely generated R-modules are rational, sharing a common denominator. The same conclusion does not hold when the top socle degree is three.

Theorem 1.1 is the main result in the paper.

Theorem 1.1. *Let $\boldsymbol{k}$ be a field and A be a standard graded Artinian Gorenstein $\boldsymbol{k}$-algebra of embedding dimension four and socle degree three. If the characteristic of $\boldsymbol{k}$ is different than two, then A has the weak Lefschetz property. If the characteristic of A is equal to two, then A has the weak Lefschetz property if and only if A is not isomorphic to*

$$\frac{\boldsymbol{k}[x,y,z,w]}{(xy, xz, xw, y^2, z^2, w^2, x^3 + yzw)}. \tag{1.1.1}$$

We first demonstrate that the $\boldsymbol{k}$-algebra A of (1.1.1) does not satisfy the weak Lefschetz property. Indeed, let $\ell = ax + by + cz + dw$ be an arbitrary nonzero linear form in A. If at least one of the parameters b, c, or d is nonzero, then let ℓ' be the nonzero linear form $\ell' = by + cz + dw \in A_1$. Observe that

$$\ell\ell' = ax(by + cz + dw) + b^2y^2 + c^2z^2 + d^2w^2 = 0 \in A.$$

If $b = c = d = 0$, then let ℓ' be the nonzero linear form $\ell' = y$ of A_1. Observe that $\ell\ell' = axy = 0$ in A. In either case, the arbitrary nonzero linear form ℓ is zero divisor on A.

In the rest of the paper we show that if A satisfies the hypotheses of Theorem 1.1, but is not isomorphic to the ring of (1.1.1), then A has the weak Lefschetz property. We use a Macaulay inverse system for A.

Let U be the vector space A_1, P be the polynomial ring $P = \operatorname{Sym}_\bullet U$, I be a homogeneous ideal of P with A isomorphic to P/I, U^* be the dual space $\operatorname{Hom}_{\boldsymbol{k}}(U, \boldsymbol{k})$ of U, and $D_\bullet U^*$ be the divided power $\boldsymbol{k}$-algebra $\bigoplus_{0\le i} D_i U^*$, with

$$D_i U^* = \operatorname{Hom}_{\boldsymbol{k}}(\operatorname{Sym}_i U, \boldsymbol{k}).$$

The rules for a divided power algebra are recorded in [16, Section 7] or [7, Appendix 2]. (In practice these rules say that $w^{(n)}$ behaves like $w^n/(n!)$ would behave if $n!$ were a unit in R.)

Macaulay duality guarantees that

$$\operatorname{ann}_{D_\bullet U^*} I$$

is a cyclic P-submodule of $D_\bullet U^*$ generated by an element in $D_3 U^*$. (Any generator $\operatorname{ann}_{D_\bullet U^*} I$ in $D_3 U^*$ is called a Macaulay inverse system for A.) Furthermore, if ϕ_3 is a Macaulay inverse system for A, then

$$I = \operatorname{ann}_P \phi_3.$$

The hypothesis that A has embedding dimension four ensures that

(1.1.2) $$\ell\phi_3 \neq 0 \quad \text{for any nonzero } \ell \text{ in } U.$$

We observe that the Macaulay inverse system for the $\boldsymbol{k}$-algebra A of (1.1.1) is

(1.1.3) $$\phi_3 = x^{*(3)} + y^*z^*w^*.$$

In section 3 we put the Macaulay inverse system into the form

(1.1.4) $$\alpha\phi_3 = x^{*(3)} + x^*\phi_{2,0} + \phi_{3,0}$$

or

(1.1.5) $$\alpha\phi_3 = x^{*(2)}y^* + x^*\phi_{2,0} + \phi_{3,0},$$

for some unit α, with $\phi_{i,0} \in D_i(\boldsymbol{k}y^* \oplus \boldsymbol{k}z^* \oplus \boldsymbol{k}w^*)$. We study various cases depending on whether ϕ_3 has form (1.1.4) or (1.1.5) and also depending on how complicated

$\phi_{2,0}$ is. We treat the four significant cases in Sections 4, 5, 6, and 7. All of our calculations employ the homomorphism

$$\Gamma_{\phi_3} : D_d U \otimes \bigwedge^d U \to \bigwedge^d U^*$$

which is introduced in section 2.B. The connection between Γ_{ϕ_3} and the weak Lefschetz property is explained in Lemma 2.9. Roughly speaking,

$$\Gamma_{\phi_3}(\ell^{(d)} \otimes x_1 \wedge \ldots \wedge x_d) = \ell x_1 \phi_3 \wedge \ldots \wedge \ell x_d \phi_3 \in \bigwedge^d U^*,$$

for $\ell, x_1, \ldots, x_d \in U$ and $\phi_3 \in U^*$. It is shown in Lemma 2.9 that if ϕ_3 is a Macaulay inverse system for A and $\ell\phi_3$ is nonzero for all nonzero ℓ in U, then

$$A \text{ has the weak Lefschetz property} \iff \Gamma_{\phi_3} \text{ is not identically zero.}$$

The hypothesis that $\ell\phi_3$ is nonzero whenever ℓ in U is nonzero is innocuous. It merely says that the embedding dimension of A is equal to the number of variables of P. If this hypothesis is not satisfied, then one can view A as a quotient of a polynomial ring with one fewer variable than P has.

Traditionally, the Lefschetz properties are studied in a graded $\boldsymbol{k}$-algebra where $\boldsymbol{k}$ is a field of characteristic zero. In particular, for example, Gondim and Zappala [13, Cor. 5.5] have proven that a standard graded Gorenstein $\boldsymbol{k}$-algebra of small codimension, with socle degree three, and presented by quadrics, has the weak Lefschetz property, provided the field $\boldsymbol{k}$ has characteristic zero. Duality is obtained using the algebra of differential operators $Q = \boldsymbol{k}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$. Every graded Artinian Gorenstein algebra A has a presentation of the form

$$A \cong Q/\operatorname{ann}_Q F, \tag{1.1.6}$$

for some n and some homogeneous polynomial $F \in \boldsymbol{k}[x_1, ..., x_n]$, where

$$\operatorname{ann}_Q F = \{\phi \in Q \mid \phi(F(x)) = 0\}.$$

The Hessian of the form F is the determinant of the square matrix $(\frac{\partial^2 F}{\partial x_i \partial x_j})$ of second order partial derivatives of F. When $\boldsymbol{k}$ has characteristic zero, Watanabe (see [29, 22]) has shown that (1.1.6) fails to have the Strong Lefschetz Property if and only if one of the non-trivial higher Hessians of F vanishes. This result has been generalized to the weak Lefschetz property using "mixed Hessians", see [13].

For the time being keep $\boldsymbol{k}$ a field of characteristic zero and let F be a homogeneous form in $\boldsymbol{k}[x_1, \ldots, x_n]$. Hesse [17, 18] believed that the Hessian of F vanishes if and only if the projective variety X, defined by F, is a cone. However, Gordan and Noether [14] proved that while Hesse's claim is true when the degree of F is 2 or $n \leq 4$, it is false for $5 \leq n$ and for forms of degree at least three. Gordan and Noether realized that X being a cone is equivalent to the condition that the partial derivatives of F are $\boldsymbol{k}$-linearly dependent, while F has vanishing Hessian if and only if the partial derivatives of F are $\boldsymbol{k}$-algebraically dependent. A class of forms F, discussed both in [14] and by Perazzo [24], with vanishing Hessian and for which $V(F)$ is not a cone, are the Perazzo forms

$$F = x_1p_1 + \cdots + x_ap_a + G \in \boldsymbol{k}[x_1,\ldots,x_a,u_1,\ldots,u_b],$$

where $p_1,\ldots,p_a$ and G are in $\boldsymbol{k}[u_1,\ldots,u_b]$, and $p_1,\ldots,p_a$ are linearly independent, but algebraically dependent. In particular, for $a+b=5$, all non-cones defined by a form with vanishing Hessian are defined by a Perazzo form. (See [14] or [30, Theorem 7.3].) The Lefschetz properties of rings defined by Perazzo forms in characteristic zero are investigated in the recent papers [12, 1].

J. Watanabe and M. de Bondt [30] have written a detailed modern argument for the Gordan-Noether Theorem. The paper [5] uses geometric techniques to give a new proof of the Gordan-Noether Theorem. In both of these papers the field $\boldsymbol{k}$ has characteristic zero.

We work in arbitrary characteristic; so we do not take literal partial derivatives. Instead we use the divided power algebra $D_\bullet U^*$ which is associated to the polynomial ring $P = \operatorname{Sym}_\bullet U$ for the vector space U over the field $\boldsymbol{k}$. We replace the "Hessian of F" with the homomorphism "Γ_{ϕ_3}" of Section 2.B.

In Section 8 we state and prove the three variable version of the Main Theorem (Theorem 1.1). Our precise formulation of the three variable version (see Lemma 8.2) is used in the inductive part of the proof of Theorem 1.1. (See the case $r=0$ in Lemma 5.3.) Furthermore, we prove the three variable version using the same argument as we use for the four variable version; except there are fewer cases and each calculation is more straightforward. The reader might want to read Section 8 as a preparation for reading the proof of Theorem 1.1.

2. NOTATION, CONVENTIONS, AND ELEMENTARY RESULTS.

2.A. **The language.**

Conventions 2.1. (a) The graded algebra $A = \bigoplus_{0\le i} A_i$ is a *standard graded A_0-algebra* if A_1 is finitely generated as an A_0-module and A is generated as an A_0-algebra by A_1.

(b) Let $\boldsymbol{k}$ be a field and $A = \bigoplus A_i$ be a standard graded $\boldsymbol{k}$-algebra. Then A has the *weak Lefschetz property* if there exists a linear form ℓ of A_1 such that the $\boldsymbol{k}$-module homomorphism

$$\mu_\ell : A_i \to A_{i+1}$$

has maximal rank for each index i, where μ_ℓ is multiplication by ℓ. (A homomorphism $\xi : V \to W$ of finitely generated $\boldsymbol{k}$-modules has *maximal rank* if $\operatorname{rank}\xi$ is equal to $\min\{\dim V, \dim W\}$.)

(c) If $A=\bigoplus_{i=0}^{\sigma}A_i$ is an Artinian standard-graded $\boldsymbol{k}$-algebra, then A is *Gorenstein with socle degree* σ if A_σ is a one dimensional vector space and every ideal of A contains A_σ.

(d) In this paper $\boldsymbol{k}$ is an arbitrary field (unless otherwise noted) and Hom, Sym, D, $\bigwedge$, $\wedge$, and $\otimes$ mean $\mathrm{Hom}_{\boldsymbol{k}}$, $\mathrm{Sym}^{\boldsymbol{k}}$, $D^{\boldsymbol{k}}$, $\bigwedge_{\boldsymbol{k}}$, $\wedge_{\boldsymbol{k}}$ and $\otimes_{\boldsymbol{k}}$, respectively.

(e) If U is a vector space over the field $\boldsymbol{k}$, then $T_\bullet U$, $\mathrm{Sym}_\bullet U$, $D_\bullet U$, and $\bigwedge^\bullet U$ are the tensor algebra, symmetric algebra, divided power algebra, and exterior algebra of U over $\boldsymbol{k}$, respectively. See, for example, [23] or [7].

(f) If M is a matrix, then $\det M=|M|$ is the determinant of M.

(g) If f is a homomorphism, then we write $\operatorname{im} f$ and $\ker f$ for the image and kernel of f, respectively.

(h) If U is a vector space, then $\dim U$ is the dimension of U as a vector space.

Conventions 2.2. (a) If U is a finite dimensional vector space over the field $\boldsymbol{k}$, then U^* represents $\mathrm{Hom}_{\boldsymbol{k}}(U,\boldsymbol{k})$ and $D_\bullet U^*$ represents the divided power algebra $\bigoplus_{i=0}^{\infty}D_iU^*$ for

$$D_iU^*=\mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}_i U,\boldsymbol{k}).$$

(b) If $x_1,\dots,x_d$ is a basis for U, then the set of monomials of degree i in $x_1,\dots,x_d$, denoted $\binom{x_1,\dots,x_d}{i}$, is a basis for the i-th symmetric power, $\mathrm{Sym}_i U$, of U and the set of homomorphisms

$$\left\{m^*\;\middle|\; m\in\binom{x_1,\dots,x_d}{i}\right\}$$

is a basis for D_iU^* where $m^*:\mathrm{Sym}_i U\to\boldsymbol{k}$ is the $\boldsymbol{k}$-module homomorphism with

$$m^*(m')=\begin{cases}1, & \text{if } m=m', \text{ and}\\ 0, & \text{if } m\neq m',\end{cases} \tag{2.2.1}$$

for m,m' in $\binom{x_1,\dots,x_d}{i}$.

(c) We make much use of the structure of $D_\bullet U^*$ as a module over $\mathrm{Sym}_\bullet U$. If $v_i\in D_U^*$ and $u_j\in\mathrm{Sym}_j U$, then u_jv_i is the element of $D_{i-j}U^*$ which sends u_{i-j} in $\mathrm{Sym}_{i-j}U$ to $v_i(u_ju_{i-j})$. In particular, if m and m' are monomials in $\mathrm{Sym}_\bullet U$ (with respect to some basis $x_1,\dots,x_d$ for U and $*$ is defined as in (2.2.1)), then

$$m'(m)^*=\begin{cases}(\frac{m}{m'})^*, & \text{if } m' \text{ divides } m,\\ 0, & \text{if } m' \text{ does not divide } m.\end{cases}$$

(d) If u^* is an element of U^*, then we write $u^{*(n)}$ for the element $(u^*)^{(n)}$ in D_nU^*.

(e) If $x_1,\dots,x_d$ is a basis for the vector space U, then $x_1^*,\dots,x_d^*$ is the dual basis for U^*. Similarly, if then $x_1^*,\dots,x_d^*$ is a basis for a vector space U^*, then $x_1,\dots,x_d$ is the dual basis for the vector space U.

(f) If $u_i\in\mathrm{Sym}_i U$ and $\phi_i\in D_iU^*$, then $u_i\phi_i=\phi_iu_i$ is an element of $\boldsymbol{k}$.

The following data is used throughout the paper.

Data 2.3. Let $\boldsymbol{k}$ be a field, U be a d-dimensional vector space over $\boldsymbol{k}$, P be the polynomial ring $P = \operatorname{Sym}_\bullet U$, ϕ_3 be a non-zero element of D_3U^*, $I = \operatorname{ann}_P(\phi_3)$, and A_{ϕ_3} be the standard graded Artinian Gorenstein $\boldsymbol{k}$-algebra $A_{\phi_3} = P/I$. The socle degree of A is three.

2.B. **The homomorphisms $\bowtie$, p_{ϕ_3}, and Γ_{ϕ_3}.**

Notation 2.4. Let $\boldsymbol{k}$ be a field, E and G be $\boldsymbol{k}$-modules, and m be a positive integer. Each pair of elements (X,Y), with $X \in D_mE$ and $Y \in \bigwedge^m G$, gives rise to an element of $\bigwedge^m(E \otimes G)$, which we denote by $X \bowtie Y$. We now give the definition of $X \bowtie Y$. Consider the composition

$$D_mE \otimes T_mG \xrightarrow{\Delta\otimes 1} T_mE \otimes T_mG \xrightarrow{\xi} \textstyle\bigwedge^m(E \otimes G),$$

where $\Delta : D_mE \to T_mE$ is co-multiplication and

$$\xi\big((x_1 \otimes \ldots \otimes x_m) \otimes (y_1 \otimes \ldots \otimes y_m)\big) = (x_1 \otimes y_1) \wedge \ldots \wedge (x_m \otimes y_m),$$

for $x_i \in E$ and $y_i \in G$. It is easy to see that the above composition factors through $D_mE \otimes \bigwedge^m G$. Let $X \otimes Y \mapsto X \bowtie Y$ be the resulting map from $D_mE \otimes \bigwedge^m G$ to $\bigwedge^m(E \otimes G)$. This map is used in [19] and is called $\langle -, - \rangle$ in [2, III.2].

Definition 2.5. Adopt Data 2.3.

(a) Define the $\boldsymbol{k}$-module homomorphism $p_{\phi_3} : \operatorname{Sym}_2 U \to U^*$ by $p_{\phi_3}(u_2) = u_2(\phi_3)$, for $u_2 \in \operatorname{Sym}_2 U$.
(b) Define the $\boldsymbol{k}$-module homomorphism $\Gamma_{\phi_3} : D_dU \otimes \bigwedge^d U \to \bigwedge^d U^*$ to be the composition

$$D_dU \otimes \textstyle\bigwedge^d U \xrightarrow{\bowtie} \bigwedge^d(U \otimes U) \xrightarrow{\bigwedge^d \text{mult}} \bigwedge^d(\operatorname{Sym}_2 U) \xrightarrow{\bigwedge^d p_{\phi_3}} \bigwedge^d U^*,$$

where

$$\text{mult} : U \otimes U \to \operatorname{Sym}_2 U$$

is multiplication in the Symmetric algebra $\operatorname{Sym}_\bullet U$.

Example 2.6. Adopt Data 2.3. Let Γ_{ϕ_3} be the $\boldsymbol{k}$-module homomorphism of Definition 2.5.(b). Let $\ell_1, \ldots, \ell_d$ be a basis for U.

If $d = 3$, then

$$\begin{aligned}
\Gamma_{\phi_3}(\ell_1^{(3)} \otimes \ell_1 \wedge \ell_2 \wedge \ell_3) &= \quad \ell_1^2\phi_3 \wedge \ell_1\ell_2\phi_3 \wedge \ell_1\ell_3\phi_3,\\
\Gamma_{\phi_3}(\ell_1^{(2)}\ell_2 \otimes \ell_1 \wedge \ell_2 \wedge \ell_3) &= \begin{cases} \ \ell_1^2\phi_3 \wedge \ell_2^2\phi_3 \wedge \ell_1\ell_3\phi_3 \\ +\ell_1^2\phi_3 \wedge \ell_1\ell_2\phi_3 \wedge \ell_2\ell_3\phi_3, \text{ and} \end{cases}\\
\Gamma_{\phi_3}(\ell_1\ell_2\ell_3 \otimes \ell_1 \wedge \ell_2 \wedge \ell_3) &= \begin{cases} \ \ell_1^2\phi_3 \wedge \ell_2^2\phi_3 \wedge \ell_3^2\phi_3 \\ +2\ell_1\ell_2\phi_3 \wedge \ell_2\ell_3\phi_3 \wedge \ell_1\ell_3\phi_3. \end{cases}
\end{aligned}$$

If $\dim U = 4$, then

$$\Gamma_{\phi_3}(\ell_1^{(4)} \otimes \ell_1 \wedge \ell_2 \wedge \ell_3 \wedge \ell_4) = \quad \ell_1^2\phi_3 \wedge \ell_1\ell_2\phi_3 \wedge \ell_1\ell_3\phi_3 \wedge \ell_1\ell_4\phi_3,$$

$$\Gamma_{\phi_3}(\ell_1^{(3)}\ell_2\otimes\ell_1\wedge\ell_2\wedge\ell_3\wedge\ell_4)=\begin{cases}\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_1\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_1\ell_2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_1\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_1\ell_2\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_2\ell_4\phi_3,\end{cases}$$

$$\Gamma_{\phi_3}(\ell_1^{(2)}\ell_2^{(2)}\otimes\ell_1\wedge\ell_2\wedge\ell_3\wedge\ell_4)=\begin{cases}\ell_1^2\phi_3\wedge\ell_1\ell_2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_2\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_2\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_1\ell_4\phi_3\\+\ell_1\ell_2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_1\ell_4\phi_3,\end{cases}$$

$$\Gamma_{\phi_3}(\ell_1^{(2)}\ell_2\ell_3\otimes\ell_1\wedge\ell_2\wedge\ell_3\wedge\ell_4)=\begin{cases}\ell_1^2\phi_3\wedge\ell_1\ell_2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_3\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_1\ell_2\phi_3\wedge\ell_3^2\phi_3\wedge\ell_2\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_3\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_2\ell_4\phi_3\\+\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_3^2\phi_3\wedge\ell_1\ell_4\phi_3\\+2\ell_1\ell_2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_1\ell_4\phi_3,\text{ and}\end{cases}$$

$$\Gamma_{\phi_3}(\ell_1\ell_2\ell_3\ell_4\otimes\ell_1\wedge\ell_2\wedge\ell_3\wedge\ell_4)=\begin{cases}\ell_1^2\phi_3\wedge\ell_2^2\phi_3\wedge\ell_3^2\phi_3\wedge\ell_4^2\phi_3\\+2\ell_1^2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_3\ell_4\phi_3\wedge\ell_2\ell_4\phi_3\\+2\ell_1\ell_3\phi_3\wedge\ell_2^2\phi_3\wedge\ell_3\ell_4\phi_3\wedge\ell_1\ell_4\phi_3\\+2\ell_1\ell_2\phi_3\wedge\ell_2\ell_4\phi_3\wedge\ell_3^2\phi_3\wedge\ell_1\ell_4\phi_3\\+2\ell_1\ell_2\phi_3\wedge\ell_2\ell_3\phi_3\wedge\ell_1\ell_3\phi_3\wedge\ell_4^2\phi_3.\end{cases}$$

Remark 2.7 is used in the proof of Observation 2.8.

Remark 2.7. Field extensions are always faithfully flat. Let V be a vector space over a field $\boldsymbol{k}$, K be a field extension of $\boldsymbol{k}$, and $v\in V$. If $v\otimes_{\boldsymbol{k}}1$ is zero in $V\otimes_{\boldsymbol{k}}K$, then v is zero in V.

Observation 2.8. *Adopt Data* 2.3. *If the homomorphism*

$$\Gamma_{\phi_3}: D_dU\otimes\textstyle\bigwedge^d U\to\bigwedge^d U^*$$

of Definition 2.5 *satisfies* $\Gamma_{\phi_3}(\ell^{(d)}\otimes\omega_U)=0$ *for all* $\ell\in U$ *and some basis element* ω_U *of* $\bigwedge^d U$*, then* Γ_{ϕ_3} *is identically zero.*

Proof. Fix a basis ω_U for $\bigwedge^d U$. The map

$$\Gamma_{\phi_3}(-\otimes\omega_U):U\to\textstyle\bigwedge^d U^*$$

is a linear transformation of vector spaces over a field. In light of Remark 2.7, it suffices to prove the assertion when $\boldsymbol{k}$ has a large number of elements.

The proof of Observation 2.8 is obtained by iterating the following claim.

Claim 2.8.1. *Let X be an element of $D_\delta U$ for some integer δ with $0\le\delta\le d$. If* $\Gamma_{\phi_3}(\ell^{(d-\delta)}X\otimes\omega_U)=0$ *for all* $\ell\in U$*, then* $\Gamma_{\phi_3}(\ell_1^{(e_1)}\ell_2^{(e_2)}X\otimes\omega_U)=0$ *for all* ℓ_1,ℓ_2 *in* U *and all nonnegative integers* e_1 *and* e_2 *with* $e_1+e_2=d-\delta$.

Proof of Claim 2.8.1. If ℓ_1 and ℓ_2 are in U and $a_1,\dots,a_{d-\delta+1}$ are distinct elements of $\boldsymbol{k}$, then

$$\Gamma_{\phi_3}((\ell_1+a_i\ell_2)^{(d-\delta)}X\otimes\omega_U)=\sum_{j=0}^{d-\delta}a_i^j\Gamma_{\phi_3}(\ell_1^{(d-\delta-j)}\ell_2^{(j)}X\otimes\omega_U). \tag{2.8.2}$$

The hypothesis guarantees that the left side of (2.8.2) is zero. It follows that product of the row vector

$$\begin{bmatrix}\Gamma_{\phi_3}(\ell_1^{(d-\delta-0)}\ell_2^{(0)}X\otimes\omega_U) & \Gamma_{\phi_3}(\ell_1^{(d-\delta-1)}\ell_2^{(1)}X\otimes\omega_U) & \dots & \Gamma_{\phi_3}(\ell_1^{(0)}\ell_2^{(d-\delta-0)}X\otimes\omega_U)\end{bmatrix}$$

and the Vandermonde matrix

$$\begin{bmatrix}1 & 1 & \dots & 1\\ a_1 & a_2 & \dots & a_{d-\delta+1}\\ \vdots & \vdots & & \vdots\\ a_1^{d-\delta} & a_2^{d-\delta} & \dots & a_{d-\delta+1}^{d-\delta}\end{bmatrix} \tag{2.8.3}$$

is zero. The Vandermonde matrix is invertible and $\Gamma_{\phi_3}(\ell_1^{(e_1)}\ell_2^{(e_2)}X\otimes\omega_U)=0$ for all ℓ_1 and ℓ_2 in U and all non-negative integers e_i with $e_1+e_2=d-\delta$. This completes the proof of Claim 2.8.1.

Now we prove Observation 2.8. First take $X=1$. The hypothesis ensures that $\Gamma_{\phi_3}(\ell^{(d)}\otimes\omega_U)=0$ for all $\ell\in U$. Apply Claim 2.8.1 to conclude that

$$\Gamma_{\phi_3}(\ell_1^{(e_1)}\ell_2^{(e_2)}\otimes\omega_U)=0 \quad \text{for all } \ell_1,\ell_2\in U \text{ and all nonnegative integers } e_1 \text{ and } e_2 \text{ with } e_1+e_2=d. \tag{2.8.4}$$

Now take $X=\ell_3^{(e_3)}$ for some $\ell_3\in U$ and some integer e_3 with $0\le e_3\le d$. Apply Claim 2.8.1, together with (2.8.4), to conclude that $\Gamma_{\phi_3}(\ell_1^{(e_1)}\ell_2^{(e_2)}\ell_3^{(e_3)}\otimes\omega_U)=0$, for all $\ell_1,\ell_2,\ell_3\in U$ and all nonnegative integers e_1, e_2, and e_e with $e_1+e_2+e_3=d$.

One finishes the proof by iterating Claim 2.8.1. □

2.C. The connection between Γ_{ϕ_3} and the weak Lefschetz property.

Lemma 2.9. *Adopt Data* 2.3 *and Definition* 2.5. *Assume $\ell\phi_3\neq 0$ for all nonzero ℓ in U. Then $A=A_{\phi_3}$ has the weak Lefschetz property if and only if Γ_{ϕ_3} is not identically zero.*

Remark 2.10. The hypothesis $\ell\phi_3\neq 0$ for all nonzero ℓ in U is harmless. It is equivalent to asserting that the degree one component of the ideal I is zero. Consequently, it is also equivalent to the hypothesis that the embedding dimension of A is equal to the vector space dimension of U.

Proof. Recall from Observation 2.8 that

$$\Gamma_{\phi_3}(\ell^{(d)}\otimes -) \text{ is zero for all } \ell\in U \iff \Gamma_{\phi_3} \text{ is identically zero.}$$

Let $x_1,\dots,x_d$ be a basis for U, ℓ be an element of U, and μ_ℓ represent the homomorphism "multiplication by ℓ". Observe that

$$\begin{aligned}
&\ell \text{ is a weak Lefschetz element in } A\\
\iff\ & \mu_\ell : A_1 \to A_2 \text{ is injective}\\
\iff\ & \mu_\ell : P_1 \to A_2 \text{ is injective,} && \text{because } I_1 = 0,\\
\iff\ & \begin{cases} \ell(\sum a_i x_i)(\phi_3) = 0, \text{ with } a_i \text{ in } \boldsymbol{k},\\ \text{only if all } a_i \text{ are zero,}\end{cases}\\
\iff\ & \begin{cases} \ell x_1(\phi_3), \ell x_2(\phi_3), \dots, \ell x_d(\phi_3) \text{ are linearly}\\ \text{independent in } U^*,\end{cases}\\
\iff\ & \Gamma_{\phi_3}(\ell^{(d)} \otimes x_1 \wedge x_2 \wedge \dots \wedge x_d) \neq 0.
\end{aligned}$$

□

Theorem 1.1, which is the main result of this paper, is an immediate consequence of Lemma 2.11, which is an immediate consequence of Lemma 2.9.

Lemma 2.11. *Adopt Data* 2.3 *with* $d = 4$. *Assume*

(a) *either the characteristic of* $\boldsymbol{k}$ *is different than two; or else, the characteristic of* $\boldsymbol{k}$ *is equal to two, but there does not exist a basis* x^*, y^*, z^*, w^* *for* U^* *with* $\phi_3 = x^{*(3)} + y^* z^* w^*$, *and*

(b) $\ell\phi_3 \neq 0$ *for all nonzero* $\ell \in U$.

Then Γ_{ϕ_3} *is not identically zero.*

The proof of Lemma 2.11 involves multiple cases and comprises the majority of this paper. The official proof is given in (3.4).

3. Put the Macaulay inverse system into a convenient form.

Ultimately, we prove a statement about an element ϕ_3 of D_3U^*, where U is a four-dimensional vector space. Our proof depends on the form of ϕ_3. There are four main cases. Two of the cases involve ϕ_3 as described in (a) of Lemma 3.1. (These two cases are distinguished by the rank r of the homomorphism $p_{\phi_{2,0}}$; see Lemma 3.2.) These two cases are treated in Propositions 4.1 and 5.1. The other two cases involve ϕ_3 as described in (b) of Lemma 3.1. These two cases are separated in 6.3 and are treated in Propositions 6.4 and 7.1.

In most characteristics, all ϕ_3 can be put in the form of (a) of Lemma 3.1; however, form (b) of Lemma 3.1 is required in characteristic three.

Lemma 3.1. *Let* $\boldsymbol{k}$ *be a field,* U *be a* d*-dimensional vector space over* $\boldsymbol{k}$, *and* ϕ_3 *be a nonzero element of* D_3U^*. *Then there exists a unit* α *of* $\boldsymbol{k}$ *and a basis* $x_1^*,\dots,x_d^*$ *for* U^* *such that*

(a) $\alpha\phi_3 = x_1^{*(3)} + x_1^*\phi_{2,0} + \phi_{3,0}$, *or*

(b) $\alpha\phi_3 = x_1^{*(2)} x_2^* + x_1^*\phi_{2,0} + \phi_{3,0}$, *or*

(c) ***k*** *has characteristic two and*

$$\phi_3 = \sum_{1\le i<j<k\le d} \alpha_{i,j,k} x_i^* x_j^* x_k^*$$

for some $\alpha_{i,j,k}$ *in* ***k***,

where $\phi_{i,0}$ *is an element of*

$$D_i(\bigoplus_{2\le j\le d} \boldsymbol{k} x_j^*).$$

Remark. The case (c) is not very interesting when $d=4$; see (3.4).

Proof. Begin with an arbitrary basis $y_1^*,\dots,y_d^*$ for U^*.

Claim 3.1.1. *After a change of basis,*

(3.1.2) $\alpha\phi_3 = x_1^{*(3)} + x_1^{*(2)}\phi_{1,0} + x_1^*\phi_{2,0} + \phi_{3,0}$, *with* $\alpha\neq 0$, *or*

(3.1.3) $\phi_3 = x_1^{*(2)}\phi_{1,0} + x_1^*\phi_{2,0} + \phi_{3,0}$, *with* $\phi_{1,0}\neq 0$, *or*

(3.1.4) $\phi_3 = \sum_{1\le i<j<k\le d} \alpha_{i,j,k} x_i^* x_j^* x_k^*$,

with $\alpha, \alpha_{i,j,k}$ *in* ***k*** *and* $\phi_{i,0}\in D_i(\bigoplus_{2\le j\le d}\boldsymbol{k}x_j^*)$.

Proof of Claim 3.1.1. Write ϕ_3 as

$$\phi_3 = \sum_{e_1+\cdots+e_d=3} \alpha_{e_1,\dots,e_d} y_1^{*(e_1)} y_2^{*(e_2)} \cdots y_{d-1}^{*(e_{d-1})} y_d^{*(e_d)},$$

with $\alpha_{e_1\dots,e_d}\in\boldsymbol{k}$. If any of the parameters

(3.1.5) $$\alpha_{0,\dots,0,3,0,\dots,0}$$

is nonzero, then ϕ_3 has the form of (3.1.2).

If the parameters of (3.1.5) are zero; but any of the parameters

(3.1.6) $$\alpha_{0,\dots,0,2,0,\dots,0,1,0\dots,0}$$

are nonzero, then ϕ_3 has the form of (3.1.3). If all of the parameters in (3.1.5) and (3.1.6) are zero, then ϕ_3 has the form of (3.1.4). This completes the proof of Claim 3.1.1.

Three observations are needed to complete the proof of Lemma 3.1. First, if ϕ_3 has the form of (3.1.2), then the change of basis $x_1^* = X^* - \phi_{1,0}$ puts ϕ_3 into the form of (a) because

$$(X^*-\phi_{1,0})^{(3)} = X^{*(3)} - X^{*(2)}\phi_{1,0} + X^*\phi_{1,0}^{(2)} - \phi_{1,0}^{(3)}.$$

Second, if ϕ_3 has the form of (3.1.3), then one may change the basis of U^* again and choose the new "x_2^*" to equal the old $\phi_{1,0}$. Third, if the characteristic of ***k*** is not two, then any nonzero element of D_3U^* of form (3.1.4) can be transformed into an element of D_3U^* of form (3.1.3). In particular, an element of form (3.1.4) in which

$x_1^*x_2^*x_3^*$ actually appears becomes an element of form (3.1.3) if one uses the basis $x_1^*, y_2^*, x_3^*, \dots, x_d^*$ for U^* with $x_2^* = y_2^* + x_1^*$ because

$$x_1^*x_1^* = 2x_1^{*(2)}$$

and this is a unit times $x_1^{*(2)}$ when the characteristic of $\boldsymbol{k}$ is not two. □

Lemma 3.2 ensures that we can record "$\phi_{2,0}$" from Lemma 3.1.(a) in an efficient manner.

Lemma 3.2. *Let U_0 be a d_0-dimensional vector space over the field $\boldsymbol{k}$ and $\phi_{2,0}$ be an element of $D_2U_0^*$. Let $p_{\phi_{2,0}} : U_0 \to U_0^*$ be the homomorphism defined by $p_{\phi_{2,0}}(\ell) = \ell\phi_{2,0}$ and let r be the rank of $p_{\phi_{2,0}}$. Then there is a basis $\ell_1, \dots, \ell_{d_0}$ for U_0 and corresponding dual basis $\ell_1^*, \dots, \ell_{d_0}^*$ for U_0^* such that $\phi_{2,0} \in D_2(\boldsymbol{k}\ell_1^* \oplus \dots \oplus \boldsymbol{k}\ell_r^*)$. In particular, if $r = 1$, then $\phi_{2,0} = a\ell_1^{*(2)}$ for some nonzero element a in $\boldsymbol{k}$; and if $r = 2$, then*

$$\phi_{2,0} = a\ell_1^{*(2)} + b\ell_1^*\ell_2^* + c\ell_2^{*(2)}, \tag{3.2.1}$$

for some elements a, b, c of $\boldsymbol{k}$ with $ac - b^2$ not equal to zero.

Proof. Let $\ell_1, \dots, \ell_{d_0}$ be a basis for U_0 such that $\ell_1\phi_{2,0}, \dots, \ell_r\phi_{2,0}$ is a basis for the image of $p_{\phi_{2,0}}$ and $\ell_{r+1}, \dots, \ell_{d_0}$ are in $\ker p_{\phi_{2,0}}$. Let $\ell_1^*, \dots, \ell_{d_0}^*$ be the corresponding dual basis for U_0^*. Write $\phi_{2,0}$ in terms of the basis

$$\{\ell_1^{*(i_1)} \dots \ell_{d_0}^{*(i_{d_0})} \mid \sum_j i_j = 2\}$$

for $D_2U_0^*$. The hypothesis that $\ell_h\phi_{2,0} = 0$ for $r+1 \le h \le 3$ ensures that $\phi_{2,0}$ is in $D_2(\boldsymbol{k}\ell_1^* \oplus \dots \oplus \boldsymbol{k}\ell_r^*)$. Furthermore, if $r = 1$, then the coefficient of $\ell_1^{*(2)}$ can not be zero; and if $r = 2$, then $\ell_1\phi_{2,0}$ and $\ell_2\phi_{2,0}$ must be linearly independent (hence, $\phi_{2,0}$ must have the form of (3.2.1) with $ac - b^2 \ne 0$.) □

Lemma 3.3 is redundant in the sense that the assertion is a consequence of Lemma 3.2 when $r = 2$. On the other hand, the constructive nature of the argument makes this statement a valuable addition to one's tool bag.

Lemma 3.3. *Let $\boldsymbol{k}$ be a field, U_0^* be a two-dimensional vector space over $\boldsymbol{k}$, and $\phi_2 \in D_2U_0^*$. If $\dim\{\ell\phi_2 \mid \ell \in U_0\}$ is at most one, then there is a basis z^*, w^* for U_0^* such that $\phi_2 = az^{*(2)}$ for some $a \in \boldsymbol{k}$.*

Proof. Let Z^*, W^* be a basis for U_0^* and Z, W be the corresponding basis for U_0. Write $\phi_2 = aZ^{*(2)} + bZ^*W^* + cW^{*(2)}$ for some $a, b, c \in \boldsymbol{k}$. The hypothesis ensures that

$$Z\phi_2 = aZ^* + bW^* \quad \text{and} \quad W\phi_2 = bZ^* + cW^*$$

are linearly dependent. It follows that $ac = b^2$.

If a, b, and c are all zero, then the conclusion holds automatically. Henceforth, we assume that at least one of the parameters a, b, or c is nonzero; so, in particular, a or c is nonzero. Without loss of generality, we assume $a \neq 0$. In this case,

$$aZ^{*(2)} + bZ^*W^* + cW^{*(2)} = a(Z^{*(2)} + \tfrac{b}{a}Z^*W^* + \tfrac{b^2}{a^2}W^{*(2)}) = a(Z^* + \tfrac{b}{a}W^*)^{(2)}$$

because $\frac{b^2}{a^2} = \frac{c}{a}$. At this point, we rename $z^* = Z^* + \frac{b}{a}W^*$ and $w^* = W^*$. The claim is established. □

3.4. The proof of Lemma 2.11. There are many cases depending upon the form of ϕ_3 in the sense of Lemma 3.1.

If ϕ_3 has the form of Lemma 3.1.(a), then let r be the rank of $p_{\phi_{2,0}}$ as described in Lemma 3.2. Of course, $0 \leq r \leq 3$. If $r = 2$, then Lemma 2.11 is established in Proposition 4.1; if $r = 1$, then Lemma 2.11 is established in Proposition 5.1; and if r is either 0 or 3, then Lemma 2.11 is established in Lemma 5.3.

If ϕ_3 has the form of Lemma 3.1.(b), then there are two cases as described in 6.3. Case 1 of 6.3 is established in Proposition 7.1 and Case 2 is established in Proposition 6.4.

If ϕ_3 has form of Lemma 3.1.(c), then $\boldsymbol{k}$ has characteristic two and

$$\phi_3 = ay^*z^*w^* + bx^*z^*w^* + cx^*y^*w^* + dx^*y^*z^*$$

for some parameters a,b,c,d from $\boldsymbol{k}$. If $\ell = ax^* + by^* + cz^* + dw^*$, then

$$\ell\phi_3 = 2(abz^*w^* + acy^*w^* + ady^*z^* + bcx^*w^* + bdx^*z^* + cdx^*y^*) = 0.$$

Thus, $\ell\phi_3 = 0$ for some nonzero $\ell \in U$. In this case, Lemma 2.11 makes no assertion about Γ_{ϕ_3}. □

4. The Macaulay inverse system has a cubic term and $r = 2$.

We prove Lemma 2.11 when ϕ_3 has the form of Lemma 3.1.(a), with $r = 2$, in the sense of Lemma 3.2. This means that, in the language of Data 2.3, there is a basis x^*, y^*, z^*, w^* for U^* so that $\phi_3 = x^{*(3)} + \phi_{2,0}x^* + \phi_{3,0}$ with $\phi_{2,0} \in D_2\boldsymbol{k}(z^*,w^*)$, $\phi_{3,0} \in D_3\boldsymbol{k}(y^*,z^*,w^*)$, and $z\phi_{2,0}$ and $w\phi_{2,0}$ linearly independent. In particular, the basis elements $x^{*(2)}y^*$, $x^{*(2)}z^*$, $x^{*(2)}w^*$, $x^*y^{*(2)}$, $x^*y^*z^*$, and $x^*y^*w^*$ of D_3U^* all appear in ϕ_3 with coefficient zero.

Proposition 4.1. *Let P_0 be a domain, U be a free P_0-module of rank four with dual module $U^* = \operatorname{Hom}_{P_0}(U,P_0)$, x,y,z,w be a basis for U with dual basis x^*,y^*,z^*,w^* for U^*. Let $\phi_3 \in D_3U^*$ have the form*

$$(4.1.1)\qquad \phi_3 = \begin{cases} x^{*(3)} + ax^*z^{*(2)} + bx^*z^*w^* + cx^*w^{*(2)} + dy^*z^{*(2)} \\ +ey^*z^*w^* + fy^*w^{*(2)} + gy^{*(2)}z^* + hy^{*(2)}w^* + iy^{*(3)} \\ +jz^{*(3)} + kz^{*(2)}w^* + lz^*w^{*(2)} + mw^{*(3)}, \end{cases}$$

for elements $a,\dots,m$ of P_0. Assume that

(a) *$ac - b^2 \neq 0$ in P_0,*

(b) $\ell\phi_3 \neq 0$ *for all nonzero* ℓ *in* U, *and*
(c) $\Gamma_{\phi_3} = 0$.

Then 2 *is equal to* 0 *in* P_0, *and, after a change of basis,* $\phi_3 = \alpha X^{*(3)} + Y^*Z^*W^*$ *for some basis* X^*, Y^*, Z^* *of* U^* *and some nonzero* $\alpha \in P_0$.

Proof. Let ω_U and ω_{U^*} represent the bases $x \wedge y \wedge z \wedge w$ and $x^* \wedge y^* \wedge z^* \wedge w^*$ of $\bigwedge^4 U$ and $\bigwedge^4 U^*$, respectively.

Claim 4.1.2. *The parameters g, h, and i are zero.*

Proof of Claim 4.1.2. We first compute

$$\begin{aligned} \Gamma_{\phi_3}(x^{(3)}y \otimes \omega_U) &= i(ac-b^2)\omega_{U^*}, \\ \Gamma_{\phi_3}(x^{(3)}z \otimes \omega_U) &= g(ac-b^2)\omega_{U^*}, \text{ and} \\ \Gamma_{\phi_3}(x^{(3)}w \otimes \omega_U) &= h(ac-b^2)\omega_{U^*}. \end{aligned} \tag{4.1.3}$$

Let ℓ be an arbitrary element of U. The expansion of $\Gamma_{\phi_3}(x^{(3)}\ell \otimes \omega_U)$ has four summands; but three of the summands involve a factor of $xy\phi_3 = 0$; consequently,

$$\begin{aligned} &\Gamma_{\phi_3}(x^{(3)}\ell \otimes \omega_U) \\ &= x^2\phi_3 \wedge y\ell\phi_3 \wedge xz\phi_3 \wedge xw\phi_3 = x^* \wedge y\ell\phi_3 \wedge (az^*+bw^*) \wedge (bz^*+cw^*) \\ &= (ac-b^2)x^* \wedge y\ell\phi_3 \wedge z^* \wedge w^*. \end{aligned}$$

Insert

$$\begin{aligned} y^2\phi_3 &= (gz^*+hw^*+iy^*) \\ yz\phi_3 &= (dz^*+ew^*+gy^*) \\ yw\phi_3 &= (ez^*+fw^*+hy^*) \end{aligned}$$

into the calculation of $\Gamma_{\phi_3}(x^{(3)}\ell \otimes \omega_U)$ in order to obtain (4.1.3). Combine the hypotheses that Γ_{ϕ_3} is identically zero, but $ac-b^2$ is nonzero in the domain P_0 to conclude that $g = h = i = 0$. This completes the proof of Claim 4.1.2.

Claim 4.1.4. *The parameter d is zero.*

Proof of Claim 4.1.4. We first compute that if $g = h = i = 0$, then

$$\begin{aligned} \Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U) &= (-cd^2+2bde-ae^2)\omega_{U^*} \\ \Gamma_{\phi_3}(z^{(4)} \otimes \omega_U) &= (ae-bd)^2\omega_{U^*} \end{aligned} \tag{4.1.5}$$

The expansion of $\Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U)$ consists of six summands; four of the summands have a factor of $xy\phi_3 = 0$ or $xz\phi_3 \wedge xz\phi_3 = 0$. It follows that

$$\begin{aligned} &\Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U) \\ &= x^2\phi_3 \wedge yz\phi_3 \wedge xz\phi_3 \wedge zw\phi_3 + x^2\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3 \wedge xw\phi_3 \end{aligned}$$

$$= \begin{cases} x^* \wedge (dz^*+ew^*) \wedge (az^*+bw^*) \wedge (bx^*+ey^*+kz^*+lw^*) \\ +x^* \wedge (dz^*+ew^*) \wedge (ax^*+dy^*+jz^*+kw^*) \wedge (bz^*+cw^*) \end{cases}$$
$$= \Big(e(bd-ae)-d(cd-eb)\Big)\omega_{U^*}$$
$$= (-cd^2+2bde-ae^2)\omega_{U^*}.$$

Also, one computes

$$\begin{aligned} &\Gamma_{\phi_3}(z^{(4)} \otimes \omega_U) \\ &= xz\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3 \\ &= (az^*+bw^*) \wedge (dz^*+ew^*) \wedge (ax^*+dy^*+jz^*+kw^*) \wedge (bx^*+ey^*+kz^*+lw^*) \\ &= (ae-bd)(z^* \wedge w^*) \wedge (ae-bd)(x^* \wedge y^*) \\ &= (ae-bd)^2\omega_{U^*}. \end{aligned}$$

Both formulas of (4.1.5) have been established. The hypothesis that $\Gamma_{\phi_3}=0$ ensures that

$$0=-cd^2+2bde-ae^2 \quad \text{and} \quad 0=(ae-bd)^2.$$

It follows that

$$0=a(-cd^2+2bde-ae^2)+(ae-bd)^2=d^2(b^2-ac).$$

The ring P_0 is a domain and b^2-ac is not zero. We conclude that $d=0$. This completes the proof of Claim 4.1.4.

In Lemma 4.2 we prove that if $d=g=h=i=0$, then

(4.1.6) $$\Gamma_{\phi_3}(xz^{(3)} \otimes \omega_U) = -e^2j\omega_{U^*},$$

(4.1.7) $$\Gamma_{\phi_3}(xyzw \otimes \omega_U) = 2e^3\omega_{U^*},$$

(4.1.8) $$\Gamma_{\phi_3}(xz^{(2)}w \otimes \omega_U) = (e^2k-2efj)\omega_{U^*},$$

(4.1.9) $$\Gamma_{\phi_3}(x^{(2)}w^{(2)} \otimes \omega_U) = (-ce^2+2bef-af^2)\omega_{U^*},$$

(4.1.10) $$\Gamma_{\phi_3}(xyw^{(2)} \otimes \omega_U) = e^2f\omega_{U^*},$$

(4.1.11) $$\Gamma_{\phi_3}(xzw^{(2)} \otimes \omega_U) = (e^2l-jf^2)\omega_{U^*},$$

(4.1.12) $$\Gamma_{\phi_3}(z^{(2)}w^{(2)} \otimes \omega_U) = (-2ace^2+2abef+a^2f^2)\omega_{U^*},$$

(4.1.13) $$\Gamma_{\phi_3}(xw^{(3)} \otimes \omega_U) = (-f^2k+2efl-e^2m)\omega_{U^*}, \text{ and}$$

(4.1.14) $$\Gamma_{\phi_3}(w^{(4)} \otimes \omega_U) = (bf-ce)^2\omega_{U^*}.$$

Claim 4.1.15. *The following assertions hold*:

(a) $ae=0$,
(b) $ej=0$,
(c) $2e=0$,
(d) $ek=0$,
(e) $ce^2+af^2=0$,

(f) $ef=0$,
(g) $-f^2j+e^2\ell=0$,
(h) $af=0$,
(i) $-f^2k-e^2m=0$, *and*
(j) $ce-bf=0$.

Proof of Claim 4.1.15. In the proof of Claim 4.1.4, we saw that $(ae-bd)^2=0$ and $d=0$. The ring P_0 is a domain. It follows that $ae=0$. Assertion (b) is a consequence of (4.1.6) and the hypothesis that P_0 is a domain; (c) follows from (4.1.7); (d) from (4.1.8) and (c); (e) from (4.1.9) and (c); (f) from (4.1.10); (g) from (4.1.11); (h) from (4.1.12) and (c); (i) from (4.1.13) and (c); and (j) from (4.1.14). This completes the proof of Claim 4.1.15.

Claim 4.1.16. *The parameter e is not equal to zero.*

Proof of Claim 4.1.16. This proof is by contradiction: suppose $e=0$. There are two cases: either $f=0$ or $f\neq 0$.

If $f=0$, then, $d=e=f=g=h=i=0$, ϕ_3 is equal to

$$x^{*(3)}+ax^*z^{*(2)}+bx^*z^*w^*+cx^*w^{*(2)}+jz^{*(3)}+kz^{*(2)}w^*+lz^*w^{*(2)}+mw^{*(3)},$$

and $y\phi_3=0$, which violates the hypothesis that $\ell\phi_3$ is nonzero for all nonzero ℓ in U.

If $e=0$ and $f\neq 0$, then, according to Claim 4.1.15, $af=bf=0$. Thus,

$$a=b=0;$$

however, the ambient hypothesis guarantees that $b^2-ac\neq 0$. This completes the proof of Claim 4.1.16.

Use the fact that $e\neq 0$, together with Claims 4.1.15, 4.1.4, and 4.1.2 to see that

$$2=a=c=d=f=g=h=i=j=k=l=m=0.$$

In this case two is equal to zero in P_0 and

$$\phi_3=x^{*(3)}+bx^*z^*w^*+ey^*z^*w^*=x^{*(3)}+(bx^*+ey^*)z^*w^*,$$

which has the form $\phi_3=X^{*(3)}+Y^*Z^*W^*$ for some basis X^*, Y^*, Z^*, W^* of U^*. □

Lemma 4.2. *If ϕ_3 is given in* (4.1.1) *with $d=g=h=i=0$, then the assertions of* (4.1.6) *to* (4.1.14) *all hold.*

Proof. We prove (4.1.6). The expansion of $\Gamma_{\phi_3}(xz^{(3)}\otimes\omega_U)$ consists of four summands. Three of the summands contain a factor of $xy\phi_3=0$, or $xz\phi_3\wedge xz\phi_3=0$, or

$$\bigwedge^3\boldsymbol{k}(xw\phi_3,\ yz\phi_3,\ xz\phi_3,\ yw\phi_3)\subseteq\bigwedge^3(P_0z^*\oplus P_0w^*)=0. \tag{4.2.1}$$

It follows that

$$\Gamma_{\phi_3}(xz^{(3)}\otimes\omega_U)=x^2\phi_3\wedge yz\phi_3\wedge z^2\phi_3\wedge zw\phi_3$$

$$= x^* \wedge ew^* \wedge (ax^* + jz^* + kw^*) \wedge (bx^* + ey^* + kz^* + lw^*)$$
$$= x^* \wedge ew^* \wedge jz^* \wedge ey^* = -e^2 j\omega_{U^*}.$$

We prove (4.1.7). The expansion of $\Gamma_{\phi_3}(xyzw \otimes \omega_U)$ consists of twenty-four summands. Twenty-two of the summands have a factor of

$$y^2\phi_3 = 0,\ \ xy\phi_3 = 0,\ \ yz\phi_3 \wedge yz\phi_3 = 0,\ \ xz\phi_3 \wedge xz\phi_3 = 0,\ \ xw\phi_3 \wedge xw\phi_3 = 0,\ \ \text{or}$$
$$yw\phi_3 \wedge yw\phi_3,\ \ \text{or}\ \ (4.2.1).$$

The other two summands are equal. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(xyzw \otimes \omega_U) &= 2x^2\phi_3 \wedge yw\phi_3 \wedge yz\phi_3 \wedge zw\phi_3\\ &= 2x^* \wedge (ez^* + fw^*) \wedge ew^* \wedge (bx^* + ey^* + kz^* + lw^*)\\ &= 2x^* \wedge ez^* \wedge ew^* \wedge ey^*\\ &= 2e^3\omega_{U^*}.\end{aligned}$$

We prove (4.1.8). The expansion of $\Gamma_{\phi_3}(xz^{(2)}w \otimes \omega_U)$ consists of twelve summands. Ten of the summands contain a factor of

$$xy\phi_3 = 0, \quad zw\phi_3 \wedge zw\phi_3 = 0, \quad xz\phi_3 \wedge xz\phi_3 = 0, \quad xw\phi_3 \wedge xw\phi_3 = 0, \quad (4.2.1), \quad \text{or}$$

$$\bigwedge\nolimits^4 \boldsymbol{k}(xz\phi_3,\ yw\phi_3,\ z^2\phi_3,\ xw\phi_3) \subseteq \bigwedge\nolimits^4 (P_0x^* \oplus P_0z^* \oplus P_0w^*) = 0. \tag{4.2.2}$$

It follows that

$$\begin{aligned}&\Gamma_{\phi_3}(xz^{(2)}w \otimes \omega_U)\\ &= x^2\phi_3 \wedge yw\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3 + x^2\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3 \wedge w^2\phi_3\\ &= \begin{cases} \ \ x^* \wedge (ez^* + fw^*) \wedge (ax^* + jz^* + kw^*) \wedge (bx^* + ey^* + kz^* + lw^*)\\ +x^* \wedge ew^* \wedge (ax^* + jz^* + kw^*) \wedge (cx^* + fy^* + lz^* + mw^*)\end{cases}\\ &= (e^2k - 2efj)\omega_{U^*}.\end{aligned}$$

We prove (4.1.9). The expansion of $\Gamma_{\phi_3}(x^{(2)}w^{(2)} \otimes \omega_U)$ consists of six summands. Four of the summands have a factor of $xy\phi_3 = 0$ or $xw\phi_3 \wedge xw\phi_3 = 0$. Thus,

$$\begin{aligned}&\Gamma_{\phi_3}(x^{(2)}w^{(2)} \otimes \omega_U)\\ &= x^2\phi_3 \wedge yw\phi_3 \wedge xz\phi_3 \wedge w^2\phi_3 + x^2\phi_3 \wedge yw\phi_3 \wedge zw\phi_3 \wedge xw\phi_3\\ &= \begin{cases} \ \ x^* \wedge (ez^* + fw^*) \wedge (az^* + bw^*) \wedge (cx^* + fy^* + lz^* + mw^*)\\ +x^* \wedge (ez^* + fw^*) \wedge (bx^* + ey^* + kz^* + lw^*) \wedge (bz^* + cw^*)\end{cases}\\ &= \Big(f(eb - fa) - e(ec - bf)\Big)\omega_{U^*} = (-ce^2 + 2bef - af^2)\omega_{U^*}.\end{aligned}$$

We prove (4.1.10). The expansion of $\Gamma_{\phi_3}(xyw^{(2)} \otimes \omega_U)$ consists of twelve summands. Eleven of the summands have a factor of

$$y^2\phi_3 = 0, \quad yw\phi_3 \wedge yw\phi_3 = 0, \quad xy\phi_3 = 0, \quad \text{or} \quad xw\phi_3 \wedge xw\phi_3 = 0.$$

It follows that

$$\begin{aligned}\Gamma_{\phi_3}(xyw^{(2)}\otimes\omega_U) &= x^2\phi_3\wedge yw\phi_3\wedge yz\phi_3\wedge w^2\phi_3\\ &= x^*\wedge(ez^*+fw^*)\wedge ew^*\wedge(cx^*+fy^*+lz^*+mw^*)\\ &= e^2f\omega_{U^*}\end{aligned}$$

We prove (4.1.11). The expansion of $\Gamma_{\phi_3}(xzw^{(2)}\otimes\omega_U)$ consists of twelve summands. Ten of the summands have a factor of

$$xy\phi_3=0,\quad zw\phi_3\wedge zw\phi_3=0,\quad xz\phi_3\wedge xz\phi_3=0,\quad xw\phi_3\wedge xw\phi_3=0\quad\text{or}\quad(4.2.1).$$

It follows that

$$\begin{aligned}&\Gamma_{\phi_3}(xzw^{(2)}\otimes\omega_U)\\ &= x^2\phi_3\wedge yz\phi_3\wedge zw\phi_3\wedge w^2\phi_3+x^2\phi_3\wedge yw\phi_3\wedge z^2\phi_3\wedge w^2\phi_3\\ &= \begin{cases}x^*\wedge ew^*\wedge(bx^*+ey^*+kz^*+lw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\\ +x^*\wedge(ez^*+fw^*)\wedge(ax^*+jz^*+kw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\end{cases}\\ &= \Big(e(el-fk)+f(ek-jf)\Big)\omega_{U^*}=(e^2\ell-jf^2)\omega)U^*.\end{aligned}$$

We prove (4.1.12). The expansion of $\Gamma_{\phi_3}(z^{(2)}w^{(2)}\otimes\omega_U)$ consists of six summands. Two of the summands have a factor of $zw\phi_3\wedge zw\phi_3=0$. It follows that

$$\begin{aligned}&\Gamma_{\phi_3}(z^{(2)}w^{(2)}\otimes\omega_U)\\ &= \begin{cases}xz\phi_3\wedge yz\phi_3\wedge zw\phi_3\wedge w^2\phi_3\\ +xz\phi_3\wedge yw\phi_3\wedge z^2\phi_3\wedge w^2\phi_3\\ +xw\phi_3\wedge yz\phi_3\wedge z^2\phi_3\wedge w^2\phi_3\\ +xw\phi_3\wedge yw\phi_3\wedge z^2\phi_3\wedge zw\phi_3\end{cases}\\ &= \begin{cases}(az^*+bw^*)\wedge ew^*\wedge(bx^*+ey^*+kz^*+lw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\\ +(az^*+bw^*)\wedge(ez^*+fw^*)\wedge(ax^*+jz^*+kw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\\ +(bz^*+cw^*)\wedge ew^*\wedge(ax^*+jz^*+kw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\\ +(bz^*+cw^*)\wedge(ez^*+fw^*)\wedge(ax^*+jz^*+kw^*)\wedge(bx^*+ey^*+kz^*+lw^*)\end{cases}\\ &= \Big(ae(bf-ec)+af(af-be)+afbe+ae(bf-ce)\Big)\omega_{U^*}\\ &= (2abef-2ace^2+a^2f^2)\omega)U^*.\end{aligned}$$

We prove (4.1.13). The expansion of $\Gamma_{\phi_3}(xw^{(3)}\otimes\omega_U)$ consists of four summands. Three of the summands have a factor of $xw\phi_3\wedge xw\phi_3=0$, $xy\phi_3=0$, or (4.2.1). It follows that

$$\begin{aligned}&\Gamma_{\phi_3}(xw^{(3)}\otimes\omega_U)\\ &= x^2\phi_3\wedge yw\phi_3\wedge zw\phi_3\wedge w^2\phi_3\\ &= x^*\wedge(ez^*+fw^*)\wedge(bx^*+ey^*+kz^*+lw^*)\wedge(cx^*+fy^*+lz^*+mw^*)\end{aligned}$$

$$= \begin{vmatrix} 0 & e & f \\ e & k & l \\ f & l & m \end{vmatrix} \omega_{U^*}$$
$$= (-f^2k + 2efl - e^2m)\omega_{U^*}.$$

We prove (4.1.14). Observe that

$$\begin{aligned}
\Gamma_{\phi_3}(w^{(4)} \otimes \omega_U) &= xw\phi_3 \wedge yw\phi_3 \wedge zw\phi_3 \wedge w^2\phi_3 \\
&= \begin{cases} (bz^* + cw^*) \wedge (ez^* + fw^*) \wedge (bx^* + ey^* + kz^* + lw^*) \\ \wedge (cx^* + fy^* + lz^* + mw^*) \end{cases} \\
&= (bf - ce)z^* \wedge w^* \wedge (bf - ce)x^* \wedge y^* \\
&= (bf - ce)^2 \omega_{U^*}.
\end{aligned}$$

□

5. The Macaulay inverse system has a cubic term and $r = 1$.

In this section $\phi_3 = x^{*(3)} + \phi_{2,0}x^* + \phi_{3,0}$, with $\phi_{i,0} \in D_iU_0^*$, and the rank of

$$p_{\phi_{2,0}} : U_0 \to U_0^*$$

is equal to one, where $U^* = \boldsymbol{k}x^* \oplus U_0^*$ and $p_{\phi_{2,0}}(\ell_0) = \ell_0(\phi_{2,0}) \in U_0^*$, for $\ell_0 \in U_0$. We prove that if $\ell\phi_3$ is nonzero for all nonzero ℓ in U, then Γ_{ϕ_3} is not identically zero (and, therefore, A_{ϕ_3} has the weak Lefschetz property by Lemma 2.9.)

According to Lemma 3.2, the hypothesis about the rank of $p_{\phi_{2,0}}$ ensures that there is a basis y^*, z^*, w^* for U^* such that $\phi_{2,0} = az^{*(2)}$ for some unit a in $\boldsymbol{k}$. In particular, the basis elements $x^{*(2)}y^*$, $x^{*(2)}z^*$, $x^{*(2)}w^*$, $x^*y^{*(2)}$, $x^*y^*z^*$, $x^*y^*w^*$, $x^*z^{*(2)}$, $x^*z^*w^*$, and $x^*w^{*(2)}$ of D_3U^* appear in ϕ_3 with coefficient zero. Thus, ϕ_3 has the form of (5.1.1).

Proposition 5.1. *Let P_0 be a domain, U be a free P_0-module of rank four with dual module $U^* = \operatorname{Hom}_{P_0}(U, P_0)$, x, y, z, w be a basis for U with dual basis x^*, y^*, z^*, w^* for U^*. Let $\phi_3 \in D_3U^*$ have the form*

$$\phi_3 = \begin{cases} x^{*(3)} + ax^*z^{*(2)} + dy^*z^{*(2)} + ey^*z^*w^* \\ +fy^*w^{*(2)} + gy^{*(2)}z^* + hy^{*(2)}w^* + iy^{*(3)} \\ +jz^{*(3)} + kz^{*(2)}w^* + lz^*w^{*(2)} + mw^{*(3)}, \end{cases} \tag{5.1.1}$$

for parameters $a, \dots, m$ in P_0. Assume that

(a) *$a \neq 0$ in P_0 and*
(b) *$\ell\phi_3 \neq 0$ for all nonzero ℓ in U.*

Then Γ_{ϕ_3} is not identically zero.

Proof. The proof is by contradiction. We assume that Γ_{ϕ_3} is identically zero; we prove that $y\phi_3$ and $w\phi_3$ are linearly dependent. (Of course, this contradicts the hypothesis that $\ell\phi_3$ is nonzero whenever ℓ is a nonzero element of U). We show that

$$y\phi_3 = gy^*z^* + hy^*w^* + iy^{*(2)} + dz^{*(2)} + ez^*w^* + fw^{*(2)} \quad \text{and}$$
$$w\phi_3 = ey^*z^* + fy^*w^* + hy^{*(2)} + kz^{*(2)} + lz^*w^* + mw^{*(2)}$$

are linearly dependent by showing that the 2×2 minors of

$$\begin{bmatrix} g & h & i & d & e & f \\ e & f & h & k & \ell & m \end{bmatrix} \tag{5.1.2}$$

are all zero. The 2×2 minors of (5.1.2) are

$$\begin{array}{lll} G_0 = fg - eh, & G_1 = gh - ei, & G_2 = h^2 - fi, \\ G_3 = gk - de, & G_4 = hk - df, & G_5 = ik - dh, \\ G_6 = gl - e^2, & G_7 = hl - ef, & G_8 = il - eh, \\ G_9 = dl - ek, & G_{10} = gm - ef, & G_{11} = hm - f^2, \\ G_{12} = im - fh, & G_{13} = dm - fk, \text{ and} & G_{14} = em - fl. \end{array}$$

Define $F_0, \dots, F_8$ to be the following elements of P_0:

$$\begin{array}{ll} F_0 = fi - h^2 & F_1 = fg - 2eh + il \\ F_2 = gl - e^2, & F_3 = im - fh, \\ F_4 = gm - 2ef + hl, & F_5 = hm - f^2, \end{array}$$
$$F_6 = de^2 - d^2f + fgj - 2ehj + 2dhk - ik^2 - dgl + ijl,$$
$$F_7 = gjl - e^2j + 2dek - gk^2 - d^2l, \text{ and}$$
$$F_8 = gjm - 2efj + e^2k + 2dfk - hk^2 + hjl - gkl - d^2m.$$

Let ω_U and ω_{U^*} represent the bases $x\wedge y\wedge z\wedge w$ and $x^*\wedge y^*\wedge z^*\wedge w^*$ of $\bigwedge^4 U$ and $\bigwedge^4 U^*$, respectively. In Lemma 5.2 we show that

$$\Gamma_{\phi_3}(x^{(2)}y^{(2)}\otimes\omega_U) = aF_0\omega_{U^*}, \tag{5.1.3}$$
$$\Gamma_{\phi_3}(x^{(2)}yz\otimes\omega_U) = aF_1\omega_{U^*}, \tag{5.1.4}$$
$$\Gamma_{\phi_3}(x^{(2)}z^{(2)}\otimes\omega_U) = aF_2\omega_{U^*}, \tag{5.1.5}$$
$$\Gamma_{\phi_3}(x^{(2)}yw)\otimes\omega_U) = aF_3\omega_{U^*}, \tag{5.1.6}$$
$$\Gamma_{\phi_3}(x^{(2)}zw\otimes\omega_U) = aF_4\omega_{U^*}, \tag{5.1.7}$$
$$\Gamma_{\phi_3}(x^{(2)}w^{(2)}\otimes\omega_U) = aF_5\omega_{U^*}, \tag{5.1.8}$$
$$\Gamma_{\phi_3}(xyz^{(2)}\otimes\omega_U) = F_6\omega_{U^*}, \tag{5.1.9}$$
$$\Gamma_{\phi_3}(xz^{(3)}\otimes\omega_U) = F_7\omega_{U^*}, \text{ and} \tag{5.1.10}$$
$$\Gamma_{\phi_3}(xz^{(2)}w\otimes\omega_U) = F_8\omega_{U^*}. \tag{5.1.11}$$

The element a of the domain P_0 is nonzero. We have assumed that Γ_{ϕ_3} is identically zero. We conclude that

$$F_0 = F_1 = F_2 = F_3 = F_4 = F_5 = F_6 = F_7 = F_8 = 0.$$

Straightforward calculation show that

$$\begin{aligned}
G_0^2 &= -glF_0 + fgF_1 - h^2F_2, & G_1^2 &= -g^2F_0 + giF_1 - i^2F_2,\\
G_2 &= -F_0, & G_3^2 &= -d^2F_2 - g(F_7 - jF_2),\\
G_4^2 &= -k^2F_0 + f(-dF_2 + jF_1 - F_6), & G_5^2 &= -d^2F_0 + i(-dF_2 + jF_1 - F_6),\\
G_6 &= F_2, & G_7^2 &= -f^2F_2 + hlF_4 - glF_5,\\
G_8^2 &= -glF_0 + ilF_1 - h^2F_2, & G_9^2 &= -k^2F_2 - l(F_7 - jF_2),\\
G_{10}^2 &= -f^2F_2 + gmF_4 - glF_5, & G_{11} &= F_5,\\
G_{12} &= F_3, & G_{13}^2 &= -k^2F_5 - m(F_8 - jF_4 + kF_2), \text{ and}\\
G_{14}^2 &= -m^2F_2 + lm(F_4) - l^2F_5.
\end{aligned}$$

It follows that $G_i = 0$ for $0 \le i \le 14$; all 2×2 minors of (5.1.2) are zero; the elements $y\phi_3$ and $w\phi_3$ of U^* are linearly independent, and there exists a nonzero element ℓ of $\boldsymbol{k}y \oplus \boldsymbol{k}w \subseteq U$ with $\ell\phi_3 = 0$. This contradicts the ambient hypothesis. The proof is complete. □

Lemma 5.2. *In the notation of Proposition* 5.1*, the formulas* (5.1.3) *to* (5.1.11) *all hold.*

Proof. We prove (5.1.3). The expansion of $\Gamma_{\phi_3}(x^{(2)}y^{(2)} \otimes \omega_U)$ has six summands. Five of the summands have a factor of $xy\phi_3 = 0$ or $xw\phi_3 = 0$. Thus,

$$\begin{aligned}
\Gamma_{\phi_3}(x^{(2)}y^{(2)} \otimes \omega_U) &= x^2\phi_3 \wedge y^2\phi_3 \wedge xz\phi_3 \wedge yw\phi_3\\
&= x^* \wedge (gz^* + hw^* + iy^*) \wedge az^* \wedge (ez^* + fw^* + hy^*)\\
&= a(if - h^2)\omega_{U^*} = aF_0\omega_{U^*}.
\end{aligned}$$

We prove (5.1.4). The expansion of $\Gamma_{\phi_3}(x^{(2)}yz \otimes \omega_U)$ has twelve summands. Ten of the summands have a factor of $xy\phi_3 = 0$ or $xw\phi_3 = 0$. Thus,

$$\begin{aligned}
\Gamma_{\phi_3}(x^{(2)}yz \otimes \omega_U) &= x^2\phi_3 \wedge y^2\phi_3 \wedge xz\phi_3 \wedge zw\phi_3 + x^2\phi_3 \wedge yz\phi_3 \wedge xz\phi_3 \wedge yw\phi_3\\
&= \begin{cases} x^* \wedge (gz^* + hw^* + iy^*) \wedge az^* \wedge (ey^* + kz^* + lw^*)\\ +x^* \wedge (dz^* + ew^* + gy^*) \wedge az^* \wedge (ez^* + fw^* + hy^*)\end{cases}\\
&= \Big(a(il - eh) + a(gf - eh)\Big)\omega_{U^*}\\
&= a(fg - 2eh + il)\omega_{U^*} = aF_1\omega_{U^*}.
\end{aligned}$$

We prove (5.1.5). The expansion of $\Gamma_{\phi_3}(x^{(2)}z^{(2)}\otimes\omega_U)$ has six summands. Five of the summands have a factor of $xy\phi_3=0$ or $xw\phi_3=0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}z^{(2)}\otimes\omega_U)&=x^2\phi_3\wedge yz\phi_3\wedge xz\phi_3\wedge zw\phi_3\\&=x^*\wedge(dz^*+ew^*+gy^*)\wedge az^*\wedge(ey^*+kz^*+lw^*)\\&=a(gl-e^2)\omega_{U^*}=aF_2\omega_{U^*}.\end{aligned}$$

We prove (5.1.6). The expansion of $\Gamma_{\phi_3}(x^{(2)}yw\otimes\omega_U)$ has twelve summands. Eleven of the summands have a factor of $xy\phi_3=0$, $xw\phi_3=0$, or $yw\phi_3\wedge yw\phi_3=0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}yw\otimes\omega_U)&=x^2\phi_3\wedge y^2\phi_3\wedge xz\phi_3\wedge w^2\phi_3\\&=x^*\wedge(gz^*+hw^*+iy^*)\wedge az^*\wedge(fy^*+lz^*+mw^*)\\&=a(im-fh)\omega_{U^*}=aF_3\omega_{U^*}.\end{aligned}$$

We prove (5.1.7). The expansion of $\Gamma_{\phi_3}(x^{(2)}zw\otimes\omega_U)$ has twelve summands. Ten of the summands have a factor of $xy\phi_3=0$ or $xw\phi_3=0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}zw\otimes\omega_U)&=x^2\phi_3\wedge yz\phi_3\wedge xz\phi_3\wedge w^2\phi_3+x^2\phi_3\wedge yw\phi_3\wedge xz\phi_3\wedge zw\phi_3\\&=\begin{cases}x^*\wedge(dz^*+ew^*+gy^*)\wedge az^*\wedge(fy^*+lz^*+mw^*)\\+x^*\wedge(ez^*+fw^*+hy^*)\wedge az^*\wedge(ey^*+kz^*+lw^*)\end{cases}\\&=\Big(a(gm-ef)+a(hl-ef)\Big)\omega_{U^*}\\&=aF_4\omega_{U^*}.\end{aligned}$$

We prove (5.1.8). The expansion of $\Gamma_{\phi_3}(x^{(2)}w^{(2)}\otimes\omega_U)$ has six summands. Five of the summands have a factor of $xy\phi_3=0$ or $xw\phi_3=0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}w^{(2)}\otimes\omega_U)&=x^2\phi_3\wedge yw\phi_3\wedge xz\phi_3\wedge w^2\phi_3\\&=x^*\wedge(ez^*+fw^*+hy^*)\wedge az^*\wedge(fy^*+lz^*+mw^*)\\&=a(hm-f^2)\omega_{U^*}\\&=aF_5\omega_{U^*}.\end{aligned}$$

We prove (5.1.9). The expansion of $\Gamma_{\phi_3}(xyz^{(2)}\otimes\omega_U)$ has twelve summands. Ten of the summands have a factor of $xy\phi_3=0$, $xw\phi_3=0$, $xz\phi_3\wedge xz\phi_3=0$, or $yz\phi_3\wedge yz\phi_3$. Thus,

$$\begin{aligned}&\Gamma_{\phi_3}(xyz^{(2)}\otimes\omega_U)\\&=x^2\phi_3\wedge y^2\phi_3\wedge z^2\phi_3\wedge zw\phi_3+x^2\phi_3\wedge yz\phi_3\wedge z^2\phi_3\wedge yw\phi_3\\&=\begin{cases}x^*\wedge(gz^*+hw^*+iy^*)\wedge(ax^*+dy^*+jz^*+kw^*)\wedge(ey^*+kz^*+lw^*)\\+x^*\wedge(dz^*+ew^*+gy^*)\wedge(ax^*+dy^*+jz^*+kw^*)\wedge(ez^*+fw^*+hy^*)\end{cases}\\&=\left(\det\begin{bmatrix}i&g&h\\d&j&k\\e&k&l\end{bmatrix}+\det\begin{bmatrix}g&d&e\\d&j&k\\h&e&f\end{bmatrix}\right)\omega_{U^*}\end{aligned}$$

$$= (de^2 - d^2f + fgj - 2ehj + 2dhk - ik^2 - dgl + ijl)\omega_{U^*}$$
$$= F_6\omega_{U^*}.$$

We prove (5.1.10). The expansion of $\Gamma_{\phi_3}(xz^{(3)} \otimes \omega_U)$ has four summands. Three of the summands have a factor of $xy\phi_3 = 0$, $xw\phi_3 = 0$, or $xz\phi_3 \wedge xz\phi_3 = 0$. Thus,

$$\begin{aligned}
&\Gamma_{\phi_3}(xz^{(3)} \otimes \omega_U)\\
&= x^2\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3\\
&= x^* \wedge (dz^* + ew^* + gy^*) \wedge (ax^* + dy^* + jz^* + kw^*) \wedge (ey^* + kz^* + lw^*)\\
&= \det \begin{bmatrix} g & d & e \\ d & j & k \\ e & k & l \end{bmatrix} \omega_{U^*}\\
&= (-e^2j + 2dek - gk^2 - d^2l + gjl)\omega_{U^*}\\
&= F_7\omega_{U^*}.
\end{aligned}$$

We prove (5.1.11). The expansion of $\Gamma_{\phi_3}(xz^{(2)}w \otimes \omega_U)$ has twelve summands. Ten of the summands have a factor of

$$xy\phi_3 = 0, \quad xw\phi_3 = 0, \quad zw\phi_3 \wedge zw\phi_3 = 0, \quad \text{or} \quad xz\phi_3 \wedge xz\phi_3 = 0.$$

Thus,

$$\begin{aligned}
&\Gamma_{\phi_3}(xz^{(2)}w \otimes \omega_U)\\
&= x^2\phi_3 \wedge yw\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3 + x^2\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3 \wedge w^2\phi_3\\
&= \begin{cases} x^* \wedge (ez^* + fw^* + hy^*) \wedge (ax^* + dy^* + jz^* + kw^*) \wedge (ey^* + kz^* + lw^*) \\ +x^* \wedge (dz^* + ew^* + gy^*) \wedge (ax^* + dy^* + jz^* + kw^*) \wedge (fy^* + lz^* + mw^*) \end{cases}\\
&= \left(\det \begin{bmatrix} h & e & f \\ d & j & k \\ e & k & l \end{bmatrix} + \det \begin{bmatrix} g & d & e \\ d & j & k \\ f & l & m \end{bmatrix} \right) \omega_{U^*}\\
&= (-2efj + e^2k + 2dfk - hk^2 + hjl - gkl - d^2m + gjm)\omega_{U^*}\\
&= F_8\omega_{U^*}.
\end{aligned}$$

□

Lemma 5.3. *Lemma* 2.11 *holds when* ϕ_3 *has the form of Lemma* 3.1.(a) *and* $r = 0$ *or* $r = 3$.

Proof. First we consider the case $r = 0$. In this case, $\phi_3 = x^{*(3)} + \phi_{3,0}$ with $\phi_{3,0} \in U_0^*$ and $\boldsymbol{k}x^* \oplus U_0^* = U^*$. Furthermore, either the characteristic of $\boldsymbol{k}$ is not two; or else, the characteristic of $\boldsymbol{k}$ is two but there does not exist a basis y^*, z^*, w^* for U_0^* with $\phi_{3,0}$ is equal to $y^*z^*w^*$.

Fix some basis y^*, z^*, w^* for U_0^*. Let x, y, z, w be the basis for U which is dual to the basis x^*, y^*, z^*, w^* for U^*. Let

$\omega_{U_0^*}$ be the basis $y^* \wedge z^* \wedge w^*$ of $\bigwedge^3 U_0^*$,

ω_{U_0} be the basis $y \wedge z \wedge w$ of $\bigwedge^3 U_0$,

ω_{U^*} be the basis $x^*\wedge y^*\wedge z^*\wedge w^*$ of $\bigwedge^4 U^*$, and

ω_U be the basis $x\wedge y\wedge z\wedge w$ of $\bigwedge^4 U$.

According to Lemma 8.2 there is an element $\theta\in D_3U_0$ so that $\Gamma_{\phi_{3,0}}(\theta\otimes\omega_{U_0})=\omega_{U_0^*}$. The fact that $x\phi_{3,0}=0$ and $\ell x^{*(3)}=0$ for $\ell\in U_0$ ensures that

$$\Gamma_{\phi_3}(x\theta\otimes\omega_U)=x^2\phi_3\wedge\Gamma_{\phi_{3,0}\otimes\omega_{U_0}}(\theta)=x^*\wedge\omega_{U_0^*}=\omega_{U^*}.$$

Conclude that $\Gamma_{\phi_3}\neq 0$.

Now we consider the case $r=3$. In this case, $\phi_3=\alpha x^{*(3)}+x^*\phi_{2,0}+\phi_{3,0}$, with $\phi_{i,0}$ in $D_iU_0^*$ and $\ell_0\phi_{2,0}$ not zero for $\ell_0\in U_0=\boldsymbol{k}y\oplus\boldsymbol{k}z\oplus\boldsymbol{k}w$. It follows that $y\phi_{2,0}$, $z\phi_{2,0}$, and $w\phi_{2,0}$ are linearly independent in U_0^*. Hence, $y\phi_{2,0}\wedge z\phi_{2,0}\wedge w\phi_{2,0}$ is nonzero in $\bigwedge^3 U_0^*$ and

$$\Gamma_{\phi_3}(x^4\otimes\omega_U)=\alpha x^*\wedge y\phi_{2,0}\wedge z\phi_{2,0}\wedge w\phi_{2,0}$$

is nonzero in $\bigwedge^4 U^*$. □

6. THE MACAULAY INVERSE SYSTEM DOES NOT HAVE A CUBIC TERM, ONE CASE.

In this section ϕ_3 does not have any terms of the form $\ell^{*(3)}$ with $\ell^*\in U^*$, but ϕ_3 can be written in the form

$$\phi_3=x^{*(2)}y^*+\phi_{2,0}x^*+\phi_{3,0},$$

with $\phi_{i,0}\in D_iU_0^*$, where $U^*=\boldsymbol{k}x^*\oplus U_0^*$ and y^* is a nonzero element of U_0^*. (This case can be avoided in most characteristics, but is necessary in characteristic three.) In Propositions 6.4 and 7.1 we prove that if $\ell\phi_3$ is nonzero for all nonzero ℓ in U, then Γ_{ϕ_3} is not identically zero (and, therefore, A_{ϕ_3} has the weak Lefschetz property by Lemma 2.9.) The proof of Propositions 6.4 and 7.1 proceed like the proof of Proposition 5.1: we assume that Γ_{ϕ_3} is identically zero and we exhibit a nonzero linear form ℓ in U with $\ell\phi_3=0$.

The hypothesis that Γ_{ϕ_3} is identically zero imposes further constraints on the form of ϕ_3 that we identify before beginning the proof of Proposition 6.4.

Lemma 6.1. *Let x^* and y^* be linearly independent elements in the four dimensional vector space U^* over the field $\boldsymbol{k}$ and let*

$$\phi_3=x^{*(2)}y^*+\phi_{2,0}x^*+\phi_{3,0}$$

be an element of D_3U^* with $\phi_{i,0}\in D_iU_0^*$ for $U^*=\boldsymbol{k}x^*\oplus U_0^*$ and $y^*\in U_0^*$. Assume that $\phi_{3,0}$ does not have any terms of the form $\ell^{*(3)}$ for $\ell^*\in U_0^*$. If Γ_{ϕ_3} is identically zero, then there exists a basis y^*,z^*,w^* for U_0^* such that

(i) *$\phi_{2,0}$ is in the subspace of U_0^* spanned by $z^{*(2)}$, $y^{*(2)}$, y^*z^*, and y^*w^*, and*
(ii) *$\phi_{3,0}$ does not involve $z^*w^{*(2)}$.*

Remark 6.2. Assertion (i) can also be written $\phi_{2,0}$ does not involve either z^*w^* or $w^{*(2)}$ and assertion (ii) can also be written $\phi_{3,0}$ is in the subspace of $D_3U_0^*$ spanned by $y^{*(2)}z^*$, $y^{*(2)}w^*$, $y^*z^{*(2)}$, $y^*z^*w^*$, $y^*w^{*(2)}$, and $z^{*(2)}w^*$.

Cases 6.3. Once we prove Lemma 6.1, then there are two cases.

Case 1. The case where $\phi_{2,0}$ is in the subspace of U_0^* spanned by $y^{*(2)}$, y^*z^*, and y^*w^* is treated in Proposition 7.1.

Case 2. The case where $\phi_{2,0}$ also involves $z^{*(2)}$ is treated in Proposition 6.4.

Proof. We prove (i). Let y^*, Z^*, W^* be any basis for U_0^*. We are given a, b, c in $\boldsymbol{k}$ with

$$\phi_3 = x^{*(2)}y^* + ax^*Z^{*(2)} + bx^*Z^*W^* + cx^*W^{*(2)} + \phi_3',$$

with ϕ_3' in the subspace of $D_3U_0^*$ spanned by $x^*y^{*(2)}$, $x^*y^*Z^*$, $x^*y^*W^*$, $y^{*(2)}Z^*$, $y^{*(2)}W^*$, $y^*Z^{*(2)}$, $y^*Z^*W^*$, $y^*W^{*(2)}$, $Z^{*(2)}W^*$, and $Z^*W^{*(2)}$. Let ω_U and ω_{U^*} be the bases $x\wedge y\wedge Z\wedge W$ and $x^*\wedge y^*\wedge Z^*\wedge W^*$ of U and U^*, respectively. Observe that

$$\begin{aligned}\Gamma_{\phi_3}(x^{(4)}\otimes\omega_U) &= x^2\phi_3\wedge xy\phi_3\wedge xZ\phi_3\wedge xW\phi_3\\ &= y^*\wedge(x^*+\theta)\wedge(\alpha_1y^*+aZ^*+bW^*)\wedge(\alpha_2y^*+bZ^*+cW^*)\\ &= (ac-b^2)\omega_{U^*},\end{aligned}$$

for some $\theta\in U_0^*$ and some α_1 and α_2 in $\boldsymbol{k}$. The hypothesis that Γ_{ϕ_3} is identically zero guarantees that

$$ac-b^2=0.$$

Apply Lemma 3.3 and choose a new basis z^*, w^* for $\boldsymbol{k}Z^*\oplus\boldsymbol{k}W^*$ so that

$$ax^*Z^{*(2)}+bx^*Z^*W^*+cx^*W^{*(2)} \text{ is in } \boldsymbol{k}x^*z^{*(2)}.$$

We have established (i).

We prove (ii). At this point

$$\phi_3 = \begin{cases} x^{*(2)}y^* + dx^*y^{*(2)} + ey^{*(2)}z^* + fy^{*(2)}w^*\\ +gx^*z^{*(2)} + hy^*z^{*(2)} + iz^{*(2)}w^* + ky^*w^{*(2)} + lz^*w^{*(2)}\\ +mx^*y^*z^* + nx^*y^*w^* + py^*z^*w^*,\end{cases} \tag{6.3.1}$$

for parameters $d,\dots,p$ from $\boldsymbol{k}$. Let ω_U and ω_{U^*} be the bases $x\wedge y\wedge z\wedge w$ and $x^*\wedge y^*\wedge z^*\wedge w^*$ of U and U^*, respectively. The expansion of $\Gamma_{\phi_3}(x^{*(2)}w^{*(2)}\otimes\omega_U)$ involves six summands. Five of the summands have a factor of $xw\phi_3\wedge xw\phi_3=0$,

$$x^2\phi_3\wedge xw\phi_3\subseteq \textstyle\bigwedge^2(\boldsymbol{k}y^*)=0, \text{ or } \bigwedge^3\boldsymbol{k}(x^2\phi_3,xz\phi_3,xw\phi_3,w^2\phi_3)\subseteq\bigwedge^3(\boldsymbol{k}y^*\oplus\boldsymbol{k}z^*)=0.$$

Thus,

$$\begin{aligned}
&\Gamma_{\phi_3}(x^{*(2)}w^{*(2)}\otimes\omega_U)\\
&=x^2\phi_3\wedge xy\phi_3\wedge zw\phi_3\wedge w^2\phi_3\\
&=(y^*\wedge(x^*+dy^*+mz^*+nw^*)\wedge(iz^*+lw^*+py^*)\wedge(ky^*+lz^*)\\
&=l^2\omega_{U^*}.
\end{aligned}$$

The hypothesis that Γ_{ϕ_3} is identically zero guarantees that $l=0$; and this completes the proof of (ii). □

We treat Case 2 of 6.3. The Macaulay inverse system for this case is the ϕ_3 of (6.3.1) with l set equal to zero. We have recorded this Macaulay inverse system as (6.4.1).

Proposition 6.4. *Let P_0 be a domain, U be a free P_0-module of rank four with dual module $U^*=\operatorname{Hom}_{P_0}(U,P_0)$, x,y,z,w be a basis for U with dual basis x^*,y^*,z^*,w^* for U^*. Let $\phi_3\in D_3U^*$ have the form*

$$\phi_3=\begin{cases}\ \ ax^{*(2)}y^*+dx^*y^{*(2)}+ey^{*(2)}z^*+fy^{*(2)}w^*\\ +gx^*z^{*(2)}+hy^*z^{*(2)}+iz^{*(2)}w^*+ky^*w^{*(2)}\\ +mx^*y^*z^*+nx^*y^*w^*+py^*z^*w^*,\end{cases}\tag{6.4.1}$$

for parameters $a,\dots,p$ from P_0. Assume that

(a) *$a\neq 0$ in P_0,*
(b) *$g\neq 0$ in P_0, and*
(c) *$\ell\phi_3\neq 0$ for all nonzero ℓ in U.*

Then Γ_{ϕ_3} is not identically zero.

Proof. The proof is by contradiction. We assume that Γ_{ϕ_3} is identically zero; we prove that the elements

$$\begin{aligned}x\phi_3&=ax^*y^*+dy^{*(2)}+my^*z^*+ny^*w^*+gz^{*(2)}\text{ and}\\ w\phi_3&=nx^*y^*+fy^{*(2)}+py^*z^*+ky^*w^*+iz^{*(2)}\end{aligned}$$

of U^* are linearly dependent. We do this by showing that the 2×2 minors of the matrix

$$\begin{bmatrix}a&d&m&n&g\\ n&f&p&k&i\end{bmatrix}\tag{6.4.2}$$

are all zero.

In Lemma 6.5 we calculate $\Gamma_{\phi_3}(-\otimes\omega_U)$ for five elements of D_3U. The assumption that Γ_{ϕ_3} is identically zero, together with the hypothesis that $ag\neq 0$ guarantees that n^2-ak, $ai-gn$, $im-gp$, $gk-in$, and

$$I_0=-af^2g-d^2gk-adhk+dkm^2+2dfgn+dhn^2-2dmnp+adp^2\tag{6.4.3}$$

all are zero. In Lemma 6.6 we show that $dk-fn$ is also zero.

The entries a and g of matrix 6.4.2 are nonzero. We know that columns 1 and 4 are linearly dependent, as are columns 1 and 5, columns 3 and 5, columns 4 and 5, and columns 2 and 4. If $n \neq 0$, then column 4 is a non-zero multiple of column 1 and therefore columns 1 and 2 are linearly dependent, indeed, in this case, column 1 is a basis for the column space of the matrix (6.4.2).

On the other hand, if $n = 0$, then the fact that columns 1,3,4,5 are all multiples of column 1 and n is the bottom entry of column 1 forces $n = p = i = k = 0$. When $n = p = i = k = 0$, then $\Gamma_{\phi_3}(xy^{(3)} \otimes \omega_U) = I_0\omega_{U^*}$ (from Lemma 6.5) becomes $-af^2g = 0$. The parameters a and g are nonzero; hence $f = 0$ and once again the matrix (6.4.2) has rank one.

Thus, in every case, $x\phi_3$ and $w\phi_3$ are linearly dependent and there is a nonzero linear form $\ell = \alpha x + \beta w$ with $\ell\phi_3 = 0$, which is a contradiction. □

Lemma 6.5. *Retain the notation of Proposition* 6.4. *Let* ω_U *and* ω_{U^*} *be the basis elements* $x \wedge y \wedge z \wedge w$ *and* $x^* \wedge y^* \wedge z^* \wedge w^*$ *of* $\bigwedge^4 U$ *and* $\bigwedge^4 U^*$*, respectively. Then the following statements hold*:

(a) $\Gamma_{\phi_3}(x^{(3)}y \otimes \omega_U) = ag(n^2 - ak)\omega_{U^*}$,
(b) $\Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U) = (ai - gn)^2\omega_{U^*}$,
(c) $\Gamma_{\phi_3}(z^{(4)} \otimes \omega_U) = (im - gp)^2\omega_{U^*}$,
(d) $\Gamma_{\phi_3}(z^{(2)}w^{(2)} \otimes \omega_U) = (gk - in)^2\omega_{U^*}$, *and*
(e) $\Gamma_{\phi_3}(xy^{(3)} \otimes \omega_U) = I_0\omega_{U^*}$, *where* I_0 *is given in* (6.4.3).

Proof. (a) The expansion of $\Gamma_{\phi_3}(x^{(3)}y \otimes \omega_U)$ has four summands. Three of the summands have a factor of $xy\phi_3 \wedge xy\phi_3 = 0$ or $x^2\phi_3 \wedge xw\phi_3 \in \bigwedge^2 P_0y^* = 0$. It follows that

$$\begin{aligned}
&\Gamma_{\phi_3}(x^{(3)}y \otimes \omega_U)\\
&= x^2\phi_3 \wedge xy\phi_3 \wedge xz\phi_3 \wedge yw\phi_3\\
&= ay^* \wedge (ax^* + dy^* + mz^* + nw^*) \wedge (gz^* + my^*) \wedge (fy^* + kw^* + nx^* + pz^*)\\
&= -ag(ak - n^2)\omega_{U^*}.
\end{aligned}$$

(b) The expansion of $\Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U)$ has six summands. Four of the summands contain a factor of $xz\phi_3 \wedge xz\phi_3 = 0$, or $x^2\phi_3 \wedge xw\phi_3 \in \bigwedge^2 \boldsymbol{k}y^* = 0$, or

$$x^2\phi_3 \wedge xz\phi_3 \wedge zw\phi_3 \in \bigwedge^3(\boldsymbol{k}y^* \oplus \boldsymbol{k}z^*) = 0.$$

It follows that

$$\begin{aligned}
&\Gamma_{\phi_3}(x^{(2)}z^{(2)} \otimes \omega_U)\\
&= \begin{cases} x^2\phi_3 \wedge xy\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3\\ +xz\phi_3 \wedge xy\phi_3 \wedge z^2\phi_3 \wedge xw\phi_3 \end{cases}\\
&= \begin{cases} ay^* \wedge (ax^* + dy^* + mz^* + nw^*) \wedge (gx^* + hy^* + iw^*) \wedge (iz^* + py^*)\\ +(gz^* + my^*) \wedge (ax^* + dy^* + mz^* + nw^*) \wedge (gx^* + hy^* + iw^*) \wedge ny^* \end{cases}
\end{aligned}$$

$$= \Big(ai(ai-ng)-gn(ai-gn)\Big)\omega_{U^*} = (ai-ng)^2\omega_{U^*}.$$

(c) One computes that

$$\begin{aligned}
&\Gamma_{\phi_3}(z^{(4)}\otimes\omega_U)\\
&= xz\phi_3\wedge yz\phi_3\wedge z^2\phi_3\wedge zw\phi_3\\
&= (gz^*+my^*)\wedge(ey^*+hz^*+mx^*+pw^*)\wedge(gx^*+hy^*+iw^*)\wedge(iz^*+py^*)\\
&= (mi-gp)^2\omega_{U^*}.
\end{aligned}$$

(d) The expansion of $\Gamma_{\phi_3}(z^{(2)}w^{(2)}\otimes\omega_U)$ has six summands. Four of the summands have a factor of $zw\phi_3\wedge zw\phi_3=0$, or

$$xz\phi_3\wedge zw\phi_3\wedge w^2\phi_3\in\bigwedge^3(\boldsymbol{k}y^*\oplus\boldsymbol{k}z^*)=0,\qquad\text{or}\qquad xw\phi_3\wedge w^2\phi_3\in\bigwedge^2\boldsymbol{k}y^*=0.$$

It follows that

$$\begin{aligned}
&\Gamma_{\phi_3}(z^{(2)}w^{(2)}\otimes\omega_U)\\
&= xz\phi_3\wedge yw\phi_3\wedge z^2\phi_3\wedge w^2\phi_3+xw\phi_3\wedge yw\phi_3\wedge z^2\phi_3\wedge zw\phi_3\\
&= \begin{cases}(gz^*+my^*)\wedge(fy^*+kw^*+nx^*+pz^*)\wedge(gx^*+hy^*+iw^*)\wedge ky^*\\ +ny^*\wedge(fy^*+kw^*+nx^*+pz^*)\wedge(gx^*+hy^*+iw^*)\wedge(iz^*+py^*)\end{cases}\\
&= -kg(ni-gk)\omega_{U^*}+ni(ni-gk)\omega_{U^*}\\
&= (ni-gk)^2\omega_{U^*}.
\end{aligned}$$

(e) The expansion of $\Gamma_{\phi_3}(xy^{(3)}\otimes\omega_U)$ has four summands. One of the summands has a factor of $xy\phi_3\wedge xy\phi_3=0$. Thus,

$$\begin{aligned}
&\Gamma_{\phi_3}(xy^{(3)}\otimes\omega_U)\\
&= \begin{cases}x^2\phi_3\wedge y^2\phi_3\wedge yz\phi_3\wedge yw\phi_3\\ +xy\phi_3\wedge y^2\phi_3\wedge xz\phi_3\wedge yw\phi_3\\ +xy\phi_3\wedge y^2\phi_3\wedge yz\phi_3\wedge xw\phi_3\end{cases}\\
&= \begin{cases}\begin{cases}ay^*\wedge(dx^*+ez^*+fw^*)\wedge(ey^*+hz^*+mx^*+pw^*)\\ \wedge(fy^*+kw^*+nx^*+pz^*)\end{cases}\\ +\begin{cases}(ax^*+dy^*+mz^*+nw^*)\wedge(dx^*+ez^*+fw^*)\wedge(gz^*+my^*)\\ \wedge(fy^*+kw^*+nx^*+pz^*)\end{cases}\\ +\begin{cases}(ax^*+dy^*+mz^*+nw^*)\wedge(dx^*+ez^*+fw^*)\\ \wedge(ey^*+hz^*+mx^*+pw^*)\wedge ny^*\end{cases}\end{cases}\\
&= \left(-a\det\begin{bmatrix}d&e&f\\m&h&p\\n&p&k\end{bmatrix}+\det\begin{bmatrix}a&d&m&n\\d&0&e&f\\0&m&g&0\\n&f&p&k\end{bmatrix}+n\det\begin{bmatrix}a&m&n\\d&e&f\\m&h&p\end{bmatrix}\right)\omega_{U^*}\\
&= (-af^2g-d^2gk-adhk+dkm^2+2dfgn+dhn^2-2dmnp+adp^2)\omega_{U^*}
\end{aligned}$$

$= I_0\omega_{U^*}.$ □

Lemma 6.6. *The parameters of Proposition* 6.4 *satisfy*

$$ag^2(dk-fn)^2 \in (n^2-ak, ai-gn, im-gp).$$

Proof. Recall that I_0, from (6.4.3), is equal to zero. A straightforward calculation shows that

$$\begin{aligned}&-gn^2I_0\\&+(-d^2g^2k+2dfg^2n-dim^2n+dghn^2)(n^2-ak)\\&+(-dkm^2n+dmn^2p)(ai-gn)\\&+(dmn^3-adn^2p)(im-gp)\end{aligned}$$

is equal to $ag^2(dk-fn)^2$. □

7. THE MACAULAY INVERSE SYSTEM DOES NOT HAVE A CUBIC TERM, THE OTHER CASE.

We treat Case 1 from 6.3. The statement is the same as the statement of Proposition 6.4, except now "g" is equal to zero.

Proposition 7.1. *Let P_0 be a domain, U be a free P_0-module of rank four with dual module $U^* = \mathrm{Hom}_{P_0}(U,P_0)$, x,y,z,w be a basis for U with dual basis x^*,y^*,z^*,w^* for U^*. Let $\phi_3 \in D_3U^*$ have the form*

$$\phi_3 = \begin{cases} \ ax^{*(2)}y^* + dx^*y^{*(2)} + ey^{*(2)}z^* + fy^{*(2)}w^* \\ +hy^*z^{*(2)} + iz^{*(2)}w^* + ky^*w^{*(2)} \\ +mx^*y^*z^* + nx^*y^*w^* + py^*z^*w^*, \end{cases} \tag{7.1.1}$$

for parameters $a,\dots,p$ from P_0. Assume that

(a) $a \neq 0$ *in P_0, and*

(b) $\ell\phi_3 \neq 0$ *for all nonzero ℓ in U.*

Then Γ_{ϕ_3} is not identically zero.

Proof. The proof is by contradiction. We suppose that Γ_{ϕ_3} is identically zero. We prove that there is a nonzero element ℓ of U with $\ell\phi_3 = 0$.

We first show that $i = 0$. Fix the bases

$$\omega_U = x\wedge y\wedge z\wedge w \quad \text{and} \quad \omega_{U^*} = x^*\wedge y^*\wedge z^*\wedge w^*$$

of $\bigwedge^4 U$ and $\bigwedge^4 U^*$, respectively. The expansion of $\Gamma_{\phi_3}(x^{(2)}z^{(2)}\otimes\omega_U)$ has six summands; however five of the summands have a factor of $xz\phi_z\wedge xz\phi_3 = 0$ or a factor from

$$\textstyle\bigwedge^2(\boldsymbol{k}x^2\phi_3 + \boldsymbol{k}xz\phi_3 + \boldsymbol{k}xw\phi_3) \subseteq \bigwedge^2 \boldsymbol{k}y^* = 0.$$

Thus,

$$\Gamma_{\phi_3}(x^{(2)}z^{(2)}\otimes\omega_U)$$

$$\begin{aligned}&= x^2\phi_3 \wedge xy\phi_3 \wedge z^2\phi_3 \wedge zw\phi_3\\ &= (ay^*)\wedge(ax^*+dy^*+mz^*+nw^*)\wedge(hy^*+iw^*)\wedge(iz^*+py^*)\\ &= a^2i^2\omega_{U^*}.\end{aligned}$$

The parameter a is not zero. We have assumed that Γ_{ϕ_3} is identically zero. We conclude that $i=0$.

Ultimately, we prove that the elements

$$\begin{aligned} x\phi_3 &= ax^*y^* + dy^{*(2)} + my^*z^* + ny^*w^*\\ z\phi_3 &= mx^*y^* + ey^{*(2)} + hy^*z^* + py^*w^*\\ w\phi_3 &= nx^*y^* + fy^{*(2)} + py^*z^* + ky^*w^*\end{aligned}$$

of D_2U^* are linearly dependent. We do this by showing that the maximal minors of the matrix

$$\begin{bmatrix} a & d & m & n\\ m & e & h & p\\ n & f & p & k\end{bmatrix}$$

are zero. We view the above matrix as the submatrix of the symmetric matrix

$$M = \begin{bmatrix} a & d & m & n\\ d & 0 & e & f\\ m & e & h & p\\ n & f & p & k\end{bmatrix}, \tag{7.1.2}$$

which is obtained by deleting row 2. In Lemma 7.2 we prove that

$$\Gamma_{\phi_3}(x^{(2)}y^{(2)}\otimes\omega_U) = -a\det M[2;2]\omega_{U^*} \quad\text{and}\quad \Gamma_{\phi_3}(y^{(4)}\otimes\omega_U) = \det M\omega_{U^*},$$

where

(7.1.3) $M[r_1,\dots,r_s;c_1,\dots,c_t]$ is the submatrix of M obtained by deleting rows $r_1,\dots,r_s$ and columns $c_1,\dots,c_t$.

We have assumed that Γ_{ϕ_3} is identically zero and that a is nonzero. We conclude that $M[2;2]$ and M both have determinant zero. Observe that

$$\begin{aligned}(\det M[2;1])^2 &= \det M[1;1]\det M[2;2] - \det M[1,2;1,2]\det M,\\ (\det M[2;3])^2 &= \det M[3;3]\det M[2;2] - \det M[2,3;2,3]\det M, \text{ and}\\ (\det M[2;4])^2 &= \det M[4;4]\det M[2;2] - \det M[2,4;2,4]\det M.\end{aligned} \tag{7.1.4}$$

One can check these formulas by hand or one can use the characteristic free straightening technique of [6] to verify these formulas; see Remark 7.3. At any rate, all four maximal minors of M with row 2 deleted are zero; $x\phi_3$, $z\phi_3$, $w\phi_3$ are linearly dependent elements of D_2U^*; and $\ell\phi_3 = 0$ for some nonzero element ℓ of $\boldsymbol{k}x\oplus\boldsymbol{k}z\oplus\boldsymbol{k}w$. This contradiction completes the proof. □

Lemma 7.2. *Let ϕ_3 be the element of D_3U^* which is given in* (7.1.1) *with i equal to zero and M be the matrix of* (7.1.2). *Adopt the convention of* (7.1.3). *Then*

(a) $\Gamma_{\phi_3}(x^{(2)}y^{(2)}\otimes\omega_U) = -a(\det M[2;2])\,\omega_{U^*}$ *and*

(b) $\Gamma_{\phi_3}(y^{(4)}\otimes\omega_U) = (\det M)\,\omega_{U^*}$.

Proof. The expansion of $\Gamma_{\phi_3}(x^{(2)}y^{(2)}\otimes\omega_U)$ consists of six summands; however five of the summands have a factor of $xy\phi_3\wedge xy\phi_3 = 0$ or have a factor from

$$\textstyle\bigwedge^2 \boldsymbol{k}(x^2\phi_3, xz\phi_3, xw\phi_3) \subseteq \bigwedge^2 \boldsymbol{k}y^* = 0.$$

Thus,

$$\begin{aligned}
\Gamma_{\phi_3}(x^{(2)}y^{(2)}\otimes\omega_U) &= x^2\phi_3\wedge xy\phi_3\wedge yz\phi_3\wedge yw\phi_3\\
&= \begin{cases}(ay^*)\wedge(ax^*+dy^*+mz^*+nw^*)\\ \wedge(ey^*+hz^*+mx^*+pw^*)\wedge(fy^*+kw^*+nx^*+pz^*)\end{cases}\\
&= -a\det\begin{bmatrix} a & m & n\\ m & h & p\\ n & p & k\end{bmatrix}.
\end{aligned}$$

We also compute

$$\begin{aligned}
\Gamma_{\phi_3}(y^{(4)}\otimes\omega_U) &= xy\phi_3\wedge y^2\phi_3\wedge yz\phi_3\wedge yw\phi_3\\
&= \begin{cases}(ax^*+dy^*+mz^*+nw^*)\wedge(dw^*+ez^*+fw^*)\\ \wedge(ey^*+hz^*+mx^*+pw^*)\wedge(fy^*+kw^*+nx^*+pz^*)\end{cases}\\
&= \det\begin{bmatrix} a & d & m & n\\ d & 0 & e & f\\ m & e & h & p\\ n & f & p & k\end{bmatrix}\omega_{U^*}.
\end{aligned}$$

□

Remark 7.3. The identities of (7.1.4) about the minors of the 4×4 symmetric matrix M of (7.1.2) actually hold in a general situation. The matrix M need not be symmetric and the size of M need not be 4.

Recall the Plücker relations on the maximal minors of a matrix. Let Y be an $r\times c$ matrix, with $r\le c$. The entries of Y are in the commutative ring R. Let $a_1,\dots,a_{r-1}$ and $b_1,\dots,b_{r+1}$ be integers between 1 and c and $Y(s_1,\dots,s_\ell)$ represents the matrix whose columns are the columns $s_1,s_2,\dots,s_\ell$ of Y. Then

$$\sum_{i=1}^{r+1}\det Y(a_1,\dots,a_{r-1},b_i)\det Y(b_1,\dots,\widehat{b_i},\dots b_{r+1}) = 0, \tag{7.3.1}$$

(where $\widehat{b_i}$ means that column b_i has been deleted.) Let M be a 4×4 matrix and Y be the matrix

$$Y = [M|J],$$

where

$$J = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0\end{bmatrix}.$$

Apply (7.3.1) to the matrix $Y = [M|J]$, where M is given in (7.1.2). Take

$$(\{a_1,a_1,a_3\},\{b_1,b_2,b_3,b_4,b_5\})$$

to be

$$(\{2,3,4\},\{1,3,4,7,8\}),\ \ (\{1,2,4\},\{1,3,4,6,7\}),\ \text{ and }\ (\{1,2,3\},\{1,3,4,5,7\})$$

to obtain (7.1.4).

8. The three variables Theorem.

In this section we state and prove the three variable version of the Main Theorem (Theorem 1.1). Our precise formulation of the three variable version (see Lemma 8.2) is used in the inductive part of the proof of Theorem 1.1. (See the case $r = 0$ in Lemma 5.3.) Furthermore, we prove the three variable version using the same argument as we use for the four variable version; except there are fewer cases and each calculation is more straightforward. The reader might want to read the present section as a preparation for reading the proof of Theorem 1.1.

Theorem 8.1. *Let $\boldsymbol{k}$ be a field and A be a standard graded Artinian Gorenstein $\boldsymbol{k}$-algebra of embedding dimension three and socle degree three. If the characteristic of $\boldsymbol{k}$ is different than two, then A has the weak Lefschetz property. If the characteristic of A is equal to two, then A has the weak Lefschetz property if and only if A is not isomorphic to*

$$\frac{\boldsymbol{k}[x,y,z]}{(x^2,y^2,z^2)}. \tag{8.1.1}$$

Proof. It is clear that the ring A of (8.1.1) does not satisfy the weak Lefschetz property. Indeed, if ℓ is a nonzero linear form of A, then $\ell^2 = 0$ in A; so, in particular, $\ell : A_1 \to A_2$ is not injective. The Macaulay inverse system for A is $\phi_3 = x^*y^*z^*$.

To complete the proof of Theorem 8.1, we adopt Data 2.3, with $d = 3$, and use the method of the proof of Theorem 1.1. Let $U = A_1$ and $\phi_3 \in D_3U^*$ be a Macaulay inverse system for A. Assume that $\ell\phi_3$ is nonzero for all ℓ in U. Assume also that either the characteristic of $\boldsymbol{k}$ is not two or that the characteristic of $\boldsymbol{k}$ equals two but there does not equal a basis x^*,y^*,z^* for U^* with $\phi_3 = x^*y^*z^*$. In Lemma 8.2, we prove that $\Gamma_{\phi_3} \neq 0$. Apply Lemma 2.9 to complete the proof. □

Lemma 8.2. *Let $\boldsymbol{k}$ be a field, U be a three-dimensional vector space over $\boldsymbol{k}$, and ϕ_3 be an element of D_3U^*. Assume*

(a) *either the characteristic of $\boldsymbol{k}$ is not two, or the characteristic of $\boldsymbol{k}$ is equal to two but $\phi_3 \neq x^*y^*z^*$for any basis x^*, y^*, z^* of U^*; and*
(b) *$\ell\phi_3$ is nonzero for every nonzero ℓ in U.*

Then Γ_{ϕ_3} is not identically zero.

Proof. The proof is by contradiction. We assume that Γ_{ϕ_3} is identically zero. There are two cases. Either ϕ_3 has the form of (a) or (b) from Lemma 3.1.

We first assume that ϕ_3 has the form of Lemma 3.1.(a). In other words, there is a basis x^*, y^*, z^* for U^* so that $\phi_3 = ax^{*(3)} + x^*\phi_{2,0} + \phi_{3,0}$ with $a \in \boldsymbol{k}$, $a \neq 0$, and $\phi_{i,0} \in D_i\boldsymbol{k}(y^*, z^*)$. Thus, there are parameters $a, b, \dots, j$, in $\boldsymbol{k}$, with $a \neq 0$, such that

$$\phi_3 = \begin{cases} ax^{*(3)} + dx^*y^{*(2)} + ex^*y^*z^* + fx^*z^{*(2)} \\ +gy^{*(3)} + hy^{*(2)}z^* + iy^*z^{*(2)} + jz^{*(3)}. \end{cases} \tag{8.2.1}$$

Let x, y, z be the basis for U which is dual to the basis x^*, y^*, z^* for U^*. Let ω_U and ω_{U^*} be the bases $x \wedge y \wedge z$ and $x^* \wedge y^* \wedge z^*$ for $\bigwedge^3 U$ and $\bigwedge^3 U^*$, respectively.

Observe that

$$\begin{aligned}\Gamma_{\phi_3}(x^{(3)} \otimes \omega_U) &= x^2\phi_3 \wedge xy\phi_3 \wedge xz\phi_3 \\ &= ax^2 \wedge (dy^* + ez^*) \wedge (ey^* + fz^*) \\ &= a(df - e^2)\omega_{U^*}.\end{aligned}$$

The assumption that Γ_{ϕ_3} is identically zero, combined with the hypothesis that $a \neq 0$ yields that $df - e^2 = 0$.

Apply Lemma 3.3 and change the basis for $\boldsymbol{k}y^* \oplus \boldsymbol{k}z^*$ in order to write

$$dx^*y^{*(2)} + ex^*y^*z^* + fx^*z^{*(2)}$$

in the form $dx^*Y^{*(2)}$ with e and f equal to zero for some new basis Y^*, Z^* for $\boldsymbol{k}y^* \oplus \boldsymbol{k}z^*$.

In Lemma 8.3 we show that when ϕ_3 is given in (8.2.1) with $e = f = 0$, then

$$\begin{aligned}\Gamma_{\phi_3}(y^{(3)} \otimes \omega_U) &= -d^2i\omega_{U^*}, \\ \Gamma_{\phi_3}(x^{(2)}z \otimes \omega_U) &= adj\omega_{U^*}, \text{ and} \\ \Gamma_{\phi_3}(xy^{(2)} \otimes \omega_U) &= a(gi - h^2)\omega_{U^*}.\end{aligned}$$

Recall our assumption that Γ_{ϕ_3} is identically zero and our local hypothesis that $a \neq 0$. If $d \neq 0$, then $i = j = h = 0$,

$$\phi_3 = ax^{*(3)} + dx^*y^{*(2)} + gy^{*(3)},$$

(because e and f continue to be zero), and $z\phi_3 = 0$, which is a contradiction. Thus, $d = 0$.

In Lemma 8.3, we show that

$$\begin{aligned}\Gamma_{\phi_3}(xy^{(2)} \otimes \omega_U) &= a(gi - h^2)\omega_{U^*}, \\ \Gamma_{\phi_3}(xz^{(2)} \otimes \omega_U) &= a(hj - i^2)\omega_{U^*}, \text{ and} \\ \Gamma_{\phi_3}(xyz \otimes \omega_U) &= a(gj - hi)\omega_{U^*}.\end{aligned}$$

Continue to assume that $\Gamma_{\phi_3}=0$, $a\neq 0$, but $d=e=f=0$. It follows that $h^2=gi$, $hi=gj$, $hj=i^2$, and

$$\begin{bmatrix} g & h & i\\ h & i & j\end{bmatrix}$$

has rank at most one. Hence some nonzero linear combination of

$$y\phi_3=gy^{*(2)}+hy^*z^*+iz^{*(2)} \quad\text{and}\quad z\phi_3=hy^{*(2)}+iy^*z^*+jz^{*(2)}$$

is zero, which is a contradiction. This contradiction establishes Lemma 8.2 for ϕ_3 described in Lemma 3.1.(a).

Now we assume that ϕ_3 has the form of Lemma 3.1.(b). In other words, there is a basis x^*, y^*, z^* for U^* so that $\phi_3=bx^{*(2)}y^*+x\phi_{2,0}+\phi_{3,0}$ with $\phi_{i,0}$ in $D_i\boldsymbol{k}(y^*,z^*)$, $b\in\boldsymbol{k}$, and $b\neq 0$. Thus, there are parameters $a,b,\dots,j$, with $b\neq 0$, such that

$$\phi_3=\begin{cases} bx^{*(2)}y^*+dx^*y^{*(2)}+ex^*y^*z^*+fx^*z^{*(2)}\\ +gy^{*(3)}+hy^{*(2)}z^*+iy^*z^{*(2)}+jz^{*(3)}.\end{cases} \tag{8.2.2}$$

It is shown in Lemma 8.4 that

$$\Gamma_{\phi_3}(x^{(3)}\otimes\omega_U)=-b^2f\omega_{U^*} \quad\text{and}\quad \Gamma_{\phi_3}(x^{(2)}z\otimes\omega_U)=-b^2j\omega_{U^*}.$$

Combine the assumption that Γ_{ϕ_3} is identically zero with the local hypothesis that $b\neq 0$ in order to conclude that $f=j=0$. It is shown in Lemma 8.5 that when $f=0$, then

$$\Gamma_{\phi_3}(x^{(2)}y\otimes\omega_U)=b(e^2-bi)\omega_{U^*} \quad\text{and}\quad \Gamma_{\phi_3}(y^{(3)}\otimes\omega_U)=(\det M)\,\omega_{U^*},$$

where

$$M=\begin{bmatrix} b & d & e\\ d & g & h\\ e & h & i\end{bmatrix}. \tag{8.2.3}$$

The assumption that Γ_{ϕ_3} is identically zero, together with the local hypothesis that $b\neq 0$, yields that $e^2-bi=0$ and $\det M=0$. It follows that all three maximal minors of M with row two deleted are zero. This conclusion follows from the technique of (7.3.1) applied to the matrix

$$\begin{bmatrix} b & d & e & 0 & 0 & 1\\ d & g & h & 0 & 1 & 0\\ e & h & i & 1 & 0 & 0\end{bmatrix}.$$

Take columns $\{2,3\}$ to be $\{a_1,\dots,a_{r-1}\}$ and columns $\{1,3,5,6\}$ to be $\{b_1,\dots,b_{r+1}\}$ to see that

$$\det\begin{bmatrix} d & e\\ h & i\end{bmatrix}\det\begin{bmatrix} d & h\\ e & i\end{bmatrix}+i\det M-\det\begin{bmatrix} g & h\\ h & i\end{bmatrix}\det\begin{bmatrix} b & e\\ e & i\end{bmatrix}=0.$$

The expressions $e^2 - bi$ and $\det M$ are zero; hence, $di - he$ is also zero. Take $\{1,2\}$ to be $\{a_1, \dots, a_{r-1}\}$ and columns $\{1,3,4,5\}$ to be $\{b_1, \dots, b_{r+1}\}$ to see that

$$\det \begin{bmatrix} b & d \\ e & h \end{bmatrix} \det \begin{bmatrix} b & e \\ d & h \end{bmatrix} + b \det M - \det \begin{bmatrix} b & d \\ d & g \end{bmatrix} \det \begin{bmatrix} b & e \\ e & i \end{bmatrix} = 0;$$

hence $bh - de = 0$. At any rate, the matrix

$$\begin{bmatrix} b & d & e \\ e & h & i \end{bmatrix}$$

has rank at most one; hence

$$\begin{aligned} x\phi_3 &= bx^*y^* + dy^{*(2)} + ey^*z^* \text{ and} \\ z\phi_3 &= ex^*y^* + hy^{*(2)} + iy^*z^* \end{aligned}$$

are linearly dependent. This contradicts the hypothesis that $\ell\phi_3$ is nonzero for all nonzero ℓ in U. □

Lemma 8.3. *If ϕ_3 is given by* (8.2.1)*, with $e = f = 0$, then*

(a) $\Gamma_{\phi_3}(y^{(3)} \otimes \omega_U) = -d^2 i \omega_{U^*}$,
(b) $\Gamma_{\phi_3}(x^{(2)} z \otimes \omega_U) = adj\omega_{U^*}$,
(c) $\Gamma_{\phi_3}(xy^{(2)} \otimes \omega_U) = a(gi - h^2)\omega_{U^*}$,
(d) $\Gamma_{\phi_3}(xz^{(2)} \otimes \omega_U) = a(hj - i^2)\omega_{U^*}$, *and*
(e) $\Gamma_{\phi_3}(xyz \otimes \omega_U) = a(gj - hi)\omega_{U^*}$.

Proof. (a) We compute

$$\begin{aligned} \Gamma_{\phi_3}(y^{(3)} \otimes \omega_U) &= xy\phi_3 \wedge w^2\phi_3 \wedge yz\phi_3 \\ &= dy^* \wedge (dx^* + gy^* + hz^*) \wedge (hy^* + iz^*) \\ &= -d^2 i \omega_{U^*}. \end{aligned}$$

(b) The expansion of $\Gamma_{\phi_3}(x^{(2)} z \otimes \omega_U)$ has three summands; however two of the summands have a factor of $xz\phi_3 = 0$; hence

$$\begin{aligned} \Gamma_{\phi_3}(x^{(2)} z \otimes \omega_U) &= x^2\phi_3 \wedge xy\phi_3 \wedge z^2\phi_3 \\ &= ax^* \wedge dy^* \wedge (iy^* + jz^*) \\ &= adj\omega_{U^*}. \end{aligned}$$

(c) The expansion of $\Gamma_{\phi_3}(xy^{(2)} \otimes \omega_U)$ has three summands; however two of the summands have a factor of $xz\phi_3 = 0$ or $xy\phi_3 \wedge xy\phi_3 = 0$; hence

$$\begin{aligned} \Gamma_{\phi_3}(xy^{(2)} \otimes \omega_U) &= x^2\phi_3 \wedge y^2\phi_3 \wedge yz\phi_3 \\ &= ax^* \wedge (dx^* + gy^* + hz^*) \wedge (hy^* + iz^*) \\ &= a(gi - h^2)\omega_{U^*}. \end{aligned}$$

(d) The expansion of $\Gamma_{\phi_3}(xz^{(2)} \otimes \omega_U)$ has three summands; however two of the summands have a factor of $xz\phi_3 = 0$; hence

$$\begin{aligned}\Gamma_{\phi_3}(xz^{(2)} \otimes \omega_U) &= x^2\phi_3 \wedge yz\phi_3 \wedge z^2\phi_3\\ &= ax^* \wedge (hy^* + iz^*) \wedge (iy^* + jz^*)\\ &= a(hj - i^2)\omega_{U^*}.\end{aligned}$$

(e) The expansion of $\Gamma_{\phi_3}(xyz \otimes \omega_U)$ has six summands; however five of the summands have a factor of $xz\phi_3 = 0$, $yz\phi_3 \wedge yz\phi_3 = 0$, or $xy\phi_3 \wedge xy\phi_3 = 0$; hence

$$\begin{aligned}\Gamma_{\phi_3}(xyz \otimes \omega_U) &= x^2\phi_3 \wedge y^2\phi_3 \wedge z^2\phi_3\\ &= ax^* \wedge (dx^* + gy^* + hz^*) \wedge (iy^* + jz^*)\\ &= a(gj - hi)\omega_{U^*}.\end{aligned}$$

□

Lemma 8.4. *If* ϕ_3 *has the form of* (8.2.2)*, then*

$$\Gamma_{\phi_3}(x^{(3)} \otimes \omega_U) = -b^2 f\omega_{U^*} \quad \textit{and} \quad \Gamma_{\phi_3}(x^{(2)}z \otimes \omega_U) = -b^2 j\omega_{U^*}.$$

Proof. We compute

$$\begin{aligned}\Gamma_{\phi_3}(x^{(3)} \otimes \omega_U) &= x^2\phi_3 \wedge xy\phi_3 \wedge xz\phi_3\\ &= by^* \wedge (bx^* + dy^* + ez^*) \wedge (ey^* + fz^*)\\ &= -b^2 f\omega_{U^*}.\end{aligned}$$

The expansion of $\Gamma_{\phi_3}(x^{(2)}z \otimes \omega_U)$ has three summands; however one of the summands has a factor of $xz\phi_3 \wedge xz\phi_3 = 0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}z \otimes \omega_U) &= x^2\phi_3 \wedge yz\phi_3 \wedge xz\phi_3 + x^2\phi_3 \wedge xy\phi_3 \wedge z^2\phi_3\\ &= \begin{cases} by^* \wedge (ex^* + hy^* + iz^*) \wedge (ey^* + fz^*)\\ +by^* \wedge (bx^* + dy^* + ez^*) \wedge (fx^* + iy^* + jz^*)\end{cases}\\ &= -bef\omega_{U^*} - b(bj - ef)\omega_{U^*}\\ &= -b^2 j\omega_{U^*}.\end{aligned}$$

□

Lemma 8.5. *If* ϕ_3 *has the form of* (8.2.2)*, with* $f = 0$*, then*

$$\Gamma_{\phi_3}(x^{(2)}y \otimes \omega_U) = b(e^2 - bi)\omega_{U^*} \quad \textit{and} \quad \Gamma_{\phi_3}(y^{(3)} \otimes \omega_U) = (\det M)\,\omega_{U^*},$$

where M *is the matrix of* (8.2.3).

Proof. The expansion of $\Gamma_{\phi_3}(x^{(2)}y \otimes \omega_U)$ has three summands. Two of the summands have a factor of $xy\phi_3 \wedge xy\phi_3 = 0$ or $x^2\phi_3 \wedge xz\phi_3 \in \bigwedge^2 \boldsymbol{k}y^* = 0$. Thus,

$$\begin{aligned}\Gamma_{\phi_3}(x^{(2)}y \otimes \omega_U) &= x^2\phi_3 \wedge xy\phi_3 \wedge yz\phi_3\\ &= by^* \wedge (bx^* + dy^* + ez^*) \wedge (ex^* + hy^* + iz^*)\\ &= b(e^2 - bi)\omega_{U^*}.\end{aligned}$$

We compute

$$\begin{aligned}\Gamma_{\phi_3}(y^{(3)}\otimes\omega_U) &= xy\phi_3\wedge y^2\phi_3\wedge yz\phi_3\\ &= (bx^*+dy^*+ez^*)\wedge(dx^*+gy^*+hz^*)\wedge(ex^*+hy^*+iz^*)\\ &= \begin{vmatrix} b & d & e\\ d & g & h\\ e & h & i\end{vmatrix}\omega_{U^*}.\end{aligned}$$

□

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