Compressed local Artinian rings.

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This talk is about joint work with Liana Şega and Adela Vraciu.
These slides are available on my website.

Traditionally, the word “compressed” has been used for $\boldsymbol{k}$-algebras.

See Iarrobino (1984), Fröberg-Laksov (1984), Boij-Laksov (1994).

The first time that the concept compressed was used for rings which do not necessarily contain a field was by Rossi and Şega in 2014, where compressed local Artinian Gorenstein rings were introduced and used.

In *Poincaré series of compressed local Artinian rings with odd top socle degree*, Journal of Algebra **505** (2018),

(1) we prove “compressed local Artinian ring” is a meaningful and rich concept (even for rings which do not necessarily contain a field and are not necessarily Gorenstein), and

(2) we prove a Poincaré series result about such rings.

In this talk I focus on topic (1).

Topic (1): "Compressed local Artinian ring" is a meaningful and rich concept (even for rings which do not necessarily contain a field and are not necessarily Gorenstein).

My goal is to encourage people to reformulate theorems about compressed local Artinian $\boldsymbol{k}$-algebras as theorems about compressed local Artinian rings (which do not necessarily contain a field).

A compressed local Artinian ring R exhibits extremal behavior.

Such a ring has maximal length among all local Artinian rings with the same embedding dimension and socle polynomial.

Extremal objects exhibit special properties and deserve extra study.

The definitions.

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring.

The embedding dimension of R is $e = \dim_{\boldsymbol{k}} \mathfrak{m}/\mathfrak{m}^2$.

The socle of R is $\operatorname{socle}(R) = \{r \in R | \mathfrak{m}r = 0\}$.

The *top socle degree* of R is the maximum integer s with $\mathfrak{m}^s \neq 0$.

The *socle polynomial* of R is the formal polynomial $\sum_{i=0}^{s} c_i z^i$, where

$$c_i = \dim_{\boldsymbol{k}} \frac{\operatorname{socle}(R) \cap \mathfrak{m}^i}{\operatorname{socle}(R) \cap \mathfrak{m}^{i+1}}.$$

The first definition.

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring of embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$.

Definition. If the Hilbert function of R is given by

$$\dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \min\left\{\binom{(e-1)+i}{i},\ \sum_{\ell=i}^{s} c_\ell \binom{(e-1)+(\ell-i)}{\ell-i}\right\}, \quad \text{for } 0 \le i \le s,$$

then R is called a *compressed local Artinian ring*.

(This is the version of the definition that is most useful in practice. It gives the entire Hilbert function of R. But nobody wants a definition that is given in terms of formula but no words.)

I'll give three more equivalent definitions.

The main theorem.

Theorem. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring with embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$. Then the following statements hold.

(a) The length of R satisfies

$$\lambda_R(R) \leq \sum_{i=0}^{s} \min\left\{ \binom{(e-1)+i}{i}, \sum_{\ell=i}^{s} c_\ell \binom{(e-1)+(\ell-i)}{\ell-i} \right\}. \tag{1}$$

(b) Equality holds in (1) if and only if R is a compressed local Artinian ring.

In words: A compressed local Artinian ring R has maximal length among all local Artinian rings with the same embedding dimension and socle polynomial.

Here is my favorite way to see if a ring is compressed.

Theorem. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring with embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$. Then

$$\lambda_R(R) \leq \binom{e + \mathrm{v}(R) - 1}{\mathrm{v}(R) - 1} + \sum_{\ell=\mathrm{v}(R)}^{s} c_\ell \binom{e + \ell - \mathrm{v}(R)}{\ell - \mathrm{v}(R)}, \tag{2}$$

where

$$\mathrm{v}(R) = \inf\left\{ i \,\middle|\, \dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) < \tbinom{(e-1)+i}{i} \right\}.$$

In particular, if $(Q, \mathfrak{n})$ is a regular local ring with R equal to Q/I with $I \subseteq \mathfrak{n}^2$, then $\mathrm{v}(R) = \max\{i \mid I \subseteq \mathfrak{n}^i\}$. (Keep in mind that R is already complete.)

Furthermore R is compressed if and only if equality holds in (2).

Valuable Consequences.

Corollary. If $(R, \mathfrak{m}, \boldsymbol{k})$ is a compressed local Artinian ring with top socle degree s, then the following statements hold.

(a) If $\mathrm{v}(R) \le j \le s$, then $(0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1}$.

(b) If $1 \le j \le s+1$, then $\mathfrak{m}^j : \mathfrak{m} = \mathfrak{m}^{j-1} + \mathrm{socle}(R)$.

(c) If R is also a level ring (that is, if $\mathrm{socle}(R) = \mathfrak{m}^s$), then

$$(0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1} \qquad \text{for } 0 \le j \le s+1.$$

The fourth definition.

Corollary. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring. Then R is a compressed ring if and only if the associated graded ring R^{g} is a compressed ring and R and R^{g} have the same socle polynomial.

(This is the fourth definition of compressed local Artinian ring.)

The technical consequence.

Lemma. Let $(R,\mathfrak{m},\boldsymbol{k})$ be a compressed local Artinian ring with embedding dimension e and top socle degree s. Assume that s is odd and that $s = 2\mathrm{v}(R)-1$. Decompose the maximal ideal $\mathfrak{m}$ as the sum of two subideals $\mathfrak{m} = (x_1) + \mathfrak{m}'$ with x_1 a minimal generator of $\mathfrak{m}$ and $\mu(\mathfrak{m}') = e-1$. Then

$$x_1^{\frac{s-1}{2}}[\mathrm{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^{\frac{s+1}{2}}] = \mathfrak{m}^s.$$

Roughly speaking, this Lemma tells us that there exists $\bar{x}_1 \in \mathrm{Tor}_1$ such that

$$\bar{x}_1 \cdot \mathrm{Tor}_{e-1} = \mathrm{Tor}_e,$$

where

$$\mathrm{Tor}_\bullet = \mathrm{Tor}^P_\bullet(R,\boldsymbol{k}) = \mathrm{H}_\bullet(K^R)$$

for $R = P/I$ with R regular local and $\mathrm{edim}\, P = \mathrm{edim}\, R$ and K^R equal to the Koszul complex on ....

Application to Poincaré series.

Theorem. Let $(R,\mathfrak{m},\boldsymbol{k})$ be a compressed local Artinian ring of embedding dimension e and top socle degree s. Assume that s is odd, $5 \le s$, and

$$\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s.$$

Then the Poincaré series

$$\sum_{i=0}^{\infty} \dim_{\boldsymbol{k}} \operatorname{Tor}^i(M,\boldsymbol{k})t^i$$

of every finitely generated R-module M is a rational function.

Comments about the proof.

• In order to study compressed rings, one must have an appropriate duality theory.

• Partial derivatives provide the duality for Iarrobino.

• Fröberg and Laksov and Boij and Laksov pick a vector space V in the polynomial ring $\boldsymbol{k}[x_1,\dots,x_e]$ and use colon ideals to define an ideal I in the polynomial ring with the property that the corresponding quotient ring has socle V. The colon ideals provide the duality in these cases.

• Rossi and Şega work in a Gorenstein ring and use Gorenstein duality directly.

• Duality for us is supplied by homomorphisms from a power of the maximum ideal to the socle.

How we get duality.

Let $(R,\mathfrak{m},\boldsymbol{k})$ be a local Artinian ring with top socle degree s. If j and k are integers with $0 \le j$, $1 \le k$, and $j+k \le s+1$, then the R-module homomorphism

$$\text{mult} : \mathfrak{m}^j \cap (0 : \mathfrak{m}^k) \to \operatorname{Hom}_R\left(\mathfrak{m}^{k-1}, \operatorname{socle}(R) \cap \mathfrak{m}^{j+k-1}\right)$$

induces an injective R-module homomorphism

$$\frac{\mathfrak{m}^j \cap (0 : \mathfrak{m}^k)}{\mathfrak{m}^j \cap (0 : \mathfrak{m}^{k-1})} \to \operatorname{Hom}_R\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k}, \operatorname{socle}(R) \cap \mathfrak{m}^{j+k-1}\right), \tag{3}$$

which we also call mult.

The injections of (3) are our main tool for studying compressed rings.

The remarkable feature of the injections

$$\frac{\mathfrak{m}^j \cap (0:\mathfrak{m}^k)}{\mathfrak{m}^j \cap (0:\mathfrak{m}^{k-1})} \to \mathrm{Hom}_R\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k}, \mathrm{socle}(R) \cap \mathfrak{m}^{j+k-1}\right)$$

is that if one of them is a surjection, and all other conditions are favorable, then a whole family of these injections are surjections.

Of course, when one of these maps is a surjection, then one is promised the existence of elements with the property that multiplication by the elements acts like **predetermined** functionals.

The critical filtration.

Observe that

$$0 = \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^0)\right) \subseteq \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^1)\right) \subseteq \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^2)\right) \subseteq \cdots \qquad (4)$$

$$\cdots \subseteq \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^{s-j-1})\right) \subseteq \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^{s-j})\right)$$

$$\subseteq \left(\mathfrak{m}^j \cap (0 : \mathfrak{m}^{s-j+1})\right) = \mathfrak{m}^j$$

is a filtration of $\mathfrak{m}^j$. The proof is obtained by exhibiting an injection from each factor of filtration (4) into a vector space whose dimension is easy to calculate.

The ubiquity of compressed standard-graded Artinian $\boldsymbol{k}$-algebras.

Theorem. **[Boij-Laksov]** Let $\boldsymbol{k}$ be an infinite field, (e,s,c) be integers with $2 \le e$ and

$$1 \le c < \binom{e+s-1}{s},$$

Q be a standard-graded polynomial ring over $\boldsymbol{k}$ of embedding dimension e, $\mathcal{G}$ be the Grassmannian of subspaces of Q_s of codimension c, and $\mathcal{L}$ be the set of homogeneous ideals I of Q such that Q/I is a standard-graded Artinian $\boldsymbol{k}$-algebra with socle polynomial cz^s. Then the following statements hold.

(a) The set $\mathcal{G}$ parameterizes $\mathcal{L}$.

(b) If V is in $\mathcal{G}$, then the corresponding ideal I in $\mathcal{L}$ is generated by

$$\sum_{i=1}^{s}(V :_{Q_i} Q_{s-i}).$$

The ubiquity of compressed standard-graded Artinian $\boldsymbol{k}$-algebras.

(c) If I is in $\mathcal{L}$, then the corresponding element of $\mathcal{G}$ is I_s.

(d) There is a non-empty open subset of $\mathcal{G}$ for which the corresponding quotient Q/I is compressed.

Of course, this is a special case of the statement that compressed $\boldsymbol{k}$-algebras are generic. In the general case, Q/I does not have to be graded; the socle polynomial could be more complicated; and the parameter space could be more complicated; see Fröberg-Laksov for a more general, but also, more complicated, result.