THE RESOLUTION OF (x^N, y^N, z^N, w^N)

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ABSTRACT. Let $\mathbf{k}$ be a field of characteristic zero, $n < N$ be positive integers, P be the polynomial ring $\mathbf{k}[x,y,z,w]$, F be the homogeneous polynomial $x^n + y^n + z^n + w^n$, K be the ideal (x^N, y^N, z^N, w^N), and $\overline{P}$ be the hypersurface ring $\overline{P} = P/(F)$. We describe the minimal multi-homogeneous resolution of $\overline{P}/K\overline{P}$ by free $\overline{P}$-modules, the socle degrees of $\overline{P}/K\overline{P}$, and the minimal multi-homogeneous resolution of the Gorenstein ring $P/(K:F)$ by free P-modules. Our arguments use Stanley's theorem that every Artinian monomial complete intersection over a polynomial ring with coefficients from a field of characteristic zero has the strong Lefschetz property as well as a multi-grading on P for which both ideals K and (F) are homogeneous. The resolution of $\overline{P}/K\overline{P}$ by free $\overline{P}$-modules is obtained from a Differential Graded Algebra resolution of $P/(K:F)$ by free P-modules, together with one homotopy map. The multi-grading is used to prove that the resulting resolution is minimal.

CONTENTS

1. INTRODUCTION.

Let $\mathbf{k}$ be a field of characteristic zero, $n < N$ be positive integers, and $\overline{P}$ be the hypersurface ring

$$\overline{P} = \mathbf{k}[x,y,z,w]/(x^n + y^n + z^n + w^n).$$

We describe the minimal multi-homogeneous resolution of $\overline{P}/(x^N, y^N, z^N, w^N)\overline{P}$ by free $\overline{P}$-modules.

For an arbitrary graded algebra R, over an arbitrary field $\mathbf{k}$, with maximal homogeneous ideal $\mathfrak{m} = (x_1, \dots, x_m)$, it is very natural to ask how the bracket powers, $\mathfrak{m}^{[N]} = (x_1^N, \dots, x_m^N)$, of $\mathfrak{m}$ are related. In particular, how is the resolution of $R/\mathfrak{m}^{[N]}$ by free R-modules related to the resolution of $R/\mathfrak{m}^{[qN]}$ for various exponents N and qN? One wonders how many truly different infinite resolutions appear as q varies and one wonders what the least positive value of q is for which the infinite tail of the resolution of $R/\mathfrak{m}^{[qN]}$ is isomorphic to a shift of the infinite tail of the resolution of $R/\mathfrak{m}^{[N]}$.

An important special case of the question takes place when $\mathbf{k}$ has positive characteristic and bracket power is replaced by Frobenius power. Of course, Frobenius powers play a fundamental role in providing invariants

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(such as Hilbert-Kunz multiplicity, F-signature, and F-pure threshold) of the ring R. Much is still unknown about these invariants, even for hypersurfaces. On the other hand, there are times that an investigation of Frobenius power really amounts to an investigation of bracket power; see, for example, [11].

We focus on hypersurfaces of the form $\overline{P}=P/(f)$, where P is a polynomial ring over a field, and f is a homogeneous polynomial in P. The most interesting feature of the $\overline{P}$-resolution of $\overline{P}/\mathfrak{m}^{[N]}\overline{P}$ is the infinite tail of the resolution, which is a matrix factorization of f, see [4].

The situation has been fairly seriously studied when $P=\mathbf{k}[x,y,z]$, $\mathfrak{m}$ is the maximal ideal (x,y,z), and $\mathbf{k}$ is a field of characteristic p. If $f=x^n+y^n+z^n$, then the Betti numbers of $\overline{P}/\mathfrak{m}^{[q]}\overline{P}$ are calculated in [14] and the resolution of $\overline{P}/\mathfrak{m}^{[q]}\overline{P}$ is given in [11]. If f is a general homogeneous form of P, then the Betti numbers of $\overline{P}/\mathfrak{m}^{[q]}\overline{P}$ are calculated in [19].

The infinite tail of the resolution of $\overline{P}/\mathfrak{m}^{[N]}$ is intimately related to the socle degrees of $\overline{P}/\mathfrak{m}^{[N]}$. It is shown in [15] how the behavior of socle degrees under the application of the Frobenius homomorphism can be used to detect that a quotient ring has finite projective dimension. Furthermore, the following result is established in [14, Thm. 1.1] and [11, Thm. 8.18] and is the starting point for [19].

Theorem. *Let $\mathbf{k}$ be a field, n, N_1, and q be positive integers, $N_2=qN_1$, $\overline{P}=\mathbf{k}[x,y,z]/(x^n+y^n+z^n)$ and $A_i=\overline{P}$-module $\overline{P}/(x^{N_i},y^{N_i},z^{N_i})$. Assume that A_1 and A_2 both have infinite projective dimension over $\overline{P}$. Let $\mathbb{F}_{i,\bullet}$ be the minimal homogeneous resolution of A_i by free $\overline{P}$-modules. Then there is an integer w with $\operatorname{soc}A_{N_2}$ isomorphic to $\operatorname{soc}A_{N_1}(-w)$ as graded vector spaces if and only if the complexes $\mathbb{F}_{2,\geq 2}$ and $\mathbb{F}_{1,\geq 2}(-w)$ are isomorphic.*

Let $\mathbf{k}$ be a field of characteristic zero, $n<N$ be positive integers, $P=\mathbf{k}[x,y,z,w]$,

$$\overline{P}=\frac{P}{(x^n+y^n+z^n+w^n)},\quad A=\frac{\overline{P}}{(x^N,y^N,z^N,w^N)\overline{P}},\quad \text{and}\quad R=\frac{P}{(x^N,y^N,z^N,w^N):(x^n+y^n+z^n+w^n)}.$$

The main result in the paper is Theorem 6.2 which gives the multi-graded Betti numbers in the minimal homogeneous resolution of R by free P-modules. Theorem 6.2 is applied in Section 7 to give the multi-graded Betti numbers in the minimal homogeneous resolution of A by free $\overline{P}$-modules and to calculate the socle degrees of A.

In the case of interest, $n<N$ and n does not divide N. Indeed, if n divides N, then everything can be done over the polynomial ring $\mathbf{k}[x,y,z,w]/(x+y+z+w)$ and then be passed to $\overline{P}$ by way of a flat ring extension; see, Section 8. Furthermore, if $N<n$, then $(x^n+y^n+z^n+w^n)\subseteq(x^N,y^N,z^N,w^N)$ are nested complete intersection ideals, $(x^N,y^N,z^N,w^N)\overline{P}$ is a quasi-complete intersection ideal of $\overline{P}$, in the sense of [1], and the two-step Tate complex [12, 5] is the minimal homogeneous resolution of A by free $\overline{P}$-modules.

One of the main ingredients in the argument is the introduction of a multi-grading $\mathbf{M}$ on P for which both ideals (x^N,y^N,z^N,w^N) and $(x^n+y^n+z^n+w^n)$ are homogeneous. This multi-grading is the key to proving that the resolutions we produce are minimal resolutions.

Another significant piece of the argument is the conversion of the problem of describing generators for the ideal

$$(x^N,y^N,z^N,w^N):(x^n+y^n+z^n+w^n),$$

into the problem of finding generators for the ideals $(x^{d_1},y^{d_2},z^{d_3}):(x+y+z)^{d_4}$, where $d_i=d+\varepsilon_i$, for all choices of $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{0,1\}$ and $d=\lfloor\frac{N}{n}\rfloor$. Codimension three Gorenstein rings are understood much better than codimension four Gorenstein rings.

The resolution of A by free $\overline{P}$-modules is built from a Differential Graded Algebra resolution of R by free P-modules, together with one homotopy map. The details of this construction are given in [10].

The starting point for the present paper is Stanley's theorem that every Artinian monomial complete intersection over a polynomial ring $\mathbf{k}[x_1,\dots,x_n]$, where $\mathbf{k}$ is a field of characteristic zero, has the strong Lefschetz property.

In addition to the main theme of the paper we highlight two other results which are probably of independent interest. Lemma 5.6 provides a technique for bounding the number of linear relations on a set of homogeneous forms of the same degree in a polynomial ring in three variables over a field.

The ideals $(x^d, y^d, z^d) : (x+y+z)^d$ and $(x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}$ in $\mathbf{k}[x,y,z]$, when $\mathbf{k}$ is a field of characteristic zero, define compressed Gorenstein rings with odd socle degree. The general theory gives bounds on the graded Betti numbers of such ideals. However, in order to produce the graded Betti numbers on the nose, we found minimal generating sets for the ideals. Proposition 9.1 gives some explicit generators for these ideals. Once these explicit generators were found, we applied Lemma 5.6 in order to bound the number of linear relations on these generators. The complete minimal generating sets for these ideals are given in Proposition 5.7. It turns out that these ideals have the same graded Betti numbers as the ideals $(G_1, G_2, G_3) : G_4$ have where the G's are general forms of degrees d, d, d, d or $d, d+1, d, d+1$ in the sense of [17, Prop. 4.1].

2. The setup and the outline of the argument.

Data 2.1. Let $\mathbf{k}$ be a field, n, d, and r be positive integers, with $r < n$, $N = dn + r$, P be the standard graded polynomial ring $P = \mathbf{k}[x,y,z,w]$, I be the ideal $I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ of P, and R be the quotient ring $R = P/I$. Let $f = x+y+z+w \in P$.

Remark. In all of our important results, the field $\mathbf{k}$ of Data 2.1 has characteristic zero; however in many of our preliminary calculations, the characteristic is not relevant. Each time we use the symbol $\mathbf{k}$, we identify whether it is an arbitrary field or a field of characteristic zero.

Definition 2.2. Retain Data 2.1. Let $\mathbf{M}$ be the Abelian group $\mathbf{M} = \mathbf{Z} \times \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n$. We impose a multi-grading by $\mathbf{M}$ on the polynomial ring P. If $D \in \mathbf{Z}$ and $\bar{r}_i \in \mathbf{Z}_n$, then the $(D, \bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4)$ component of P, denoted $P_{(D,\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)}$, is the $\mathbf{k}$-span of the monomials $x^{\rho_1} y^{\rho_2} z^{\rho_3} w^{\rho_4}$, with $\rho_1 + \rho_2 + \rho_3 + \rho_4 = D$ and the image of ρ_i in $\mathbf{Z}_n$ is $\bar{r}_i$ for each i. Notice that $P_{(D,\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)}$ is equal to zero unless the image of D in $\mathbf{Z}_n$ is $\bar{r}_1 + \bar{r}_2 + \bar{r}_3 + \bar{r}_4$. If M is a $\mathbf{k}$-module which is multi-graded by $\mathbf{M}$, then let $H_M(-)$ denote the Hilbert function of M with respect to the $\mathbf{M}$-grading on M. In other words, for each element m in $\mathbf{M}$, let $H_M(m)$ denote the vector space dimension of the component of M of degree m.

In the language of 2.1 and 2.2, one can easily check that P is graded by $\mathbf{M}$ in the sense that

$$P_{m_1} \cdot P_{m_2} \subseteq P_{m_1+m_2},$$

for $m_i \in \mathbf{M}$. The ideals (x^N, y^N, z^N, w^N) and $(x^n + y^n + z^n + w^n)$ are both homogeneous under the multi-grading of 2.2. It follows that the ideal $I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ is homogeneous under the multi-grading, and the multi-grading is inherited by $R = P/I$. The Hilbert function of R with respect to this multi-grading is given in Proposition 4.2.

In Section 5, we describe the generators of I. In Proposition 5.1 we take advantage of the multi-grading to show that the generators of I can be obtained from the generators of the ideals

$$(x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}) : f, \tag{1}$$

where each ε_i is either 0 or 1. In Observation 5.2 we obtain the generators of the ideals of (1) from ideals of the form

$$(x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}) : (x+y+z)^{d+\varepsilon_4}. \tag{2}$$

The ideals of (2) are precisely the ideals that exhibit the fact that monomial complete intersections in a polynomial ring over a field of characteristic zero have the strong Lefschetz property. Much numerical information about these ideals is known. The ideals define compressed quotient rings, and therefore the number and degree of the generators are completely known in the case when the socle degree is even, (i.e., $\sum_{i=1}^4 \varepsilon_i = 1, 3$). In the case when the socle degree is odd (i.e., $\sum_{i=1}^4 \varepsilon_i = 0, 2, 4$), then the general theory provides some bounds on the generator degrees. In order to learn the precise generator degrees, we exhibit explicit generators in Proposition 5.7.

3. Notation, conventions, and preliminary results.

Let $\mathbf{Z}$ represent the ring of integers and $\mathbf{Z}_n$ represent the quotient ring $\mathbf{Z}/(n\mathbf{Z})$ for each integer n in $\mathbf{Z}$. If r is an integer, then $\bar{r}$ is the image of r in $\mathbf{Z}_n$.

Notation 3.1. For a polynomial g in a polynomial ring, let $g^{[n]}$ denote the result of replacing each variable in g by the nth power of the variable. Note that $g \to g^{[n]}$ is an injective ring homomorphism.

Example 3.2. In the language of 2.1 and 2.2, every polynomial of $P_{(D,\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)}$ has the form

$$g^{[n]}x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4},$$

where the ρ_i are integers with $0 \le \rho_i \le n-1$, the image of ρ_i in $\mathbf{Z}_n$ is $\bar{r}_i$, $D = kn + \sum_{i=1}^4 \rho_i$ for some non-negative integer k, and g is a homogeneous polynomial in P of degree k.

Observation 3.3. *Let* $\mathbf{k}$ *be an arbitrary field. Adopt the language of* 2.1 *and* 2.2. *Let* k, ρ_1, ρ_2, ρ_3, *and* ρ_4 *be non-negative integers with* $0 \le \rho_i \le n-1$, *then there is an isomorphism of* $\mathbf{k}$*-vector spaces*

$$P_k \longrightarrow P_{(nk+\rho_1+\rho_2+\rho_3+\rho_4,\bar{\rho}_1,\bar{\rho}_2,\bar{\rho}_3,\bar{\rho}_4)},$$

given by

$$g \mapsto g^{[n]}x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4};$$

where k is an element of $\mathbf{Z}$ *and* $(nk+\rho_1+\rho_2+\rho_3+\rho_4,\bar{\rho}_1,\bar{\rho}_2,\bar{\rho}_3,\bar{\rho}_4)$ *is an element of* $\mathbf{M}$.

Proof. This assertion is an immediate consequence of Example 3.2. □

The following notation makes sense in light of Observation 3.3 and it allows us to convey all of the information about an element of $\mathbf{M}$ with a minimal amount of writing.

Notation 3.4. Adopt the data of 2.1 and 2.2. Consider the homomorphism of Abelian groups

$$\mathbf{Z}^5 \to \mathbf{M},$$

which is given by

$$(k,\rho_1,\rho_2,\rho_3,\rho_4) \mapsto m_{(k,\rho_1,\rho_2,\rho_3,\rho_4)},$$

where

$$m_{(k,\rho_1,\rho_2,\rho_3,\rho_4)} = (kn+\rho_1+\rho_2+\rho_3+\rho_4,\bar{\rho}_1,\bar{\rho}_2,\bar{\rho}_3,\bar{\rho}_4) \quad \text{in } \mathbf{M}.$$

The 5-tuple of integers $(k,\underline{\rho}) = (k,\rho_1,\rho_2,\rho_3,\rho_4)$ is in *standard form* if $0 \le k$ and $0 \le \rho_i \le n-1$ for all i.

3.5. Let $\mathbf{k}$ be a field. The graded algebra $R = \bigoplus_{0 \le i} R_i$ is a *standard graded* $\mathbf{k}$*-algebra* if R_0 is equal to $\mathbf{k}$, R_1 is a finitely generated R_0-module, and R is generated by R_1 as an algebra over R_0.

3.6. If F is a module graded by the Abelian group $(G,+)$ and $a \in G$, then $F(-a)$ is the graded module with $F(-a)_b = F_{b-a}$ for all $b \in G$.

3.7. Let I be an ideal in a ring R, N be an R-module, and L and M be submodules of N. Then

$$L :_I M = \{x \in I \mid xM \subseteq L\} \quad \text{and} \quad L :_M I = \{m \in M \mid Im \subseteq L\}.$$

Any undecorated ":" means $:_R$ where R is the ambient ring.

3.8. Let $\mathbf{k}$ be an arbitrary field, R be a standard graded $\mathbf{k}$-algebra, $\mathfrak{m}$ be the maximal homogeneous ideal of R, and $M = \oplus_i M_i$ be a finitely generated graded R-module.

(a) Let $H_M(-)$ denote the *Hilbert function* of M with respect to the standard grading on M. In other words,

$$H_M(i) = \dim_{\mathbf{k}} M_i.$$

The integer $H_R(1)$ is called the *embedding dimension* of R.

(b) The relationship between the Hilbert function $H_P(-)$ of Definition 2.2 and the Hilbert function $H_P(-)$ of (a) is explained in Observation 3.3. If $(k, \underline{\rho})$ is a 5-tuple of integers in standard form (in the sense of 3.4), then

$$H_P(m_{(k,\underline{\rho})}) = H_P(k).$$

Of course, the value of $H_P(k)$ is well known:

$$H_P(k) = \binom{k+3}{k},$$

for $P = \mathbf{k}[x,y,z,w]$. In particular, $H_P(k) = 0$, if $k < 0$.

3.9. Let $(G,+)$ be an Abelian group and R be a ring which is graded by G, with R_0 equal to a field $\mathbf{k}$, $\bigoplus_{g\in G\setminus\{0\}} R_g$ equal to the maximal homogeneous ideal of R, and R finitely generated as an R_0-algebra. Let M be a finitely generated R-module which is also graded by G.

(a) The *socle* of M is the vector space

$$\operatorname{socle} M = 0 :_M \mathfrak{m}.$$

(b) If R is Artinian, then R is *Gorenstein* if $\operatorname{socle} R$ is a one-dimensional vector space over $\mathbf{k}$.

(c) There is an isomorphism of graded R-modules

$$\operatorname{socle} M \cong \bigoplus_i \mathbf{k}(-s_i),$$

for some finite set of elements $\{s_i\}$ from G. The elements $\{s_i\}$ are called the *socle degrees* of M.

(d) If $\mathfrak{a}$ is a homogeneous ideal of R with $R/\mathfrak{a}$ Artinian, local, and Gorenstein with socle degree δ, and f is a homogeneous element of R in R_d for some $d \in G$. Then $\mathfrak{a} : f$ is a homogeneous ideal of R with $R/(\mathfrak{a} : f)$ Artinian, local, and Gorenstein with socle degree $\delta - d$.

Indeed, the R-module homomorphism $R(-d) \to R$, which is given by multiplication by f induces a homogeneous injection

$$\operatorname{socle} \frac{R}{(\mathfrak{a} : f)}(-d) \longrightarrow \operatorname{socle} \frac{R}{\mathfrak{a}}.$$

(e) If $R = \mathbf{k}[x_1, \dots, x_n]$ is a polynomial ring, with each x_i a homogeneous element of R, then the socle degrees of M may be read from the back twists in a minimal homogeneous resolution

$$0 \to C_n = \oplus_{i=1}^{s} P(-\beta_i) \to \cdots \to C_0$$

of M by free R-modules.

Indeed, the computation of $\operatorname{Tor}_n^R(M, \mathbf{k})$ in each coordinate yields a graded isomorphism

$$\operatorname{socle} M \cong \bigoplus_{i=1}^{s} \mathbf{k}(\kappa - \beta_i),$$

where $\prod_i x_i \in P_\kappa$.

3.10. We use two symmetries that are associated to a graded Artinian Gorenstein algebra R over a field $\mathbf{k}$.

3.10.1. The Hilbert function $H_R(-)$ is symmetric in the sense that $H_R(i) = H_R(s-i)$, for all i, where s is the socle degree of R. This property follows from the fact that the multiplication map $R_i \otimes R_{s-i} \to R_s$ is a perfect pairing.

3.10.2. If $R = P/I$, where P is a polynomial ring over $\mathbf{k}$, then the minimal homogeneous resolution,

$$0 \to P(-\beta) = F_g \to F_{g-1} \to \cdots \to F_1 \to F_0 = P,$$

of R by free P-modules, is self-dual, in the sense that

$$\operatorname{Hom}_P(F_i, P)(-\beta) \cong F_{g-i}.$$

This property follows from the fact that $\operatorname{Ext}_g^P(R, P) \cong R(\beta)$.

3.11. The notion of a compressed Artinian algebra over a field was introduced by Iarrobino [8]. Rossi and Şega [22, Prop. 4.2] extended the definition to be meaningful for local Artinian Gorenstein rings which do not necessarily contain a field. The Rossi-Şega definition can be extended further to be meaningful for rings that are not Gorenstein [13, Def. 2.5].

Definition 3.11.1. Let $\mathbf{k}$ be an arbitrary field, R be a standard graded, Gorenstein, local Artinian $\mathbf{k}$-algebra of socle degree s and embedding dimension e with $1 < e$. If

$$H_R(i) = \min\left\{\binom{e-1+i}{e-1}, \binom{e-1+s-i}{e-1}\right\}, \tag{3}$$

for $0 \le i \le s$, then R is a *compressed* ring.

We apply Definition 3.11.1 when R is an Artinian ring of the form $R = P/I$, where $P = \mathbf{k}[x_1,\dots,x_e]$ is a standard graded polynomial ring over a field, I a homogeneous ideal of P, and $I \subseteq (x_1,\dots,x_e)^2$. In this case, equation (3) becomes

$$H_R(i) = \min\{H_P(i), H_P(s-i)\}.$$

3.12. (a) A linear transformation $\phi : V_1 \to V_2$ of finite dimensional vector spaces over the field $\mathbf{k}$ has *maximal rank* if

$$\operatorname{rank}\phi = \min\{\dim_{\mathbf{k}} V_1, \dim_{\mathbf{k}} V_2\}.$$

In particular, if ϕ has maximal rank, then ϕ is injective or surjective.

(b) Let $A = \bigoplus A_i$ be a standard graded Artinian algebra over the field $\mathbf{k}$. Then A has the *weak Lefschetz property* if there exists a linear form L of A_1 such that multiplication by L from $A_i \to A_{i+1}$ has maximal rank for each index i. In this case, L is called a *Lefschetz element* of A. The set of Lefschetz elements forms a (possibly empty) Zariski open subset of A_1. The algebra A has the *strong Lefschetz property* if there exists a linear form L of A_1 such that multiplication by L^t from $A_i \to A_{i+t}$ has maximal rank for each index i and each exponent t.

(c) If $\mathbf{k}$ is an infinite field, $P = \mathbf{k}[x_1,\dots,x_n]$ is a standard graded polynomial ring, and $A = P/I$ is defined by a complete intersection ideal generated by monomials, then A has the weak Lefschetz property if and only if $x_1+\dots+x_n$ is a Lefschetz element. See [18, Prop. 2.2].

(d) The starting point for the present paper is the theorem of Stanley [23, Thm. 2.4] that every Artinian monomial complete intersection over a polynomial ring $\mathbf{k}[x_1,\dots,x_n]$, where $\mathbf{k}$ is a field of characteristic zero, has the strong Lefschetz property. Stanley's proof used algebraic topology. Other proofs have been given by Watanabe [25, Cor. 3.5], using representations of $\mathfrak{sl}_2$, and Reid, Roberts, and Roitman [21, Thm. 10], using commutative algebra. In Observation 3.13, we appeal to one of the preliminary results from [21]; the proof, given in [21, Thm. 5], is remarkably elementary.

Observation 3.13. *Let $\mathbf{k}$ be a field of characteristic zero, $P = \mathbf{k}[x_1,\dots,x_m]$ be a standard graded polynomial ring, $d_1,\dots,d_{m+1}$ be positive integers, K be the ideal $(x_1^{d_1},\dots,x_m^{d_m})$ of P, F be the element $x_1+\dots+x_m$ in P, and J be the ideal $K : F^{d_{m+1}}$ of P. Then the following statements hold.*

(a) *The graded $\mathbf{k}$-algebra P/J has socle degree $s = \sum_{i=1}^m d_i - m - d_{m+1}$.*

(b) *If θ is a non-zero homogeneous element of*

$$\ker\left(\frac{P}{K} \xrightarrow{(x_1+\dots+x_m)^{d_{m+1}}} \frac{P}{K}(d_{m+1})\right),$$

then $\lceil \frac{s+1}{2} \rceil \le \deg\theta$.

(c) *The socle degree of P/K is $\sigma = \sum_{i=1}^m d_i - m$ and*

$$\min\{H_{P/K}(i), H_{P/K}(i+1)\} = \begin{cases} H_{P/K}(i), & \text{if } 0 \le i < \frac{\sigma}{2}, \text{ and} \\ H_{P/K}(i+1), & \text{if } \frac{\sigma}{2} \le i. \end{cases}$$

(d) *If $m = 3$ and there are integers d and ε_i with $d_i = d + \varepsilon_i$ and $\varepsilon_i \in \{0,1\}$, for $1 \le i \le 4$, then the initial degree of J is at least $\lceil \frac{s+1}{2} \rceil$ and P/J is a compressed ring.*

Proof. (a) Recall that $x_1^{d_1-1}x_2^{d_2-1}\cdots x_m^{d_m-1}$ represents a socle generator for P/K. Apply 3.9.(d) to compute the socle degree of P/J.

(b) The field $\mathbf{k}$ has characteristic zero, hence, [21, Thm. 5] applies and

$$\left\lceil \frac{(d_1+\cdots+d_m-m)-d_{m+1}+1}{2}\right\rceil \le \deg\theta.$$

Apply (a) in order to complete the proof of (b).

(c) The socle of P/K is represented by $\prod_{j=1}^m x_j^{d_j-1}$. Apply (b), with $d_{m+1}=1$, in order to see that

$$H_{P/K}(i)\le H_{P/K}(i+1)$$

for $i<\left\lceil\frac{d_1+\cdots+d_m-m}{2}\right\rceil=\lceil\frac{\sigma}{2}\rceil$. Use the symmetry $H_{P/K}(i)=H_{P/K}(\sigma-i)$ of the Hilbert function to finish the argument.

(d) We first verify that the extra hypotheses of (d) ensure that the elements $x_1^{d_1}$, $x_2^{d_2}$, and $x_3^{d_3}$ of J have degree at least $\lceil\frac{s+1}{2}\rceil$. In other words, we must show that

$$\left\lceil\frac{2d-2+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4}{2}\right\rceil \le \min\{d+\varepsilon_1,d+\varepsilon_2,d+\varepsilon_3\}. \tag{4}$$

On the other hand, it is clear that

$$\left\lceil\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4}{2}\right\rceil\le\left\lceil\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}\right\rceil\le\min\{1+\varepsilon_1,1+\varepsilon_2,1+\varepsilon_3\}.$$

Add $d-1$ to both sides to obtain (4).

Combine (b) with (4) to see that the initial degree of J is at least $\lceil\frac{s+1}{2}\rceil$. It follows that if $0\le i<\lceil\frac{s+1}{2}\rceil$, then $i\le s-i$ and

$$H_R(i)=H_P(i)=\min\{H_P(i),H_P(s-i)\}; \tag{5}$$

and, if $\lceil\frac{s+1}{2}\rceil\le i\le s$, then $s-i<\lceil\frac{s+1}{2}\rceil\le i$ and

$$H_R(i)=H_R(s-i)=H_P(s-i)=\min\{H_P(i),H_P(s-i)\}. \tag{6}$$

(The left-most equality in (6) holds because the Hilbert function of a graded Artinian Gorenstein ring is symmetric and the middle equality in (6) is a consequence of (5).) At any rate

$$H_R(i)=\min\{H_P(i),H_P(s-i)\},$$

for $0\le i\le s$ and R is a compressed ring, see Definition 3.11.1. $\square$

3.14. Let $R=\mathbf{k}[x_1,\dots,x_m]$ be a polynomial ring, I be a homogeneous ideal of R which contains a homogeneous complete intersection ideal $K=(f_1,\dots,f_m)$. The ideal $K:I$ is *linked* to I. If $\mathbb{K}$ is the Koszul complex which resolves R/K, $\mathbb{F}$ is a resolution of R/I of length g, and $\alpha:\mathbb{K}\to\mathbb{F}$ is a map of complexes which extends the identity map in degree zero, then the dual of the mapping cone of α is a resolution of $R/(K:I)$. These results are due to Peskine and Szpiro (see [20, Props. 1.3 and 2.6] or [3, Props. 5.1 and 5.1a]).

4. THE MULTI-HOMOGENEOUS HILBERT FUNCTION OF R.

In Proposition 4.2 we calculate the Hilbert function of the ring R of Data 2.1 with respect to the multi-grading $\mathbf{M}$ of Definition 2.2. Proposition 4.2 requires the base field $\mathbf{k}$ to have characteristic zero.

Lemma 4.1. *Let* $\mathbf{k}$ *be an arbitrary field. Adopt the data of* 2.1 *and* 2.2. *Let* $(k,\underline{\rho})$ *be a* 5*-tuple of integers in standard form in the sense of* 3.4 *and*

$$\varepsilon_i=\begin{cases}1, & \textit{if } \rho_i<r, \textit{ and}\\ 0, & \textit{otherwise.}\end{cases} \tag{7}$$

Then the component of I of multi-degree $m_{(k,\underline{\rho})}$ is equal to

$$\left\{x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4}g^{[n]}\,\middle|\, g \text{ is a homogeneous polynomial of degree } k \text{ in } (x^{d+\varepsilon_1},y^{d+\varepsilon_2},z^{d+\varepsilon_3},w^{d+\varepsilon_4}):_P f\right\}.$$

Proof. Let θ be an arbitrary element of $P_{m_{(k,\underline{\rho})}}$. Recall from Example 3.2 that $\theta = x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4}g^{[n]}$ for some homogeneous polynomial g in P of degree k. Observe that

$$\begin{aligned} \theta\in I \quad &\iff \quad x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4}g^{[n]}f^{[n]}\in(x^N,y^N,z^N,w^N)\\ &\iff \quad g^{[n]}f^{[n]}\in(x^N,y^N,z^N,w^N):x^{\rho_1}y^{\rho_2}z^{\rho_3}w^{\rho_4}\\ &\iff \quad g^{[n]}f^{[n]}\in(x^{N-\rho_1},y^{N-\rho_2},z^{N-\rho_3},w^{N-\rho_4}).\end{aligned}$$

Thus, $\theta\in I$ if and only if there are multi-homogeneous polynomials $c_1,\dots,c_4$ in P with

$$x^{dn+r-\rho_1}c_1+y^{dn+r-\rho_2}c_2+z^{dn+r-\rho_3}c_3+w^{dn+r-\rho_4}c_4=g^{[n]}f^{[n]}\in P_{m_{(k+1,0,0,0,0)}}.$$

The multi-degree of c_1 is

$$m_{(k+1,0,0,0,0)}-m_{(d,r-\rho_1,0,0,0)}=\begin{cases} m_{(k+1-d,\rho_1-r,0,0,0)}, & \text{if } r\le\rho_1, \text{ and}\\ m_{(k-d,n+\rho_1-r,0,0,0)}, & \text{if } \rho_1<r\end{cases}$$

in $\mathbf{M}$. If $r\le\rho_1$, then $0\le\rho_1-r<n$, and c_1 is of the form $c_1=x^{\rho_1-r}d_1^{[n]}$, for some homogeneous polynomial d_1 in P of degree $k+1-d$. If $\rho_1<r$, then $0<n+\rho_1-r<n$, and c_1 is of the form $c_1=x^{n+\rho_1-r}d_1^{[n]}$, for some homogeneous polynomial d_1 in P of degree $k-d$. Use (7) to combine the two cases as

$$x^{dn+r-\rho_1}c_1=d_1^{[n]}x^{(d+\varepsilon_1)n},$$

for some homogeneous polynomial d_1 in P of degree $k-d+1-\varepsilon_1$. The same calculation holds for the other terms. Thus, θ is in I if and only if there exist homogeneous polynomials $d_1,\dots,d_4$ in P with

$$g^{[n]}f^{[n]}=x^{(d+\varepsilon_1)n}d_1^{[n]}+y^{(d+\varepsilon_2)n}d_2^{[n]}+z^{(d+\varepsilon_3)n}d_3^{[n]}+w^{(d+\varepsilon_4)n}d_4^{[n]}.$$

Recall that the map $\phi: P\to P$, given by $\phi(g_1)=g_1^{[n]}$, for $g_1\in P$, is an injective ring homomorphism. It follows that θ is in I if and only if

$$gf\in(x^{d+\varepsilon_1},y^{d+\varepsilon_2},z^{d+\varepsilon_3},w^{d+\varepsilon_4}).$$ □

Proposition 4.2. *Let* $\mathbf{k}$ *be a field of characteristic zero. Adopt the data of* 2.1 *and* 2.2. *Let* $(k,\underline{\rho})$ *be a* 5*-tuple of integers in standard form in the sense of* 3.4. *Then*

$$H_R(m_{(k,\underline{\rho})})=\begin{cases} H_P(k+\varepsilon)-\sum\limits_{i=1}^{4}H_P(k+\varepsilon-d-\varepsilon_i)+\sum\limits_{1\le i<j\le 4}H_P(k+\varepsilon-2d-\varepsilon_i-\varepsilon_j)\\ -\sum\limits_{1\le i<j<\ell\le 4}H_P(k+\varepsilon-3d-\varepsilon_i-\varepsilon_j-\varepsilon_\ell)+H_P(k+\varepsilon-4d-\sum\limits_{i=1}^{4}\varepsilon_i),\end{cases}\tag{8}$$

where

$$\varepsilon_i=\begin{cases}1, & \text{if } \rho_i<r, \text{ and}\\ 0, & \text{otherwise,}\end{cases} \qquad \text{and} \qquad \varepsilon=\begin{cases}0, & \text{if } k<2d-2+\frac{\sum_{i=1}^4\varepsilon_i}{2}, \text{ and}\\ 1, & \text{if } 2d-2+\frac{\sum_{i=1}^4\varepsilon_i}{2}\le k.\end{cases}$$

Proof. Apply Lemma 4.1 to see that

$$H_R(m_{(k,\underline{\rho})})=H_S(k),$$

where

$$S=\frac{k[x,y,z,w]}{(x^{d+\varepsilon_1},y^{d+\varepsilon_2},z^{d+\varepsilon_3},w^{d+\varepsilon_4}):f}.$$

Let

$$C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}=\frac{k[x,y,z,w]}{(x^{d+\varepsilon_1},y^{d+\varepsilon_2},z^{d+\varepsilon_3},w^{d+\varepsilon_4})}.$$

The Hilbert function of $C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ can be found from the Koszul complex resolution. It is

$$H_{C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}}(k) = \begin{cases} H_P(k) - \sum_{i=1}^{4} H_P(k-d-\varepsilon_i) + \sum_{1\le i<j\le 4} H_P(k-2d-\varepsilon_i-\varepsilon_j) \\ -\sum_{1\le i<j<\ell\le 4} H_P(k-3d-\varepsilon_i-\varepsilon_j-\varepsilon_\ell) + H_P(k-4d-\sum_{i=1}^{4}\varepsilon_i). \end{cases} \tag{9}$$

Note that the graded component of

$$\frac{(x^{d+\varepsilon_{r_1}}, y^{d+\varepsilon_{r_2}}, z^{d+\varepsilon_{r_3}}, w^{d+\varepsilon_{r_4}}) : f}{(x^{d+\varepsilon_{r_1}}, y^{d+\varepsilon_{r_2}}, z^{d+\varepsilon_{r_3}}, w^{d+\varepsilon_{r_4}})}$$

of degree k is the kernel of the map given by multiplication by $f : [C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}]_k \to [C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}]_{k+1}$. The field $\mathbf{k}$ has characteristic zero; hence, $C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ has the weak Lefschetz property; and therefore the multiplication by f map has maximal rank. It follows that

$$H_S(k) = \min\{H_{C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}}(k), H_{C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}}(k+1)\} = \begin{cases} H_{C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}}(k) & \text{if } k < 2d-2+\frac{\sum_{i=1}^4 \varepsilon_i}{2} \\ H_{C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}}(k+1) & \text{if } 2d-2+\frac{\sum_{i=1}^4 \varepsilon_i}{2} \le k. \end{cases}$$

See Observation 3.13.(c). □

5. THE GENERATORS OF I.

Proposition 5.1. *Let* $\mathbf{k}$ *be an arbitrary field. Adopt the data of* 2.1. *Then the ideal*

$$I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$$

is generated by all the elements of the form $(x^{1-\varepsilon_1} y^{1-\varepsilon_2} z^{1-\varepsilon_3} w^{1-\varepsilon_4})^r g^{[n]}$ *with* $g \in (x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}) : f$, *for all choices of* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}$.

Proof. The ideal I is homogeneous. It is generated by elements of the form $x^{\rho_1} y^{\rho_2} z^{\rho_3} w^{\rho_4} g^{[n]}$ with

$$g \in (x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}) : f,$$

as described in Lemma 4.1. Each such $x^{\rho_1} y^{\rho_2} z^{\rho_3} w^{\rho_4} g^{[n]}$ is a multiple of $x^{s_1} y^{s_2} z^{s_3} w^{s_4} g^{[n]}$, where

$$s_i = \begin{cases} r, & \text{if } r \le \rho_i, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

□

Remark. When $\mathbf{k}$ has characteristic zero, we reduce this list to a minimal set of generators in Proposition 5.12.

Observation 5.2. *Let* $\mathbf{k}$ *be an arbitrary field,* $P = \mathbf{k}[x_1, \dots, x_t]$ *be a standard graded polynomial ring,* L *be the linear form* $\sum_{i=1}^t x_i$ *in* P, *and* $d_1, \dots, d_t$ *be positive integers. Then there is a homogeneous isomorphism of* P*-modules*

$$\frac{(x_1^{d_1}, \dots, x_{t-1}^{d_{t-1}}) : \left(\sum_{i=1}^{t-1} x_i\right)^{d_t}}{(x_1^{d_1}, \dots, x_{t-1}^{d_{t-1}})}(-d_t+1) \xrightarrow{\Phi} \frac{(x_1^{d_1}, \dots, x_t^{d_t}) : (L)}{(x_1^{d_1}, \dots, x_t^{d_t})}, \tag{10}$$

which is induced by multiplication by the polynomial

$$\frac{x_t^{d_t} - \left(-\sum_{i=1}^{t-1} x_i\right)^{d_t}}{L}.$$

Remark. The module on the left side of (10) is naturally a $\mathbf{k}[x_1, \dots, x_{t-1}]$-module. This module acquires the structure of a P-module by way of the $\mathbf{k}$-algebra homomorphism $\phi : P \to \mathbf{k}[x_1, \dots, x_{t-1}]$, with $\phi(x_i) = x_i$, for $1 \le i \le t-1$, and $\phi(x_t) = -(x_1 + \cdots + x_{t-1})$.

Proof. This result is essentially [16, Thm. 2.1]. We reproduce part of the proof. It is not difficult to see that Φ is a well-defined P-module homomorphism. We describe the inverse of Φ, which we call Ψ. If G is a homogeneous element in $(x_1^{d_1},\dots,x_t^{d_t}):L$, then

$$LG=\sum_{i=1}^{t} B_i x_i^{d_i}, \tag{11}$$

for some homogeneous $B_i\in \mathbf{k}[x_1,\dots,x_t]$. The Ψ is defined to send the class of G in the right side of (10) (denoted $[G]$) to the class of $\phi(B_t)$ in the left side of (10) (denoted $[\phi(B_t)]$). Once again, it is not difficult to see that Ψ is a well-defined homomorphism of P-modules. We compute $\Phi\circ\Psi$.

Notice that if $B\in P$, then $B-\phi(B)\in\ker(\phi)=(L)$. In particular, for each B_i in (11), there is an element B_i' in P with

$$\phi(B_i)=B_i+LB_i'. \tag{12}$$

Apply the $\mathbf{k}$-algebra homomorphism ϕ to both sides of (11) to obtain

$$0=\sum_{i=1}^{t-1}\phi(B_i)x_i^{d_i}+\phi(B_t)\left(-\sum_{i=1}^{t-1}x_i\right)^{d_t}. \tag{13}$$

It follows that

$$\begin{aligned}
(\Phi\circ\Psi)([G])=\Phi([\phi(B_t)])&=\left[\frac{x_t^{d_t}\phi(B_t)-\left(-\sum_{i=1}^{t-1}x_i\right)^{d_t}\phi(B_t)}{L}\right]\\
&=\left[\frac{\sum_{i=1}^{t}x_i^{d_i}\phi(B_i)}{L}\right], && \text{by (13)},\\
&=\left[\frac{\sum_{i=1}^{t}x_i^{d_i}B_i+L\sum_{i=1}^{t}x_i^{d_i}B_i'}{L}\right], && \text{by (12)},\\
&=\left[G+\sum_{i=1}^{t}x_i^{d_i}B_i'\right], && \text{by (11)},\\
&=[G].
\end{aligned}$$

□

Remark 5.3. Observations 5.1 and 5.2 show that in order to find generators for the ideal I of Data 2.1, it suffices to find generators for the ideals $(x^{d_1},y^{d_2},z^{d_3}):(x+y+z)^{d_4}$, where $d_i=d+\varepsilon_i$, for all choices of $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{0,1\}$.

According to Observation 3.13.(d), the ideals of Remark 5.3 all define compressed quotient rings, when the characteristic of $\mathbf{k}$ is zero. Boij has obtained significant information about the graded Betti numbers of compressed quotient rings. In Corollary 5.5, we apply Boij's results to the ideals of Remark 5.3, when the characteristic of $\mathbf{k}$ is zero. When the quotient ring has even socle degree, then the graded Betti numbers are completely described in Corollary 5.5. However, when the quotient ring has odd socle degree, then more work is required in order to give a complete description of the graded Betti numbers. This work is carried out in Proposition 5.7.

Lemma 5.4. *Let* $\mathbf{k}$ *be an arbitrary field,* Q *be the standard graded polynomial ring* $Q=\mathbf{k}[x,y,z]$, *and* J *be a homogeneous ideal in* Q *which defines a compressed Artinian quotient ring with socle degree* s. *Then the* *minimal homogeneous resolution of Q/J by free Q-modules has the form*

$$0 \to Q(-s-3) \to Q(-\tfrac{s}{2}-2)^{s+3} \to Q(-\tfrac{s}{2}-1)^{s+3} \to Q,$$

if s is even; and

$$0 \to Q(-s-3) \longrightarrow \begin{matrix} Q(-(\frac{s+1}{2})-1)^{\nu} \\ \oplus \\ Q(-(\frac{s+1}{2})-2)^{(s+3)/2} \end{matrix} \longrightarrow \begin{matrix} Q(-(\frac{s+1}{2}))^{(s+3)/2} \\ \oplus \\ Q(-(\frac{s+1}{2})-1)^{\nu} \end{matrix} \longrightarrow Q,$$

for some non-negative integer ν, if s is odd.

Proof. Let t be the initial degree of J and F be the minimal homogeneous resolution of Q/J by free Q-modules. Apply [2, Prop. 3.2] to see that the beginning of F has the form

$$Q(-t)^{b_1'} \oplus Q(-t-1)^{b_1''} \to Q,$$

for some non-negative integers b_1' and b_1''. The socle degree of Q/J is s and the codimension of Q/J is 3. It follows that the final module in F is $Q(-s-3)$; see 3.9.(e). The complex F is self-dual (see 3.10.2); hence F has the form

$$0 \to Q(-s-3) \longrightarrow \begin{matrix} Q(-s-2+t)^{b_1''} \\ \oplus \\ Q(-s-3+t)^{b_1'} \end{matrix} \longrightarrow \begin{matrix} Q(-t)^{b_1'} \\ \oplus \\ Q(-t-1)^{b_1''} \end{matrix} \longrightarrow Q, \tag{14}$$

for some integers b_1' and b_1'', with $0 < b_1'$ and $0 \le b_1''$.

The fact that Q/J is a compressed ring guarantees that $\lceil \frac{s+1}{2} \rceil \le t$; see Definition 3.11.1. The complex F is a minimal resolution. The component

$$Q(-s-3+t)^{b_1'} \to Q(-t)^{b_1'} \oplus Q(-t-1)^{b_1''}$$

of F can not be the zero map and can not be a map of constants. Thus,

$$t < s+3-t \quad \text{and} \quad \lceil \tfrac{s+1}{2} \rceil \le t < \tfrac{s+3}{2}.$$

It follows that $t = \lceil \frac{s+1}{2} \rceil$.

If b_1'' is positive, then the component

$$Q(-s-2+t)^{b_1''} \to Q(-t)^{b_1'} \oplus Q(-t-1)^{b_1''}$$

of F can not be the zero map and can not be a map of constants. Thus,

$$t < s+2-t \quad \text{and} \quad \lceil \tfrac{s+1}{2} \rceil \le t < \tfrac{s+2}{2}.$$

Of course, this is impossible if s is even. Thus, b_1'' is zero when s is even. In this case, the resolution of F is pure and resolves a Gorenstein quotient ring with a matrix of linear forms in the middle. One may apply the Herzog-Kühl formula [7, Thm. 1] or the Buchsbaum-Eisenbud Theorem [3, Thm. 2.1] to calculate that $b_1 = s+3$.

If s is odd, then

$$\begin{aligned} b_1' &= H_Q(t) - H_{Q/J}(t) \\ &= H_Q(t) - H_{Q/J}(s-t), && \text{because } Q/J \text{ is graded and Gorenstein,} \\ &= H_Q(t) - H_Q(t-1), && \text{because } Q/J \text{ is compressed and } s-t = t-1 < t, \\ &= t+1. \end{aligned}$$

Rename b_1'' to be ν. The proof is complete. □

Corollary 5.5. *Let $\mathbf{k}$ be a field of characteristic zero, d be a positive integer, and*

$$J_1 = (x^d, y^d, z^d) : (x+y+z)^d, \qquad J_2 = (x^d, y^d, z^d) : (x+y+z)^{d+1},$$
$$J_3 = (x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}, \text{ and} \qquad J_4 = (x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1}$$

be ideals of $Q = \mathbf{k}[x,y,z]$. Then

(a) *J_2 is minimally generated by $2d-1$ elements of degree $d-1$, and all of the relations on these generators are linear,*
(b) *J_4 is minimally generated by $2d+1$ elements of degree d, and all of the relations on these generators are linear,*
(c) *J_1 and J_3 are each minimally generated by d generators of degree $d-1$ and ν generators of degree d, where ν is the dimension of the vector space of linear relations on the generators of degree $d-1$.*

Proof. According to Observation 3.13.(d) and (a), each ring Q/J_i is compressed and has socle degree s_i, where $s_1 = 2d-3$, $s_2 = 2d-4$, $s_3 = 2d-3$, and $s_4 = 2d-2$. Apply Lemma 5.4 to obtain the result. □

Lemma 5.6 is a key step in the direction of obtaining more precise information about the ideals J_1 and J_3 of Corollary 5.5. Item (a) of Lemma 5.6 is a result which probably is of independent interest. It provides a technique for bounding the number of linear relations on a set of homogeneous forms of the same degree in a polynomial ring in three variables over a field.

Lemma 5.6. *Let $\mathbf{k}$ be an arbitrary field, d and m be positive integers, and $f_1,\dots,f_d$ be homogeneous forms of degree m in $\mathbf{k}[x,y,z]$. Assume that for each s, with $1\le s\le d$, f_s has degree exactly $m+1-s$ in y when viewed as an element of $(\mathbf{k}[x,z])[y]$. For each s, let $F_{s-1,s}$ in $\mathbf{k}[x,z]$ be the coefficient of y^{m+1-s} in f_s. Assume further that $F_{s-1,s}$ does not divide $F_{s,s+1}$ in $\mathbf{k}[x,z]$ for any s with $2\le s\le d-1$. Then the following statements hold.*
(a) *The vector space of linear relations on $f_1,\dots,f_d$ has dimension at most one.*
(b) *Assume that there is a non-zero linear relation on $f_1,\dots,f_d$. Let h be a homogeneous form of degree $m+1$ in $\mathbf{k}[x,y,z]$. Assume that h has degree exactly $m+1-d$ in y when viewed as an element of $(\mathbf{k}[x,z])[y]$. Let H_d in $\mathbf{k}[x,z]$ be the coefficient of y^{m+1-d} in h. Assume that H_d is not divisible by $F_{d-1,d}$, then $h\notin(f_1,\dots,f_d)$.*

Proof. The hypothesis that f_s has degree $m+1-s$ in y for each s, with $1\le s\le d$, ensures that $d\le m+1$.

We first prove (a). Write the polynomial f_s in the form

$$f_s = \sum_{\ell=0}^{m+1-s} F_{m-\ell,s}y^{\ell},$$

with $F_{i,s}$ a homogeneous form in $\mathbf{k}[x,z]$ of degree i and $F_{s-1,s}$ is not zero. Assume that there is a linear relation

$$(a_1x+b_1y+c_1z)f_1+\dots+(a_dx+b_dy+c_dz)f_d = 0, \tag{15}$$

with a_i, b_i, and c_i elements of the field $\mathbf{k}$. We will prove that the relation (15) is completely determined by the choice of b_2. View the expression on the left hand side of (15) as a polynomial in y with coefficients in $\mathbf{k}[x,z]$. For $0\le s\le d$, the coefficient of y^{m+1-s} in (15) is equal to zero; thus,

$$\begin{cases} b_1F_{s,1}=0, & \text{if } s=0,\\ (a_1x+c_1z)F_{s-1,1}+\dots+(a_sx+c_sz)F_{s-1,s}+b_1F_{s,1}+\dots+b_sF_{s,s}+b_{s+1}F_{s,s+1}=0, & \text{if } 1\le s\le d-1,\\ (a_1x+c_1z)F_{s-1,1}+\dots+(a_sx+c_sz)F_{s-1,s}+b_1F_{s,1}+\dots+b_sF_{s,s}=0, & \text{if } s=d. \end{cases} \tag{16}$$

Recall that $F_{0,1}$ is a unit in $\mathbf{k}$. It follows from equation (16), with $s=0$, that $b_1=0$. For $s=1$, equation (16) is

$$(a_1x+c_1z)F_{0,1}+b_1F_{1,1}+b_2F_{1,2}=0;$$

hence

$$(a_1x+c_1z)F_{0,1}+b_2F_{1,2}=0.$$

For any fixed choice of b_2, there is at most once choice of $(a_1,c_1)\in\mathbf{k}^2$ for which this equation holds.

Fix b_2. We claim that for every s, with $1\le s\le d-1$, the values of $a_1,c_1,\dots,a_s,c_s,b_1,\dots,b_s,b_{s+1}$ such that (16) holds are unique. For $s>1$, we prove this by induction on s. Assume that $a_1,c_1,\dots,a_{s-1},c_{s-1},b_1,\dots,b_{s-1},b_s$ are uniquely determined. We solve for a_s,c_s,b_{s+1} so that (16) holds. Let

$$G_s = (a_1x+c_1z)F_{s-1,1}+\dots+(a_{s-1}x+c_{s-1}z)F_{s-1,s-1}+b_1F_{s,1}+\dots+b_sF_{s,s}.$$

The inductive hypothesis ensures that all the coefficients of G_s have been solved for uniquely in terms of the value of b_2. Apply (16) to see that

$$G_s + b_{s+1}F_{s,s+1} = -(a_s x + c_s z)F_{s-1,s}. \tag{17}$$

Therefore the value of b_{s+1} is such that $G_s + b_{s+1}F_{s,s+1}$ is divisible by $F_{s-1,s}$. The fact that $F_{s,s+1}$ is not divisible by $F_{s-1,s}$ implies that there is a nonzero remainder when $F_{s,s+1}$ is divided by $F_{s-1,s}$. (These are homogeneous polynomials in two variable; they can be translated into polynomials in one variable $T = z/x$ by de-homogenization, so the notion of remainder makes sense.) Let R_s be this remainder. There is at most one value of b_{s+1} that makes $b_{s+1}R_s$ equal to the remainder obtained when G_s is divided by $F_{s-1,s}$. Once b_{s+1} has been found, a_s and c_s are obtained by dividing both sides of (17) by $F_{s-1,s}$.

When $s = d$, Equation (16) becomes

$$(a_1x + c_1z)F_{d-1,1} + \ldots + (a_{d-1}x + c_{d-1}z)F_{d-1,d-1} + (a_dx + c_dz)F_{d-1,d} + b_1F_{d,1} + \cdots + b_dF_{d,d} = 0.$$

Since there is at most one solution (once b_2 has been fixed) for $a_1, c_1, \ldots, a_{d-1}, c_{d-1}, b_1, \ldots, b_d$ that solve the previous equations, it follows that there are also unique values of a_d, c_d that make this term equal to zero. This concludes the proof of part (a).

For part (b), fix b_2 as above and assume that there is a (unique) solution for $a_1, c_1, \ldots, a_{d-1}, c_{d-1}, b_1, \ldots, b_{d-1}, b_d$ that satisfy (16) for all $1 \le s \le d-1$. The assumption that there is a non-zero linear relation on $f_1, \ldots, f_d$ means that for these values of $a_1, c_1, \ldots, a_{s-1}, c_{s-1}, b_1, \ldots, b_{s-1}, b_s$, there are values for a_d, c_d such that

$$(a_1x + c_1z)F_{d-1,1} + \ldots + (a_{d-1}x + c_{d-1}z)F_{d-1,d-1} + (a_dx + c_dz)F_{d-1,d} + b_1F_{d,1} + \cdots + b_dF_{d,d} = 0, \tag{18}$$

Assume, by way of contradiction, that $h \in (f_1, \ldots, f_d)$. It follows that there exist constants a_i', b_i', and c_i' with

$$(a_1'x + b_1'y + c_1'z)f_1 + \ldots + (a_d'x + b_d'y + c_d'z)f_d = h. \tag{19}$$

The degree of h in the variable y is $m+1-d$; hence the coefficients of $y^{m+1}, y^m, \ldots, y^{m+2-d}$ in (19) must be zero. Therefore, $b_1' = 0$, and $a_1', c_1', \ldots, a_{d-1}', c_{d-1}', b_1', \ldots, b_{d-1}', b_d'$ satisfy the equations (16) for all $1 \le s \le d-1$. Choose $b_2 = b_2'$. It follows that $a_i = a_i'$ and $c_i = c_i'$ for $1 \le i \le d-1$, and $b_i = b_i'$ for $1 \le i \le d$.

The coefficient of y^{m+1-d} in (19) is

$$(a_1x + c_1z)F_{d-1,d} + \ldots + (a_{d-1}x + c_{d-1}z)F_{d-1,d-1} + (a_d'x + c_d'z)F_{d-1,d} + b_1F_{d,1} + \cdots + b_dF_{d,d} = H_d$$

Compare the most recent equation with (18) in order to see that

$$H_d = ((a_d' - a_d)x + (c_d' - c_d)z)F_{d-1,d},$$

which contradicts the hypothesis that H_d is not divisible by $F_{d-1,d}$. □

Proposition 9.1 gives explicit elements of degree $d-1$ for the ideals J_1 and J_3 of Corollary 5.5. (The proof of Proposition 9.1 is fairly long, but it is self-contained and is given in Section 9.) Once these explicit generators are found one may apply Lemma 5.6 in order to bound the number of linear relations on these generators. As a consequence, one learns a complete minimal generating set for these ideals and all of the graded Betti numbers in a minimal resolution of these ideals.

It is convenient to record the polynomials $\{f_{j,\varepsilon}\}$ of Proposition 9.1 at this point. Let j, d, ε, and σ be integers with

$$1 \le j \le d, \quad 0 \le \varepsilon \le 1, \quad \text{and} \quad j-1 \le \sigma \le d-1.$$

Let

$$F_{\sigma,j,\varepsilon} = \sum_{k=0}^{j-1} (-1)^{\sigma+k} \binom{d-1-k}{d-j} \binom{d-1-\sigma+k}{k} \binom{\sigma+\varepsilon}{j-1+\varepsilon} x^k z^{\sigma-k} \tag{20}$$

and

$$f_{j,\varepsilon} = \sum_{s=j-1}^{d-1} F_{s,j,\varepsilon} y^{d-1-s}. \tag{21}$$

Proposition 5.7. *Let $\mathbf{k}$ be a field of characteristic zero and d be a positive integer. For integers j and ε, with $1 \le j \le d$ and $0 \le \varepsilon \le 1$, let $f_{j,\varepsilon}$ be the polynomial of* (21) *viewed as an element of $\mathbf{k}[x,y,z]$. Then*

$$(x^d, y^{d+\varepsilon}, z^d) : (x+y+z)^{d+\varepsilon} \text{ is minimally generated by } \begin{cases} f_{1,\varepsilon}, \dots, f_{d,\varepsilon}, & \text{if } d \text{ is odd, and} \\ f_{1,\varepsilon}, \dots, f_{d,\varepsilon}, z^d, & \text{if } d \text{ is even,} \end{cases}$$

as an ideal in $\mathbf{k}[x,y,z]$.

Proof. Fix ε. Recall that the ideal $(x^d, y^{d+\varepsilon}, z^d) : (x+y+z)^{d+\varepsilon}$ is called $J_{1+2\varepsilon}$ in Corollary 5.5. The inclusion $(f_{1,\varepsilon}, \dots, f_{d,\varepsilon}) \subseteq J_{1+2\varepsilon}$ is established in Proposition 9.1. We verify that $f_{1,\varepsilon}, \dots, f_{d,\varepsilon}$ satisfy the hypothesis in Lemma 5.6.(a). That is, we verify that $F_{s,s+1,\varepsilon}$ is not divisible by $F_{s-1,s,\varepsilon}$ for $2 \le s \le d-1$, where the F's are given in (20):

$$\begin{aligned} F_{s,s+1,\varepsilon} &= \sum_{k=0}^{s} (-1)^{s-k} \binom{d-1-k}{d-s-1} \binom{d-1-s+k}{k} x^k z^{s-k} \\ &= \left[(-1)^s \binom{d-1}{d-s-1} z^s + \dots - (d-s) \binom{d-2}{s-1} x^{s-1} z + \binom{d-1}{s} x^s \right] \quad \text{and} \\ F_{s-1,s,\varepsilon} &= \left[(-1)^{s-1} \binom{d-1}{d-s} z^{s-1} + \dots - (d-s+1) \binom{d-2}{s-2} x^{s-2} z + \binom{d-1}{s-1} x^{s-1} \right]. \end{aligned}$$

Assume, by way of contradiction, that $F_{s-1,s,\varepsilon}$ divides $F_{s,s+1,\varepsilon}$, and write

$$(ax+bz)F_{s-1,s,\varepsilon} = F_{s,s+1,\varepsilon} \tag{22}$$

with $a, b \in \mathbf{k}$. Compare the coefficients of z^s, x^s, and $x^{s-1}z$ in equation (22) and conclude that

$$b(-1)^{s-1} \binom{d-1}{d-s} = (-1)^s \binom{d-1}{d-s-1}, \tag{23}$$

$$a \binom{d-1}{s-1} = \binom{d-1}{s}, \quad \text{and} \tag{24}$$

$$-a(d-s+1) \binom{d-2}{s-2} + b \binom{d-1}{s-1} = -(d-s) \binom{d-2}{s-1}. \tag{25}$$

To complete the calculation, we use two identities about binomial coefficients which hold for all integers a and b:

$$b \binom{a}{b} = \binom{a}{b-1} (a-b+1) \quad \text{and} \tag{26}$$

$$\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}. \tag{27}$$

Apply (26) to (23) and (24) to see that

$$a = -b = \frac{d-s}{s}.$$

(The relevant numbers are non-zero because $2 \le s \le d-1$.) Multiply both sides of (25) by $-s/(d-s)$ and use (27) to see that

$$(d-s+1) \binom{d-2}{s-2} + \left(\binom{d-2}{s-2} + \binom{d-2}{s-1} \right) = s \binom{d-2}{s-1}.$$

It follows that

$$(d-s+2) \binom{d-2}{s-2} = (s-1) \binom{d-2}{s-1}.$$

Apply (26) again to obtain

$$(d-s+2) \binom{d-2}{s-2} = \binom{d-2}{s-2} (d-s).$$

Thus, $2\binom{d-2}{s-2} = 0$; which of course is a contradiction because $2 \le s \le d-1$. The claim that $F_{s,s+1,\varepsilon}$ is not divisible by $F_{s-1,s,\varepsilon}$, for $2 \le s \le d-1$, is verified.

Apply Lemma 5.6.(a) to see that the vector space of linear relations on $f_{1,\varepsilon}, \dots, f_{d,\varepsilon}$ has dimension at most one. Therefore, according to Corollary 5.5, the ideal $J_{1+2\varepsilon}$, has at most one minimal generator of degree d and either

$$\{f_{i,\varepsilon} \mid 1 \le i \le d\} \quad \text{or} \quad \{f_{i,\varepsilon} \mid 1 \le i \le d\} \cup \{h\}, \tag{28}$$

for some homogeneous form h of degree d, is a minimal generating set for $J_{1+2\varepsilon}$. However, $J_{1+2\varepsilon}$ is a grade three Gorenstein ideal, therefore, $J_{1+2\varepsilon}$ has an odd number of minimal generators (see [24] or [3, Cor. 2.2]). If d is odd, then $J_{1+2\varepsilon}$ is equal to the left hand candidate from (28). If d is even, then $J_{1+2\varepsilon}$ is equal to the right hand candidate from (28), and, according to Corollary 5.5, there is a non-zero relation on $f_{1,\varepsilon}, \dots, f_{d,\varepsilon}$. When d is even, we apply Lemma 5.6.(b) to see that $z^d \notin (f_{1,\varepsilon}, \dots, f_{d,\varepsilon})$. (It is clear that z^d is not divisible by $F_{d-1,d,\varepsilon} = z^{d-1} + \dots + x^{d-1}$). On the other hand, it is obvious that $z^d \in J_{1+2\varepsilon}$. When d is even, the vector space $[J_{1+2\varepsilon}/(f_{1,\varepsilon}, \dots, f_{d,\varepsilon})]_d$ has dimension one, and h in (28) may be taken to be z^d. The proof is complete. ☐

Corollary 5.8. *Let* $\mathbf{k}$ *be a field of characteristic zero, P be the standard graded polynomial ring* $P = \mathbf{k}[x,y,z,w]$, *d be a positive integer, and* ε_1, ε_2, ε_3, *and* ε_4 *be elements of* $\{0,1\}$. *Then the ideal*

$$(x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}) : (x+y+z+w) \tag{29}$$

of P is minimally generated by

(a) $x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}$, *and d additional generators of degree* $2d-2$, *when* $\sum_{i=1}^4 \varepsilon_i = 0$, *for* $2 \le d$;
(b) $x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}$, *and* $2d-1$ *additional generators of degree* $2d-1$, *when* $\sum_{i=1}^4 \varepsilon_i = 1$, *for* $2 \le d$;
(c) $x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}$, *and d additional generators of degree* $2d-1$, *when* $\sum_{i=1}^4 \varepsilon_i = 2$, *for* $2 \le d$; *and*
(d) $x^{d+\varepsilon_1}, y^{d+\varepsilon_2}, z^{d+\varepsilon_3}, w^{d+\varepsilon_4}$, *and* $2d$ *additional generators of degree* $2d$, *when* $\sum_{i=1}^4 \varepsilon_i = 3$.

Remark. Note that the result is symmetric in x, y, z, w.

Proof. The assertion follows from Observation 5.2, Corollary 5.5, and Proposition 5.7. When one applies the technique of Observation 5.2 to the modules of Corollary 5.5 it is important to notice that a minimal generating set of J_2 represents a minimal generating set of $J_2/(x^d, y^d, z^d)$; however, z^d never represents a minimal generator of $J_4/(x^{d+1}, y^{d+1}, z^d)$, $J_1/(x^d, y^d, z^d)$, or $J_3/(x^d, y^{d+1}, z^d)$, even though z^d is always a minimal generator of J_4 and sometimes is part of a minimal generating set of J_1 or J_3, as described in Proposition 5.7. ☐

Remark 5.9. If $d = 1$, then the elements of (a), (b), and (c) continue to generate the ideal (29); however, they do not form a minimal generating set. We return to this theme in the proof of Proposition 5.12.

Recall the data of 2.1. Proposition 5.1 and Corollary 5.8 combine to describe a list of generators for the ideal

$$I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n),$$

of the standard graded polynomial ring $P = \mathbf{k}[x,y,z,w]$, where $\mathbf{k}$ is a field of characteristic zero. Propositions 5.10 and 5.11 remove redundant elements from the list. A minimal generating set for I is given in Proposition 5.12.

Proposition 5.10. *Let* $\mathbf{k}$ *be a field of characteristic zero. Recall the data of* 2.1. *The generators of I on the list given in Proposition* 5.1 *that are obtained when* $\sum_{i=1}^4 \varepsilon_i = 1$ *are redundant.*

Proof. We show that these generators are linear combinations of the generators listed in Proposition 5.1 that are obtained when $\sum_{i=1}^4 \varepsilon_i = 2$.

Consider $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0, \varepsilon_4 = 1$. The other cases are similar. We claim that

$$(x^d, y^d, z^d, w^{d+1}) : f \subseteq \Big((x^{d+1}, y^d, z^d, w^{d+1}) : f\Big) + \Big((x^d, y^{d+1}, z^d, w^{d+1}) : f\Big). \tag{30}$$

Once we establish Claim (30), then a generator of I of the form $(xyz)^r g^{[n]}$, with $g \in (x^d, y^d, z^d, w^{d+1}) : f$, can be written as

$$(xyz)^r g^{[n]} = x^r (y^r z^r h^{[n]}) + y^r (x^r z^r k^{[n]})$$

with $h \in (x^{d+1}, y^d, z^d, w^{d+1}) : f$ and $k \in (x^d, y^{d+1}, z^d, w^{d+1}) : f$ such that $g = h + k$. In order to prove the claim, recall from Observation 5.2 that every $g \in (x^d, y^d, z^d, w^{d+1}) : f$ can be written as $g = pg_0$, plus an element of (x^d, y^d, z^d, w^{d+1}), where g_0 is in $(x^d, y^d, z^d) : (x+y+z)^{d+1}$ and p is the polynomial

$$p = \left(w^{d+1} - (-(x+y+z))^{d+1}\right)/(x+y+z+w). \tag{31}$$

It is therefore sufficient to prove

$$(x^d, y^d, z^d) : (x+y+z)^{d+1} = \left((x^{d+1}, y^d, z^d) : (x+y+z)^{d+1}\right) + \left((x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}\right). \tag{32}$$

The inclusion $\supseteq$ is obvious. We know from Corollary 5.5 that $J_2 = (x^d, y^d, z^d) : (x+y+z)^{d+1}$ is minimally generated by $2d-1$ elements of degree $d-1$; consequently, it is enough to show that the inclusion $\subseteq$ claimed in (32) holds for the graded components of degree $d-1$. We know from Corollary 5.5 that the graded component of degree $d-1$ of each ideal on the right hand side of (32) has dimension d. By the inclusion-exclusion formula for vector space dimensions, it is enough to show that the $d-1$ graded component of

$$\left((x^{d+1}, y^d, z^d) : (x+y+z)^{d+1}\right) \cap \left((x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}\right) \tag{33}$$

has dimension at most one. Observe that the ideal of (33) is equal to

$$(x^{d+1}, y^{d+1}, x^d y^d, z^d) : (x+y+z)^{d+1}. \tag{34}$$

Let u, v be two elements in (34) of degree $d-1$. Observe that

$$u(x+y+z)^{d+1} \equiv \alpha x^d y^d \quad \text{and} \quad v(x+y+z)^{d+1} \equiv \beta x^d y^d \mod (x^{d+1}, y^{d+1}, z^d),$$

for some $\alpha, \beta \in \mathbf{k}$. Therefore, $\beta u - \alpha v \in (x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1}$. We know from Corollary 5.5 that the ideal $J_4 = (x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1}$ is generated in degree d, and therefore $\beta u - \alpha v = 0$ as desired. $\square$

Proposition 5.11. *Let $\mathbf{k}$ be a field of characteristic zero. Recall the data of* 2.1. *The generators of I on the list given in Proposition* 5.1 *that are obtained when $\sum_{i=1}^4 \varepsilon_i = 3$ are redundant.*

Proof. We show that these generators are linear combinations of the generators listed in Proposition 5.1 that are obtained when $\sum_{i=1}^4 \varepsilon_i = 2$ and $\sum_{i=1}^4 \varepsilon_i = 4$.

Consider $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = 1, \varepsilon_3 = 0$. The other cases are similar. We claim that

$$(x^{d+1}, y^{d+1}, z^d, w^{d+1}) : f \subseteq x[(x^d, y^{d+1}, z^d, w^{d+1}) : f] + (x^{d+1}, y^{d+1}, z^{d+1}, w^{d+1}) : f. \tag{35}$$

Once we establish Claim (35), then a generator of I of the form $z^r g^{[n]}$, with $g \in (x^{d+1}, y^{d+1}, z^d, w^{d+1}) : f$ can be written as

$$z^r g^{[n]} = x^{n-r}(x^r z^r h^{[n]}) + z^r k^{[n]},$$

with

$$h \in (x^d, y^{d+1}, z^d, w^{d+1}) : f \quad \text{and} \quad k \in (x^{d+1}, y^{d+1}, z^{d+1}, w^{d+1}) : f.$$

In order to prove the claim, recall from Observation 5.2 that every $g \in (x^{d+1}, y^{d+1}, z^d, w^{d+1}) : f$ can be written as $g = pg_0$, plus an element of $(x^{d+1}, y^{d+1}, z^d, w^{d+1})$, where $g_0 \in (x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1}$ and p is given in (31). It is therefore sufficient to prove

$$(x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1} = \begin{cases} x[(x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}] \\ +(x^{d+1}, y^{d+1}, z^{d+1}) : (x+y+z)^{d+1} \end{cases} \tag{36}$$

The inclusion $\supseteq$ is obvious. We know from Corollary 5.5 that $J_4 = (x^{d+1}, y^{d+1}, z^d) : (x+y+z)^{d+1}$ is minimally generated by $2d+1$ elements of degree d; consequently, it suffices to show that the inclusion $\subseteq$ claimed in (36) holds for the graded components of degree d.

The first ideal on the right hand side of (36) is xJ_3, whose degree d component has dimension d, and the second ideal on the right hand side of (36) is the version of J_1 obtained when $d+1$ is used in the role of d, and therefore its degree d component has dimension $d+1$. We count vector space dimension. It suffices to show that

$$x[(x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}] \cap [(x^{d+1}, y^{d+1}, z^{d+1}) : (x+y+z)^{d+1}] \tag{37}$$

$x^N \in P_{m_{(d,r,0,0,0)}}$,			
$y^N \in P_{m_{(d,0,r,0,0)}}$,			
$z^N \in P_{m_{(d,0,0,r,0)}}$,			
$w^N \in P_{m_{(d,0,0,0,r)}}$,			
$(xyzw)^r a_i^{[n]} \in P_{m_{(2d-2,r,r,r,r)}}$,	for $1 \le i \le d$,	$a_i \in (x^d,y^d,z^d,w^d):f$,	and $\deg(a_i)=2d-2$,
$(zw)^r b_{1i}^{[n]} \in P_{m_{(2d-1,0,0,r,r)}}$,	for $1 \le i \le d$,	$b_{1i} \in (x^{d+1},y^{d+1},z^d,w^d):f$	and $\deg(b_{1i})=2d-1$,
$(yw)^r b_{2i}^{[n]} \in P_{m_{(2d-1,0,r,0,r)}}$,	for $1 \le i \le d$,	$b_{2i} \in (x^{d+1},y^d,z^{d+1},w^d):f$	and $\deg(b_{2i})=2d-1$,
$(xw)^r b_{3i}^{[n]} \in P_{m_{(2d-1,r,0,0,r)}}$,	for $1 \le i \le d$,	$b_{3i} \in (x^d,y^{d+1},z^{d+1},w^d):f$	and $\deg(b_{3i})=2d-1$,
$(yz)^r b_{4i}^{[n]} \in P_{m_{(2d-1,0,r,r,0)}}$,	for $1 \le i \le d$,	$b_{4i} \in (x^{d+1},y^d,z^d,w^{d+1}):f$	and $\deg(b_{4i})=2d-1$,
$(xz)^r b_{5i}^{[n]} \in P_{m_{(2d-1,r,0,r,0)}}$,	for $1 \le i \le d$,	$b_{4i} \in (x^d,y^{d+1},z^d,w^{d+1}):f$	and $\deg(b_{5i})=2d-1$,
$(xy)^r b_{6i}^{[n]} \in P_{m_{(2d-1,r,r,0,0)}}$,	for $1 \le i \le d$,	$b_{6i} \in (x^d,y^d,z^{d+1},w^{d+1}):f$	and $\deg(b_{6i})=2d-1$,
$c_i^{[n]} \in P_{m_{(2d,0,0,0,0)}}$,	for $1 \le i \le d+1$,	$c_i \in (x^{d+1},y^{d+1},z^{d+1},w^{d+1}):f$,	and $\deg(c_i)=2d$.

TABLE 1. The multi-homogeneous generating set for I as described in Proposition 5.12.

has no elements of degree d.

On the other hand, this assertion is obvious. Any homogeneous element of (37) of degree d has the form xu with u a homogeneous element of $(x^d,y^{d+1},z^{d+1}):(x+y+z)^{d+1}$ of degree $d-1$, because, x, y^{d+1}, z^{d+1} is a regular sequence in $\mathbf{k}[x,y,z]$. It follows from Corollary 5.5 (or [21, Thm. 5]) that $u=0$. $\square$

Proposition 5.12. *Adopt the data of* 2.1*,* 2.2*, and* 3.4*, with* $\mathbf{k}$ *a field of characteristic zero. The generators on the list given in Proposition* 5.1 *that are obtained when* $\sum_{i=1}^{4}\varepsilon_i$ *is equal to* $0,2$ *and* 4*, together with* x^N,y^N,z^N,w^N*, form a minimal set of generators for the ideal I.*

Proof. Apply Propositions 5.8, 5.10, and 5.11 to see that the multi-homogeneous elements listed in Table 1 generate I. The polynomials a_i, b_{hi}, and c_i in P, from Table 1, all are homogeneous. The polynomials $a_1,\dots,a_d$ are linearly independent; as are the polynomials $b_{h1},\dots,b_{hd}$, for each h, and the polynomials $c_1,\dots,c_{d+1}$. We prove that the list of generators given in Table 1 is a minimal generating set for I.

Notice that if m_1 and m_2 are distinct elements of $\mathbf{M}$ selected from

$$\begin{Bmatrix} m_{(2d-2,r,r,r,r)}, & m_{(2d-1,0,0,r,r)}, & m_{(2d-1,0,r,0,r)}, & m_{(2d-1,r,0,0,r)}, \\ m_{(2d-1,0,r,r,0)}, & m_{(2d-1,r,0,r,0)}, & m_{(2d-1,r,r,0,0)}, & m_{(2d,0,0,0,0)} \end{Bmatrix}, \tag{38}$$

then $P_{m_1-m_2}=0$. For example,

$$\begin{aligned} &P_{m_{(2d,0,0,0,0)}-m_{(2d-1,r,r,0,0)}} = P_{m_{(-1,n-r,n-r,0,0)}} = 0, \\ &P_{m_{(2d,0,0,0,0)}-m_{(2d-2,r,r,r,r)}} = P_{m_{(-2,n-r,n-r,n-r,n-r)}} = 0, \\ &P_{m_{(2d-1,r,r,0,0)}-m_{(2d-1,r,0,r,0)}} = P_{m_{(-1,0,r,n-r,0)}} = 0, \text{ and} \\ &P_{m_{(2d-1,r,r,0,0)}-m_{(2d-2,r,r,r,r)}} = P_{m_{(-1,0,0,n-r,n-r)}} = 0. \end{aligned}$$

As a consequence, any relation, on the generators of Table 1, which has a non-zero constant coefficient, can only involve the generators, x^N, y^N, z^N, w^N and generators from one of the multi-degrees from the list (38).

Proposition 5.8 establishes that

5.12.1. $x^d,y^d,z^d,w^d,a_1,\dots,a_d$ is a minimal generating set for the ideal $(x^d,y^d,z^d,w^d,a_1,\dots,a_d)$, provided $2 \le d$;

5.12.2. $x^d,y^d,z^d,w^d,b_{h1},\dots,b_{hd}$ is a minimal generating set for the ideal $(x^d,y^d,z^d,w^d,b_{h1},\dots,b_{hd})$, for each h, provided $2 \le d$; and

5.12.3. $x^d,y^d,z^d,w^d,c_1,\dots,c_{d+1}$ is a minimal generating set for the ideal $(x^d,y^d,z^d,w^d,c_1,\dots,c_{d+1})$, for $1 \le d$.

It quickly follows that none of the generators from Table 1 is redundant, except possibly when $d=1$. Indeed, for example, if

$$\lambda_1 x^N + \lambda_2 y^N + \lambda_3 z^N + \lambda_4 w^N + (xyzw)^r a^{[n]} = 0 \tag{39}$$

in $P_{m_{(2d-2,r,r,r,r)}}$ is a multi-homogeneous equation with

$$\lambda_1 \in P_{m_{(d-2,0,r,r,r)}}, \quad \lambda_2 \in P_{m_{(d-2,r,0,r,r)}}, \quad \lambda_3 \in P_{m_{(d-2,r,r,0,r)}}, \quad \lambda_4 \in P_{m_{(d-2,r,r,r,0)}},$$

and some $\mathbf{k}$-linear combination of a of $a_1,\dots,a_d$ in P_{2d-2}, then

$$\lambda_1 = y^r z^r w^r \mu_1^{[n]}, \quad \lambda_2 = x^r z^r w^r \mu_2^{[n]}, \quad \lambda_3 = x^r y^r w^r \mu_3^{[n]}, \quad \text{and} \quad \lambda_4 = x^r y^r z^r \mu_4^{[n]},$$

for some $\mu_i \in P_{d-2}$. The polynomial ring P is a domain; hence (39) implies

$$\mu_1^{[n]}(x^d)^{[n]} + \mu_2^{[n]}(y^d)^{[n]} + \mu_3^{[n]}(z^d)^{[n]} + \mu_4^{[n]}(w^d)^{[n]} + a^{[n]} = 0.$$

The homomorphism $g \mapsto g^{[n]}$ is an injection. Apply (5.12.1) to conclude that the coefficients of a as a $\mathbf{k}$-linear combination of $a_1,\dots,a_d$ are all zero.

If $d=1$, then the polynomial a_1 of Table 1 is $a_1 = 1$. It is not true that $x,y,z,w,1$ is a minimal generating set for the ideal $(x,y,z,w,1)$; but it is true, and obvious, that $x^{n+r},y^{n+r},z^{n+r},w^{n+r},(xyzw)^r$ is a minimal generating set for the ideal $(x^{n+r},y^{n+r},z^{n+r},w^{n+r},(xyzw)^r)$. Also, if $d=1$, then the polynomial b_{11} of Table 1 is $b_{11} = x-y$. It is not true that $x,y,z,w,x-y$ is a minimal generating set for the ideal $(x,y,z,w,x-y)$; but it is true, and obvious, that $x^{n+r},y^{n+r},z^{n+r},w^{n+r},z^r w^r(x^n-y^n)$ is a minimal generating set for the ideal $(x^{n+r},y^{n+r},z^{n+r},w^{n+r},z^r w^r(x^n-y^n))$. When $d=1$, one can treat b_{h1}, for all h in the same manner. □

6. THE MAIN THEOREM.

Adopt the language of 2.1 and 2.2 with $\mathbf{k}$ a field of characteristic zero. Theorem 6.2 gives the multi-graded Betti numbers in the minimal homogeneous resolution of P/I by free P-modules.

Notation 6.1. In the statement of Theorem 6.2, each of the symbols e_1, e_2, e_3, and e_4, is either the integer 0 or the integer 1. The sums $\oplus_{\sum e_i=0}$, $\oplus_{\sum e_i=1}$, $\oplus_{\sum e_i=2}$, $\oplus_{\sum e_i=3}$, and $\oplus_{\sum e_i=4}$, have 1, 4, 6, 4, and 1 summand respectively. In particular, for example, the sum $\oplus_{\sum e_i=2}$ has six summands; these summands correspond to (e_1,e_2,e_3,e_4) equal to $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, and $(0,0,1,1)$.

The notation $m_{(k,\rho_1,\rho_2,\rho_3,\rho_4)}$ is explained in 3.4.

Theorem 6.2. *Let $\mathbf{k}$ be a field of characteristic zero, n, d, and r be positive integers, with $r<n$, N be the integer $N = dn+r$, P be the standard graded polynomial ring $P = \mathbf{k}[x,y,z,w]$, I be the ideal*

$$I = (x^N,y^N,z^N,w^N) : (x^n+y^n+z^n+w^n)$$

of P, and R be the quotient ring $R = P/I$. Give P the multi-grading of Definition 2.2. Then the minimal multi-homogeneous resolution of R by free P-modules has the form

$$F: \quad 0 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0,$$

with $F_0 = P$,

$$F_1 = \begin{cases} \bigoplus_{\sum e_i=1} P(-m_{(d,re_1,re_2,re_3,re_4)}) \\ \oplus \bigoplus_{\sum e_i=4} P(-m_{(2d-2,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=0} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{d+1}, \end{cases} \qquad F_3 = \begin{cases} \bigoplus_{\sum e_i=3} P(-m_{(3d-1,re_1,re_2,re_3,re_4)}) \\ \oplus \bigoplus_{\sum e_i=0} P(-m_{(2d+1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=4} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^{d+1}, \end{cases}$$

$$F_2 = \begin{cases} \bigoplus_{\sum e_i=3} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^{2d+1} \\ \oplus \bigoplus_{\sum e_i=1} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{2d+1}, \end{cases} \quad \text{and} \quad F_4 = \bigoplus_{\sum e_i=4} P(-m_{(4d-1,re_1,re_2,re_3,re_4)}).$$

Remark. If one ignores the $\mathbf{M}$-grading on P and merely views P as a standard graded polynomial ring, then the minimal homogeneous resolution F of Theorem 6.2 is $F_0 = P$,

$$F_1 = P(-(nd+r))^4 \oplus P(-(2nd-2n+4r))^d \oplus P(-(2nd-n+2r))^{6d} \oplus P(-2nd)^{d+1},$$
$$F_2 = P(-(2nd-n+3r))^{8d+4} \oplus P(-(2nd+r))^{8d+4},$$
$$F_3 = P(-(3nd-n+3r))^4 \oplus P(-(2nd+n))^d \oplus P(-(2nd+2r))^{6d} \oplus P(-(2nd-n+4r))^{d+1}, \text{ and}$$
$$F_4 = P(-(4nd-n+4r)).$$

Proof. The module F_1 may be read from Proposition 5.12. The socle of $P/(x^N, y^N, z^N, w^N)$ is represented by $x^{N-1}y^{N-1}z^{N-1}w^{N-1}$; hence, the multi-degree of the socle of $P/(x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ is

$$\delta_0 = m_{(4d-1,r-1,r-1,r-1,r-1)};$$

see 3.9.(d). Apply 3.9.(e) to see that $F_4 = P(-\delta)$, where

$$\begin{aligned}\delta &= \text{the multi-degree of } xyzw + \delta_0\\ &= m_{(0,1,1,1,1)} + m_{(4d-1,r-1,r-1,r-1,r-1)} = m_{(4d-1,r,r,r,r)}.\end{aligned}$$

(40) The resolution F is self-dual, see 3.10.2. If $F_1 = \oplus P(-\beta_i)$ and $F_4 = P(-\delta)$, then $F_3 = \oplus P(-\delta + \beta_i)$.

The situation is straightforward. We know the multi-graded Hilbert function of all of the modules in the exact sequence

$$0 \to F_4 \to F_3 \to F_2 \to F_1 \to P \to R \to 0,$$

except for the module F_2. However the multi-graded Hilbert function is additive on exact sequences; so we can solve for the multi-graded Hilbert function of F_2 and then describe F_2 as a multi-graded free P-module. In particular, rank is additive on exact sequences and R has rank zero; hence

(41) $$F_2 \text{ is a free } P\text{-module of rank } 16d + 8.$$

(Recall that the rank of the P-module M is the dimension of the vector space $K \otimes_P M$ where K is the quotient field of P.)

The only difficulty comes from the fact that it is difficult to manipulate the formula of Proposition 4.2. On the other hand, it is easy to use Proposition 4.2 to compute $H_R(m)$ for any $m \in \mathbf{M}$; consequently, we will determine the structure of F_2 by evaluating the equation

(42) $$H_{F_2}(m) = H_R(m) - H_P(m) + H_{F_1}(m) + H_{F_3}(m) - H_{F_4}(m),$$

which holds for all $m \in \mathbf{M}$, at a few carefully chosen $m \in \mathbf{M}$.

Let

$$\mathcal{S} = \left\{ (\beta, \underline{\gamma}) \;\middle|\; \begin{array}{l} (\beta,\underline{\gamma}) \text{is a 5-tuple of integers in standard form in the sense of 3.4} \\ \text{and } P(-m_{b,\underline{\gamma}}) \text{ is a summand of } F_2 \end{array} \right\}$$

and, for each $(\beta, \underline{\gamma}) \in \mathcal{S}$, let

$$\#_{(\beta,\underline{\gamma})} \text{ equal the number of summands of } P(-m_{(\beta,\underline{\gamma})}) \text{ in } F_2.$$

Thus,

(43) $$F_2 = \bigoplus_{(\beta,\underline{\gamma})\in\mathcal{S}} P(-m_{(\beta,\underline{\gamma})})^{\#(\beta,\underline{\gamma})}.$$

It is necessary to determine the elements of $\mathcal{S}$ and the value of $\#_{(\beta,\underline{\gamma})}$ for each element $(\beta, \underline{\gamma})$ of $\mathcal{S}$. To that end, we evaluate $H_{F_2}(m)$ at various $m \in \mathbf{M}$. Let $(k, \underline{\rho})$ be a five-tuple of integers in standard form. Recall the description of F_2 which is given in (43). Observe that

(44) $$H_{F_2}(m_{(k,\underline{\rho})}) = \sum_{(\beta,\underline{\gamma})\in\mathcal{S}} \#_{(\beta,\underline{\gamma})} H_P(m_{(k,\underline{\rho})} - m_{(\beta,\underline{\gamma})})$$

Use 3.8.(b) to convert the Hilbert function which corresponds to the multi-grading $\mathbf{M}$ on P into the Hilbert function that corresponds to the standard grading on P. In particular, if $(\ell,\underline{\sigma})$ is a five-tuple of integers in standard form in the sense of 3.4, then

$$H_P(m_{(\ell,\underline{\sigma})}) = H_P(\ell).$$

It follows that the expression $H_P(m_{(k,\underline{\rho})} - m_{(\beta,\underline{\gamma})})$, which appears in (44), is equal to $H_P(k-\beta-E)$, where E is the cardinality of the set $\{i \mid \rho_i < \gamma_i\}$.

For each fixed triple of integers (β, s, u), with $0 \le \beta$ and $0 \le s, u \le 4$, we count

$$\begin{aligned}
\nu_{\beta,s} &= \sum_{\{\underline{\gamma} \mid (\beta,\underline{\gamma})\in\mathcal{S} \text{ and exactly } s \text{ of the } \gamma_i \text{ satisfy } 0<\gamma_i\}} \#_{(\beta,\underline{\gamma})},\\
\nu'_{\beta,s} &= \sum_{\{\underline{\gamma} \mid (\beta,\underline{\gamma})\in\mathcal{S} \text{ and exactly } s \text{ of the } \gamma_i \text{ satisfy } r\le\gamma_i\}} \#_{(\beta,\underline{\gamma})},\\
\mu_{\beta,u} &= \sum_{\{\underline{\gamma} \mid (\beta,\underline{\gamma})\in\mathcal{S} \text{ and exactly } u \text{ of the } \gamma_i \text{ satisfy } r<\gamma_i\}} \#_{(\beta,\underline{\gamma})}, \text{ and}\\
\mu'_{\beta,u} &= \sum_{\{\underline{\gamma} \mid (\beta,\underline{\gamma})\in\mathcal{S} \text{ and exactly } u \text{ of the } \gamma_i \text{ satisfy } n-1<\gamma_i\}} \#_{(\beta,\underline{\gamma})}.
\end{aligned} \tag{45}$$

Notice that for each fixed β, s_0, and u_0,

$$\sum_{s=s_0}^{4} \nu'_{\beta,s} \le \sum_{s=s_0}^{4} \nu_{\beta,s} \quad \text{and} \quad \sum_{u=u_0}^{4} \mu'_{\beta,u} \le \sum_{u=u_0}^{4} \mu_{\beta,u}. \tag{46}$$

Furthermore, $\mu'_{\beta,u}$ is equal to zero unless $u=0$.

First calculation. We evaluate (42) at $m_{(k,0,0,0,0)}$ and $m_{(k,r-1,r-1,r-1,r-1)}$, for $k \le 2d+2$, to learn that

$$\begin{aligned}
\nu'_{\beta,s} &= \nu_{\beta,s} = 0, && \text{if } \beta+s \le 2d,\\
\sum_{\beta+s=2d+1} \nu'_{\beta,s} &= \sum_{\beta+s=2d+1} \nu_{\beta,s} = 8d+4, &&\\
\sum_{\beta+s=2d+2} \nu'_{\beta,s} &= \sum_{\beta+s=2d+2} \nu_{\beta,s} = 8d+4, && \text{and}\\
\nu'_{\beta,s} &= \nu_{\beta,s} = 0, && \text{if } 2d+3 \le \beta+s.
\end{aligned} \tag{47}$$

Observe that

$$\begin{aligned}
H_{F_1}(m_{(k,r-1,r-1,r-1,r-1)}) = H_{F_1}(m_{(k,0,0,0,0)}) &= \begin{cases} 4H_P(k-d-1)+dH_P(k-2d-2)\\ +6dH_P(k-2d-1)+(d+1)H_P(k-2d), \end{cases}\\
H_{F_3}(m_{(k,r-1,r-1,r-1,r-1)}) = H_{F_3}(m_{(k,0,0,0,0)}) &= \begin{cases} 4H_P(k-3d-2)+dH_P(k-2d-1)\\ +6dH_P(k-2d-2)+(d+1)H_P(k-2d-3), \text{ and} \end{cases}\\
H_{F_4}(m_{(k,r-1,r-1,r-1,r-1)}) = H_{F_4}(m_{(k,0,0,0,0)}) &= H_P(k-4d-3).
\end{aligned}$$

Use Proposition 4.2 with $\varepsilon_i = 1$ for all i and

$$\varepsilon = \begin{cases} 0, & \text{if } k \le 2d-1,\\ 1, & \text{if } 2d \le k, \end{cases}$$

to calculate

$$\begin{aligned}
&H_R(m_{(k,r-1,r-1,r-1,r-1)}) = H_R(m_{(k,0,0,0,0)})\\
&= \begin{cases} H_P(k)-4H_P(k-d-1)+6H_P(k-2d-2)-4H_P(k-3d-3)+H_P(k-4d-4), & \text{if } k\le 2d-1, \text{ and}\\ H_P(k+1)-4H_P(k-d)+6H_P(k-2d-1)-4H_P(k-3d-2)+H_P(k-4d-3), & \text{if } 2d\le k. \end{cases}
\end{aligned}$$

For $k \le 2d-1$, the left side of equation (42) is

$$\sum_{\beta}\sum_{s=0}^{4} \nu_{\beta,s} H_P(k-\beta-s), \qquad \text{when evaluated at } m_{(k,0,0,0,0)}, \text{ and} \tag{48}$$

$$\sum_{\beta}\sum_{s=0}^{4} \nu'_{\beta,s} H_P(k-\beta-s), \qquad \text{when evaluated at } m_{(k,r-1,r-1,r-1,r-1)};$$

and the right side is

$$(d+1)H_P(k-2d-3)+(7d+6)H_P(k-2d-2)+7dH_P(k-2d-1)+(d+1)H_P(k-2d)=0,$$

when evaluated at either multi-index. (Keep in mind that $H_P(j)=0$ for $j<0$.) It follows that

$$\begin{aligned} 0 &= H_P(0)\Big(\sum_{\beta+s=k}\nu_{\beta,s}\Big)+H_P(1)\Big(\sum_{\beta+s=k-1}\nu_{\beta,s}\Big)+\cdots \\ &= H_P(0)\Big(\sum_{\beta+s=k}\nu'_{\beta,s}\Big)+H_P(1)\Big(\sum_{\beta+s=k-1}\nu'_{\beta,s}\Big)+\cdots \end{aligned}$$

and by induction one concludes that $\nu_{\beta,s}=0=\nu'_{\beta,s}$ whenever $\beta+s\le 2d-1$. For $k=2d$, the left side of equation (42) is

$$H_P(0)\Big(\sum_{\beta+s=2d}\nu_{\beta,s}\Big), \qquad \text{when evaluated at } m_{(k,0,0,0,0)}, \text{ and}$$

$$H_P(0)\Big(\sum_{\beta+s=2d}\nu'_{\beta,s}\Big), \qquad \text{when evaluated at } m_{(k,r-1,r-1,r-1,r-1)};$$

and the right hand side is

$$H_P(2d+1)-H_P(2d)-4H_P(d)+4H_P(d-1)+(d+1)=\binom{2d+3}{2}-4\binom{d+2}{2}+d+1=0,$$

when evaluated at either multi-index. Therefore $\nu'_{\beta,s}=\nu_{\beta,s}=0$ whenever $\beta+s=2d$. (Recall that the Hilbert function that corresponds to the standard grading on P is well known; see 3.8.(b). We also used the Pascal triangle identity for binomial coefficients (27).) For $k=2d+1$, the left side of equation (42) is

$$H_P(0)\Big(\sum_{\beta+s=2d+1}\nu_{\beta,s}\Big)+H_P(1)\Big(\sum_{\beta+s=2d}\nu_{\beta,s}\Big), \qquad \text{when evaluated at } m_{(k,0,0,0,0)}, \text{ and}$$

$$H_P(0)\Big(\sum_{\beta+s=2d+1}\nu'_{\beta,s}\Big)+H_P(1)\Big(\sum_{\beta+s=2d}\nu'_{\beta,s}\Big), \qquad \text{when evaluated at } m_{(k,r-1,r-1,r-1,r-1)};$$

and the right hand side is

$$H_P(2d+2)-H_P(2d+1)-4H_P(d+1)+4H_P(d)+(7d+6)+4(d+1)=8d+4,$$

when evaluated at either multi-index. It follows that

$$\sum_{\beta+s=2d+1}\nu_{\beta,s}=8d+4=\sum_{\beta+s=2d+1}\nu'_{\beta,s}.$$

For $k=2d+2$, the left hand side of equation (42)

$$\Big(\sum_{\beta+s=2d+1}\nu_{\beta,s}\Big)H_P(1)+\Big(\sum_{\beta+s=2d+2}\nu_{\beta,s}\Big)H_P(0)=4(8d+4)+\sum_{\beta+s=2d+2}\nu_{\beta,s},$$

when evaluated at $m_{(k,0,0,0,0)}$, and

$$\Big(\sum_{\beta+s=2d+1}\nu'_{\beta,s}\Big)H_P(1)+\Big(\sum_{\beta+s=2d+2}\nu'_{\beta,s}\Big)H_P(0)=4(8d+4)+\sum_{\beta+s=2d+2}\nu'_{\beta,s},$$

when evaluated at $m_{(k,r-1,r-1,r-1,r-1)}$;

and the right hand side is

$$\begin{aligned}&H_P(2d+3)-H_P(2d+2))-4(H_P(d+2)-H_P(d+1))+7dH_P(0)+(7d+6)H_P(1)+(d+1)H_P(2)\\&=40d+20,\end{aligned}$$

when evaluated at either multi-index. Thus,

$$\sum_{\beta+s=2d+2}\nu_{\beta,s}=8d+4=\sum_{\beta+s=2d+2}\nu'_{\beta,s}.$$

Recall from (41) that the rank of F_2 is $16d+8$; consequently, $\nu'_{\beta,s}=\nu_{\beta,s}=0$ whenever $2d+3\le\beta+s$. This completes the first calculation.

Observe that the first calculation has the following consequences.

6.2.1. If $(\beta,\underline{\gamma})$ is in $\mathcal{S}$, then β plus the number of non-zero γ_i is either $2d+1$ or $2d+2$.

6.2.2. If $(\beta,\underline{\gamma})$ is in $\mathcal{S}$, $2d-3\le\beta\le 2d+2$.

6.2.3. Every positive γ_i that occurs in a direct summand of F_2 must be at least r.

Assertion 6.2.1 is obvious now that (47) has been established; 6.2.2 is a consequence of 6.2.1 because there are four γ_i. The fact that

$$\sum_{\beta+s=k}\nu'_{\beta,s}=\sum_{\beta+s=k}\nu_{\beta,s},$$

for each $k\in\{2d+1,2d+2\}$, together with (46), ensures that $\nu'_{\beta,s}=\nu_{\beta,s}$ for all β and s. Thus, every positive γ_i that occurs in a direct summand of F_2 must be at least r, and this is 6.2.3.

The parameters $\mu_{\beta,u}$ and $\mu'_{\beta,u}$ are defined in (45). It follows from (6.2.2) that

$$\mu_{\beta,u}\text{ and }\mu'_{\beta,u}\text{ are both zero unless }2d-3\le\beta+u\le 2d+2. \tag{49}$$

Second calculation. We evaluate (42) at $m_{(k,r,r,r,r)}$ and $m_{(k,n-1,n-1,n-1,n-1)}$, for $2d-3\le k\le 2d$, to learn that

$$\begin{aligned}&\mu'_{\beta,u}=\mu_{\beta,u}=0, &&\text{if }2d-3\le\beta+u\le 2d-2,\\ &\sum_{\beta+u=2d-1}\mu'_{\beta,u}=\sum_{\beta+u=2d-1}\mu_{\beta,u}=8d+4, &&\\ &\sum_{\beta+u=2d}\mu'_{\beta,u}=\sum_{\beta+u=2d}\mu_{\beta,u}=8d+4, &&\text{and}\\ &\mu'_{\beta,u}=\mu_{\beta,u}=0, &&\text{if }2d+1\le\beta+u\le 2d+2.\end{aligned} \tag{50}$$

Use 3.8.(b) to compute that

$$\begin{aligned}H_{F_1}(m_{(k,r,r,r,r)})=H_{F_1}(m_{(k,n-1,n-1,n-1,n-1)})&=\begin{cases}4H_P(k-d)+dH_P(k-2d+2)\\+6dH_P(k-2d+1)+(d+1)H_P(k-2d),\end{cases}\\ H_{F_3}(m_{(k,r,r,r,r)})=H_{F_3}(m_{(k,n-1,n-1,n-1,n-1)})&=\begin{cases}dH_P(k-2d-1)+6dH_P(k-2d)\\+(d+1)H_P(k-2d+1)+4H_P(k-3d+1),\text{ and}\end{cases}\\ H_{F_4}(m_{(k,r,r,r,r)})=H_{F_4}(m_{(k,n-1,n-1,n-1,n-1)})&=H_P(k-4d+1).\end{aligned}$$

Apply (44) and (49) to see that

$$\begin{aligned}
H_{F_2}(m_{(k,r,r,r,r)}) &= \sum_{\{(\beta,\underline{\gamma})\in\mathcal{S}\}} \#_{(\beta,\underline{\gamma})} H_P(m_{(k,r,r,r,r)} - m_{(\beta,\underline{\gamma})}) \\
&= \sum_{u=0}^{4} \sum_{\{(\beta,\underline{\gamma})\in\mathcal{S} \mid\ |\{i|r<\gamma_i\}|=u\}} \#_{(\beta,\underline{\gamma})} H_P(k-\beta-u) \\
&= \sum_{u=0}^{4} \sum_{\beta} \mu_{(\beta,u)} H_P(k-\beta-u) \\
&= \sum_{U=2d-3}^{2d+2} \Big(\sum_{\beta+u=U} \mu_{(\beta,u)} \Big) H_P(k-U).
\end{aligned}$$

A similar calculation yields

$$H_{F_2}(m_{(k,n-1,n-1,n-1,n-1)}) = \sum_{U=2d-3}^{2d+2} \Big(\sum_{\beta+u=U} \mu'_{(\beta,u)} \Big) H_P(k-U).$$

Use Proposition 4.2 with $\varepsilon_i = 0$ for all i and

$$\varepsilon = \begin{cases} 0, & \text{if } k \le 2d-3, \\ 1, & \text{if } 2d-2 \le k, \end{cases}$$

to calculate

$$H_R(m_{(k,r,r,r,r)}) = H_R(m_{(k,n-1,n-1,n-1,n-1)}) = \begin{cases} H_P(k+\varepsilon) - 4H_P(k-d+\varepsilon) + 6H_P(k-2d+\varepsilon) \\ -4H_P(k-3d+\varepsilon) + H_P(k-4d+\varepsilon). \end{cases}$$

Equation (42) now yields

$$\begin{aligned}
&\sum_{U=2d-3}^{2d+2} \big(\sum_{\beta+u=U} \mu_{\beta,u} \big) H_P(k-U) \\
&= H_{F_2}(m_{(k,r,r,r,r)}) \\
&= H_R(m_{(k,r,r,r,r)}) - H_P(m_{(k,r,r,r,r)}) + H_{F_1}(m_{(k,r,r,r,r)}) + H_{F_3}(m_{(k,r,r,r,r)}) - H_{F_4}(m_{(k,r,r,r,r)}) \\
&= \begin{cases} \ \ H_P(k+\varepsilon) - H_P(k) - 4H_P(k-d+\varepsilon) + 4H_P(k-d) + dH_P(k-2d+2) \\ +6H_P(k-2d+\varepsilon) + (7d+1)H_P(k-2d+1) + (7d+1)H_P(k-2d) \\ +dH_P(k-2d-1) - 4H_P(k-3d+\varepsilon) + 4H_P(k-3d+1) + H_P(k-4d+\varepsilon) - H_P(k-4d+1). \end{cases}
\end{aligned} \tag{51}$$

In a similar manner,

$$\begin{aligned}
&\sum_{U=2d-3}^{2d+2} \big(\sum_{\beta+u=U} \mu'_{\beta,u} \big) H_P(k-U) \\
&= H_{F_2}(m_{(k,n-1,n-1,n-1,n-1)}) \\
&= \begin{cases} \ \ H_R(m_{(k,n-1,n-1,n-1,n-1)}) - H_P(m_{(k,n-1,n-1,n-1,n-1)}) + H_{F_1}(m_{(k,n-1,n-1,n-1,n-1)}) \\ +H_{F_3}(m_{(k,n-1,n-1,n-1,n-1)}) - H_{F_4}(m_{(k,n-1,n-1,n-1,n-1)}) \end{cases} \\
&= H_R(m_{(k,r,r,r,r)}) - H_P(m_{(k,r,r,r,r)}) + H_{F_1}(m_{(k,r,r,r,r)}) + H_{F_3}(m_{(k,r,r,r,r)}) - H_{F_4}(m_{(k,r,r,r,r)}),
\end{aligned} \tag{52}$$

which is given in (51). When $k = 2d-3$, then (51) and (52) become

$$\sum_{\beta+u=2d-3} \mu'_{\beta,u} = \sum_{\beta+u=2d-3} \mu_{\beta,u} = 0;$$

hence, $\mu'_{\beta,u} = \mu_{\beta,u} = 0$ for $\beta + u = 2d-3$. When $k = 2d-2$, then (51) and (52) become

$$\sum_{\beta+u=2d-2} \mu'_{\beta,u} = \sum_{\beta+u=2d-2} \mu_{\beta,u} = H_P(2d-1) - H_P(2d-2) - 4H_P(d-1) + 4H_P(d-2) + dH_P(0) = 0.$$

It follows that if $\beta + u = 2d-2$, then $\mu'_{\beta,u} = \mu_{\beta,u} = 0$. When $k = 2d-1$, then (51) and (52) become

$$\begin{aligned}\sum_{\beta+u=2d-1} \mu'_{\beta,u} = \sum_{\beta+u=2d-1} \mu_{\beta,u} &= H_P(2d) - H_P(2d-1) - 4H_P(d) + 4H_P(d-1) + dH_P(1) + (7d+7)H_P(0) \\ &= 8d+4.\end{aligned}$$

When $k = 2d$, then (51) and (52) become

$$\begin{aligned}(8d+4)H_P(1) + \sum_{\beta+u=2d} \mu'_{\beta,u} &= (8d+4)H_P(1) + \sum_{\beta+u=2d} \mu_{\beta,u} \\ &= \begin{cases} H_P(2d+1) - H_P(2d) - 4H_P(d+1) \\ -4H_P(d) + dH_P(2) + (7d+7)H_P(1) + (7d+1)H_P(0) \end{cases} \\ &= 40d+20.\end{aligned}$$

It follows that

$$\sum_{\beta+u=2d} \mu_{\beta,u} = \sum_{\beta+u=2d} \mu'_{\beta,u} = 8d+4.$$

Recall from (41) that the rank of F_2 is $16d+8$; consequently, $\mu'_{\beta,u} = \mu_{\beta,u} = 0$ whenever $2d+1 \le \beta + u$. This completes the second calculation.

Observe that the second calculation has the following consequences.

6.2.4. The parameters $\mu_{\beta,u}$ and $\mu'_{\beta,u}$ are equal for all β and u.

6.2.5. The values of γ_i that occur in the direct summands of F_2 are either 0 or r.

6.2.6. There are $8d+4$ direct summands of F_2 with $\beta = 2d-1$, and $8d+4$ direct summands with $\beta = 2d$.

Assertion 6.2.4 is an immediate consequence of (50) and (46). The parameter γ_i is required to satisfy $0 \le \gamma_i \le n-1$; thus, $\mu'_{\beta,u}$ is zero whenever u is positive. Apply 6.2.4 to conclude that $\mu_{\beta,u}$ is zero whenever u is positive and all γ_i that occur in the direct summands of F_2 are less than or equal to r. On the other hand, 6.2.3 ensures that all non-zero γ_i are at least r. Assertion 6.2.5 has been established. We have seen that $\mu_{\beta,u}$ is zero unless $u = 0$ and that $\mu_{\beta_0,0}$ is equal to the number of summands of F_2 with $\beta = \beta_0$. (The definition of μ is given in (45).) Assertion 6.2.6 now follows from (50).

Third calculation. We evaluate $H_{F_2}(m_{(k,r,r,0,0)})$ for $2d-1 \le k$. For $w \in \{0,1,2\}$ and β in $\{2d-1, 2d\}$, let $\psi_{\beta,w}$ denote the number of summands in F_2 that have the fixed value of β, and satisfy

$$|\{i \mid 3 \le i \le 4 \text{ and } \gamma_i = r\}| = w$$

The parameter $\beta + w$ satisfies $2d-1 \le \beta + w \le 2d+2$. Observe that

$$\begin{aligned} H_{F_1}(m_{(k,r,r,0,0)}) &= 2H_P(k-d) + 2H_P(k-d-1) + dH_P(k-2d+1) + (6d+1)H_P(k-2d) + dH_P(k-2d-1) \\ H_{F_3}(m_{(k,r,r,0,0)}) &= \begin{cases} dH_P(k-2d) + (6d+1)H_P(k-2d-1) + dH_P(k-2d-2) \\ +2H_P(k-3d-1) + 2H_P(k-3d), \end{cases} \\ H_{F_4}(m_{(k,r,r,0,0)}) &= H_P(k-4d-1).\end{aligned}$$

Use Proposition 4.2 to calculate $H_R(m_{(k,r,r,0,0)})$, for $2d-1 \le k$. Observe that $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon_4 = 1$, $\varepsilon = 1$, and

$$H_R(m_{(k,r,r,0,0)}) = \begin{cases} H_P(k+1) - 2H_P(k-d) - 2H_P(k-d+1) + H_P(k-2d-1) + 4H_P(k-2d) \\ +H_P(k-2d+1) - 2H_P(k-3d) - 2H_P(k-3d-1) + H_P(k-4d-1). \end{cases}$$

The right hand side of equation (42), evaluated at $m_{(k,r,r,0,0)}$, with $2d-1\le k$, is

$$\begin{cases} H_P(k+1)-H_P(k)-2H_P(k-d+1)+2H_P(k-d-1)+(d+1)H_P(k-2d+1)\\ +(7d+5)H_P(k-2d)+(7d+2)H_P(k-2d-1)+dH_P(k-2d-2).\end{cases}$$

The left hand side of equation (42) evaluated at $m_{(k,r,r,0,0)}$ is

$$\sum_{W=2d-1}^{2d+2}\Big(\sum_{\beta+w=W}\psi_{\beta,w}\Big)H_P(k-W)$$

When $k=2d-1$, equation (42), evaluated at $m_{(k,r,r,0,0)}$, becomes

$$\sum_{\beta+w=2d-1}\psi_{\beta,w}=H_P(2d)-H_P(2d-1)-2H_P(d)+2H_P(d-2)+d+1=0$$

Thus, there are no summands of F_2 that have $\beta=2d-1$ and $\gamma_3=\gamma_4=0$. The same calculation applies to any choice of two out of $\gamma_1,\dots,\gamma_4$. Therefore, for every summand of F_2 with $\beta=2d-1$, and every choice of $\{i,j\}\subseteq\{1,2,3,4\}$, we must have $\gamma_i=r$ or $\gamma_j=r$. It follows that every such summand of F_2 has at least three of $\gamma_1,\dots,\gamma_4$ equal to r.

Recall from 6.2.1 that if $P(-m_{(\beta,\underline{\gamma})})$ is a summand of F_2 then β plus the number of non-zero γ_i is either $2d+1$ or $2d+2$. Therefore, it is not possible for $\gamma_1,\dots,\gamma_4$ all to be non-zero. Due to symmetry and the fact that there are $8d+4$ summands with $\beta=2d-1$, it follows that there are $2d+1$ summands of F_2 with $\beta=2d-1$ and $\gamma_i=\gamma_j=\gamma_\ell=r$ for each choice of a three element subset $\{i,j,\ell\}$ of $\{1,2,3,4\}$ (and the remaining γ equal to zero).

Furthermore, 6.2.1 also yields that the $8d+4$ summands of F_2 that have $\beta=2d$ must satisfy β plus the number of non-zero γ_i is equal to $2d+1$. In other words, for each such summand, there is exactly one of $\gamma_1,\dots,\gamma_4$ equal to r. By symmetry, it follows that there are $2d+1$ summands with $\beta=2d$, $\gamma_i=r$ and $\gamma_j=0$ for all $j\in\{1,2,3,4\}\setminus\{i\}$, for each choice of $i=1,2,3,4$.

This concludes the proof of the theorem. □

7. Consequences of the main theorem.

Adopt the language of 2.1 and 2.2 with $\mathbf{k}$ a field of characteristic zero. Corollary 7.1 gives the multi-graded Betti numbers in the minimal homogeneous resolution of $A=\overline{P}/(x^N,y^N,z^N,w^N)\overline{P}$ by free $\overline{P}$-modules, for

$$\overline{P}=P/(x^n+y^n+z^n+w^n).$$

Corollary 7.2 gives the resolution of A by free P-module, as well as the socle degrees of A.

The notation $m_{(k,\underline{\rho})}$ is described in 3.4 and the notation involving (e_1,e_2,e_3,e_4) is described in 6.1.

Corollary 7.1. *Let $\mathbf{k}$ be a field of characteristic zero, n, d, and r be positive integers, with $r<n$, N be the integer $N=dn+r$, P be the standard graded polynomial ring $P=\mathbf{k}[x,y,z,w]$, and $\overline{P}$ be the quotient ring $P/(x^n+y^n+z^n+w^n)$. Give P the multi-grading of Definition 2.2. Then the minimal multi-homogeneous resolution of $\overline{P}/(x^N,y^N,z^N,w^N)\overline{P}$ by free $\overline{P}$-modules has the form*

$$G:\quad \cdots\to G_4\to G_3\to G_2\to G_1\to G_0,$$

with $G_0=\overline{P}$, $G_1=\bigoplus_{\sum e_i=1}\overline{P}(-m_{(d,re_1,re_2,re_3,re_4)})$,

$$G_2=\begin{cases}\bigoplus_{\sum e_i=4}\overline{P}(-m_{(2d-1,re_1,re_2,re_3,re_4)})^{d}\\ \oplus\bigoplus_{\sum e_i=2}\overline{P}(-m_{(2d,re_1,re_2,re_3,re_4)})^{d+1}\\ \oplus\bigoplus_{\sum e_i=0}\overline{P}(-m_{(2d+1,re_1,re_2,re_3,re_4)})^{d+1},\end{cases}$$

$$G_i=\begin{cases}\bigoplus_{\sum e_i=3}\overline{P}(-m_{(2d+\frac{i-3}{2},re_1,re_2,re_3,re_4)})^{2d+1}\\ \oplus\bigoplus_{\sum e_i=1}\overline{P}(-m_{(2d+\frac{i-1}{2},re_1,re_2,re_3,re_4)})^{2d+1},\end{cases}\qquad \textit{for } i \textit{ odd with } 3\le i, \textit{ and}$$

$$G_i = \begin{cases} \bigoplus_{\sum e_i=4} \overline{P}(-m_{(2d+\frac{i-4}{2},re_1,re_2,re_3,re_4)})^{2d+1} \\ \oplus \bigoplus_{\sum e_i=2} \overline{P}(-m_{(2d+\frac{i-2}{2},re_1,re_2,re_3,re_4)})^{2d+1} \\ \oplus \bigoplus_{\sum e_i=0} \overline{P}(-m_{(2d+\frac{i}{2},re_1,re_2,re_3,re_4)})^{2d+1}, \end{cases} \qquad \textit{for } i \textit{ even with } 4 \le i.$$

Remark. If one ignores the **M**-grading on P and merely views P as a standard graded polynomial ring, then the minimal homogeneous resolution G of Corollary 7.1 is $G_0 = \overline{P}$, $G_1 = \overline{P}(-(nd+r))^4$,

$$G_2 = \overline{P}(-(2nd-n+4r))^d \oplus \overline{P}(-(2nd+2r))^{6d+6} \oplus \overline{P}(-(2nd+n))^{d+1},$$
$$G_i = \overline{P}(-(2nd+\tfrac{(i-3)n}{2}+3r))^{8d+4} \oplus \overline{P}(-(2nd+\tfrac{(i-1)n}{2}+r))^{8d+4}, \quad \text{for } i \text{ odd with } 3 \le i, \text{ and}$$
$$G_i = \overline{P}(-(2nd+\tfrac{(i-4)n}{2}+4r))^{2d+1} \oplus \overline{P}(-(2nd+\tfrac{(i-2)n}{2}+2r))^{12d+6} \oplus \overline{P}(-(2nd+\tfrac{in}{2}))^{2d+1},$$
for i even with $4 \le i$.

Proof. We apply [10, Thm. 9.1, part 2]. Let F be the minimal multi-homogeneous resolution of

$$R = P/(x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$$

by free P-modules which is given in Theorem 6.2, $\bar{\ }$ be the functor $\overline{P} \otimes_P -$, and $\underline{0}$ represent the four-tuple of integers $(0,0,0,0)$. Let K_1 and F_{12} be the summands

$$K_1 = \bigoplus_{\sum e_i = 1} P(-m_{(d,re_1,re_2,re_3,re_4)}) \quad \text{and} \quad F_{12} = \begin{cases} \bigoplus_{\sum e_i=4} P(-m_{(2d-2,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=0} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{d+1} \end{cases}$$

of F_1, F_{32} be the summand

$$F_{32} = \begin{cases} \bigoplus_{\sum e_i=0} P(-m_{(2d+1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=4} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^{d+1} \end{cases}$$

of F_3, and K_i be $\bigwedge^i K_1$. Notice that the differential of F sends K_1 onto the complete intersection ideal (x^N, y^N, z^N, w^N) and that degree considerations show that F_{32} is the annihilator of K_1 in the perfect pairing $F_1 \otimes F_3 \to F_4$ which is induced by the multi-homogeneous Differential Graded (DG) Algebra structure on F. Let $\alpha_1 : K_1 \to F_1$ be the inclusion map, $\bigwedge^\bullet K_1$ be the Koszul complex, $\alpha_\bullet : \bigwedge^\bullet K_1 \to F$ be a multi-homogeneous map of DG complexes which extends the identity map in degree zero and α_1 in degree one. The multi-homogeneous maps $\beta_i : F_i \to K_i$ are defined in [10] in such a way that the multi-degree of $\beta_i \circ \alpha_i$ is equal to the multi-degree of multiplication by $x^n+y^n+z^n+w^n$, which is $m_{(1,\underline{0})}$. In particular, $K_4 = P(-m_{(4d,r,r,r,r)})$, $F_4 = P(-(m_{(4d-1,r,r,r,r)}))$, the multi-homogeneous version of β_4 is

$$\beta_4 : F_4 \to K_4(m(1,\underline{0})),$$

and the multi-homogeneous version of [10, (3.2.1)] is

$$\begin{tikzcd} 0 \arrow[r] & K_4 \arrow[r] \arrow[d, "\alpha_4"] & K_3 \arrow[r] \arrow[d, "\alpha_3"] & K_2 \arrow[r] \arrow[d, "\alpha_2"] & K_1 \arrow[r] \arrow[d, "\alpha_1"] & P \arrow[d, "\alpha_0", "="'] \\ 0 \arrow[r] & F_4 \arrow[r] \arrow[d, "\beta_4"] & F_3 \arrow[r] \arrow[d, "\beta_3"] & F_2 \arrow[r] \arrow[d, "\beta_2"] & F_1 \arrow[r] \arrow[d, "\beta_1"] & P \arrow[d, "\beta_0"] \\ 0 \arrow[r] & K_4(m_{(1,\underline{0})}) \arrow[r] & K_3(m_{(1,\underline{0})}) \arrow[r] & K_2(m_{(1,\underline{0})}) \arrow[r] & K_1(m_{(1,\underline{0})}) \arrow[r] & P(m_{(1,\underline{0})}). \end{tikzcd}$$

Apply [10, Thm. 9.1, part 2] to obtain the multi-homogeneous resolution

$$G: \quad \cdots \to G_2 \to G_1 \to G_0 \tag{53}$$

of $\overline{P}/(x^N,y^N,z^N,w^N)\overline{P}$ by free $\overline{P}$-modules, where the modules of G are

$$G_i = \begin{cases} \overline{P}, & \text{if } i=0,\\ \overline{K_1}, & \text{if } i=1,\\ \overline{F_{1,2}}(-m_{(1,\underline{0})}) \oplus \overline{K_2}, & \text{if } i=2,\\ \overline{F_2}(-m_{(\frac{i-1}{2},\underline{0})}), & \text{if } 3\le i \text{ and } i \text{ is odd, and}\\ \overline{F_{12}}(-m_{(\frac{i}{2},\underline{0})}) \oplus \overline{K_2}(-m_{(\frac{i}{2}-1,\underline{0})}) \oplus \overline{F_{32}}(-m_{(\frac{i}{2}-1,\underline{0})}), & \text{if } 4\le i \text{ and } i \text{ is even.} \end{cases}$$

No shifts are given in [10]; however, the differentials are explicitly given and the shifts can be calculated from the differentials.

Theorem 9.1 in [10] does not guarantee that the resulting resolution is minimal; however, now that we have determined the multi-homogeneous shifts in G, it is quite obvious that G is a minimal resolution. Indeed, each shift which appears in each G_j has the form $m_{(k,re_1,re_2,re_3,re_4)}$ where the parity of $\sum e_i$ is equal to the parity of j. □

The notation $m_{(k,\underline{\rho})}$ is described in 3.4 and the notation involving (e_1,e_2,e_3,e_4) is described in 6.1.

Corollary 7.2. *Let* $\mathbf{k}$ *be a field of characteristic zero, n, d, and r be positive integers, with $r<n$, N be the integer $N=dn+r$, P be polynomial ring $P=\mathbf{k}[x,y,z,w]$, and A be the almost complete intersection ring*

$$\frac{P}{(x^N,y^N,z^N,w^N,x^n+y^n+z^n+w^n)}.$$

Give P and A the multi-grading of Definition 2.2*. Then the minimal multi-homogeneous resolution of A by free P-modules has the form*

$$0\to L_4\to L_3\to L_2\to L_1\to L_0,$$

where $L_0=P$,

$$L_1 = \begin{cases} P(-m_{(1,0,0,0,0)}) \\ \oplus \bigoplus_{\sum e_i=1} P(-m_{(d,re_1,re_2,re_3,re_4)}), \end{cases} \qquad L_2 = \begin{cases} \bigoplus_{\sum e_i=1} P(-m_{(d+1,re_1,re_2,re_3,re_4)}) \\ \oplus \bigoplus_{\sum e_i=4} P(-m_{(2d-1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{d+1} \\ \oplus \bigoplus_{\sum e_i=0} P(-m_{(2d+1,re_1,re_2,re_3,re_4)})^{d+1}, \end{cases}$$

$$L_3 = \begin{cases} \bigoplus_{\sum e_i=1} P(-m_{(2d+1,re_1,re_2,re_3,re_4)})^{2d+1} \\ \oplus \bigoplus_{\sum e_i=3} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{2d+1}, \end{cases} \qquad L_4 = \begin{cases} \bigoplus_{\sum e_i=0} P(-m_{(2d+2,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=2} P(-m_{(2d+1,re_1,re_2,re_3,re_4)})^d \\ \oplus \bigoplus_{\sum e_i=4} P(-m_{(2d,re_1,re_2,re_3,re_4)})^{d+1}, \end{cases}$$

and the socle of A is isomorphic as a multi-graded P-module to

$$\begin{aligned} &\mathbf{k}(-m_{(2d-2,n-1,n-1,n-1,n-1)})^d \\ \oplus&\mathbf{k}(-m_{(2d-1,n-1,n-1,r-1,r-1)})^d \oplus \mathbf{k}(-m_{(2d-1,n-1,r-1,n-1,r-1)})^d \oplus \mathbf{k}(-m_{(2d-1,n-1,r-1,r-1,n-1)})^d \\ \oplus&\mathbf{k}(-m_{(2d-1,r-1,n-1,n-1,r-1)})^d \oplus \mathbf{k}(-m_{(2d-1,r-1,n-1,r-1,n-1)})^d \oplus \mathbf{k}(-m_{(2d-1,r-1,r-1,n-1,n-1)})^d \\ \oplus&\mathbf{k}(-m_{(2d,r-1,r-1,r-1,r-1)})^{d+1}. \end{aligned}$$

Remark. If one ignores the $\mathbf{M}$-grading on P and merely views P as a standard graded polynomial ring, then the minimal homogeneous resolution L of Corollary 7.2 is $L_0=P$,

$$\begin{aligned} L_1 &= P(-n)\oplus P(-(nd+r))^4,\\ L_2 &= P(-(nd+n+r))^4 \oplus P(-(2nd-n+4r))^d \oplus P(-(2nd+2r))^{6d+6} \oplus P(-(2nd+n))^{d+1},\\ L_3 &= P(-(2nd+n+r))^{8d+4} \oplus P(-(2nd+3r))^{8d+4},\\ L_4 &= P(-(2nd+2n))^d \oplus P(-(2nd+n+2r))^{6d} \oplus P(-(2nd+4r))^{d+1}, \end{aligned}$$

and the socle of A is isomorphic to

$$\mathbf{k}(-(2dn+2n-4))^d \oplus \mathbf{k}(-(2dn+n+2r-4))^{6d} \oplus \mathbf{k}(-(2dn+4r-4))^{d+1}.$$

Proof. The ideals $(x^N, y^N, z^N, w^N, x^n + y^n + z^n + w^n)$ and $(x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ of P are linked by the complete intersection (x^N, y^N, z^N, w^N). Use the resolution of Theorem 6.2 and the technique of linkage (see 3.14) to resolve A and use 3.9.(e) to read the socle degrees of A from the resolution of A. □

We offer two examples of Corollaries 7.1 and 7.2.

Example 7.3. Let $\mathbf{k}$ be a field of characteristic zero and d and r be positive integers. Let $n = 2r$, N be the integer $N = dn + r$, P be polynomial ring $P = \mathbf{k}[x, y, z, w]$, and A be the almost complete intersection ring

$$\frac{P}{(x^N, y^N, z^N, w^N, x^n + y^n + z^n + w^n)}.$$

Then A is a level algebra with socle isomorphic to

$$\mathbf{k}(-(4rd + 4r - 4))^{8d+1}$$

and the minimal homogeneous resolution of A by free $\overline{P}$-modules is pure and has the form

$$G: \quad \cdots \to G_4 \to G_3 \to G_2 \to G_1 \to G_0,$$

with

$$G_0 = \overline{P}, \quad G_1 = \overline{P}(-(2rd + r))^4, \quad G_2 = \overline{P}(-(4rd + 2r))^{8d+7}, \quad \text{and} \quad G_i = \overline{P}(-(4rd + ir))^{16d+8},$$

for $3 \le i$. The matrix factorization of $x^n + y^n + z^n + w^n$ which comprises the infinite tail of G consists of two matrices of homogeneous forms of degree r.

Example 7.4. Let $\mathbf{k}$ be a field of characteristic zero and n, d, r, and c be positive integers, with $r < n$. Let

$$N = dn + r, \quad n' = cn, \quad r' = cr, \quad \text{and} \quad N' = cN.$$

Let P be the standard graded polynomial ring $P = \mathbf{k}[x, y, z, w]$, $\overline{P}$ and $\overline{P}'$ be the hypersurface rings

$$\overline{P} = P/(x^n + y^n + z^n + w^n) \quad \text{and} \quad \overline{P}' = P/(x^{n'} + y^{n'} + z^{n'} + w^{n'}),$$

and A and A' be the almost complete intersection rings

$$A = \frac{\overline{P}}{(x^N, y^N, z^N, w^N)\overline{P}} \quad \text{and} \quad A' = \frac{\overline{P}'}{(x^{N'}, y^{N'}, z^{N'}, w^{N'})\overline{P}}.$$

The ring homomorphism $P \to P$, which sends g to $g^{[c]}$, is a flat homomorphism which induces a flat ring homomorphism $\overline{P} \to \overline{P}'$. Indeed, P is a free module over the subring $\mathbf{k}[x^c, y^c, z^c, w^c]$ and $\overline{P}'$ is a free module over the subring

$$\frac{\mathbf{k}[x^c, y^c, z^c, w^c]}{(x^{n'} + y^{n'} + z^{n'} + w^{n'})}.$$

Furthermore, $A \otimes_{\overline{P}} \overline{P}' \cong A'$. It follows that if the socle of A is isomorphic to $\bigoplus_i \mathbf{k}(-\sigma_i)$, then the socle of A' is isomorphic to

$$\bigoplus_i \mathbf{k}(-(c\sigma_i + 4c - 4)); \tag{54}$$

and if G is the minimal homogeneous resolution of A by free $\overline{P}$-modules, then $G \otimes_{\overline{P}} \overline{P}'$ is the minimal homogeneous resolution of A' by free $\overline{P}'$-modules. In particular, if the minimal homogeneous resolution of A by free $\overline{P}$-modules has the form

$$\cdots \to \bigoplus_j \overline{P}(-\beta_{ij}) \to \dots,$$

then the minimal homogeneous resolution of A' by free $\overline{P}'$-modules has the form

$$\cdots \to \bigoplus_j \overline{P}'(-c\beta_{ij}) \to \dots. \tag{55}$$

It is easy to see that the formulas of Corollaries 7.1 and 7.2 satisfy (54) and (55). The phenomenon of this example is illustrated in Example 7.3; it also appears in Section 8.

8. The case when N is a multiple of n.

In this section we study the ideal (x^N,y^N,z^N,w^N) in the standard graded polynomial ring $\mathbf{k}[x,y,z,w]$ when $N=dn$ for some integer d with $2\le d$. This situation is fundamentally different than the situation of Sections 6 and 7 where N is not a multiple of n. In the present situation, the ideal $(x^N,y^N,z^N,w^N):(x^n+y^n+z^n+w^n)$ has $d+4$ generators rather than the $8d+5$ generators of Theorem 6.2, for $d=\lfloor\frac{N}{n}\rfloor$; the socle dimension of $\overline{P}/(x^N,y^N,z^N,w^N)\overline{P}$ is d in the present section, but $8d+1$ in Corollary 7.2; and the projective dimension of the $\overline{P}$-module $\overline{P}/(x^N,y^N,z^N,w^N)\overline{P}$ is finite in the present section, but infinite in Corollary 7.1.

Proposition 8.1. *Let $\mathbf{k}$ be a field of characteristic zero, n and d be integers, with $1\le n$ and $2\le d$, $N=dn$, P be the standard graded polynomial ring $P=\mathbf{k}[x,y,z,w]$, R, A, and $\overline{P}$ be the quotient rings:*

$$\begin{aligned} R&=P/(x^N,y^N,z^N,w^N):(x^n+y^n+z^n+w^n),\\ A&=P/(x^N,y^N,z^N,w^N,x^n+y^n+z^n+w^n),\mathit{and}\\ \overline{P}&=P/(x^n+y^n+z^n+w^n).\end{aligned}$$

Then the following statements hold.

(a) *The minimal homogeneous resolution of R by free P-modules has the form*

$$0\to P(-(4dn-n))\to \begin{matrix}P(-(3dn-n))^4\\ \oplus\\ P(-(2dn+n))^d\end{matrix}\to \begin{matrix}P(-(2dn-n))^{d+3}\\ \oplus\\ P(-(2dn))^{d+3}\end{matrix}\to \begin{matrix}P(-dn)^4\\ \oplus\\ P(-(2dn-2n))^d\end{matrix}\to P.$$

(b) *The minimal homogeneous resolution of A by free P-modules has the form*

$$0\to P(-(2dn+2n))^d\to \begin{matrix}P(-(2dn+n))^{d+3}\\ \oplus\\ P(-(2dn))^d\end{matrix}\to \begin{matrix}P(-2dn)^3\\ \oplus\\ P(-(dn+n))^4\\ \oplus\\ P(-(2dn-n))^d\end{matrix}\to \begin{matrix}P(-dn)^4\\ \oplus\\ P(-n)\end{matrix}\to P.$$

(c) *The socle of A is isomorphic to $\mathbf{k}(-(2dn+2n-4))^d$.*

(d) *The $\overline{P}$-module A has finite projective dimension and the minimal homogeneous resolution of A by free $\overline{P}$-modules has the form*

$$0\to \overline{P}(-(2dn+n))^d\to \begin{matrix}\overline{P}(-2dn)^3\\ \oplus\\ \overline{P}(-(2dn-n))^d\end{matrix}\to \overline{P}(-dn)^4\to \overline{P}.$$

Proof. The ring homomorphism $P\to P$, which sends each element g of P to $g^{[n]}$, is flat. Indeed, P is a free $\mathbf{k}[x^n,y^n,z^n,w^n]$-module. In a similar manner, the ring homomorphism

$$P/(x+y+z+w)\to\overline{P},$$

which sends the class of g in $P/(x+y+z+w)$ to the class of $g^{[n]}$ in $\overline{P}$, is a flat homomorphism. Indeed, $\overline{P}$ is a free $\mathbf{k}[x^n,y^n,z^n,w^n]/(x^n+y^n+z^n+w^n)$-module. Furthermore, the ideals

$$(x^N,y^N,z^N,w^N)\quad\text{and}\quad (x^N,y^N,w^N,z^N):(x^n+y^n+z^n+w^n)\quad\mathit{in}\quad \overline{P},$$

are the images of

$$(x^d,y^d,z^d,w^d)\quad\text{and}\quad (x^d,y^d,w^d,z^d):(x^d+y^d+z^d+w^d)\quad\mathit{in}\quad \frac{P}{(x+y+z+w)},$$

under the homomorphism $(-)^{[n]}:P/(x+y+z+w)\to\overline{P}$. Consequently, in order to prove (a), (b), and (d), as stated, it suffices to prove the corresponding assertions when $n=1$. Of course, (c) is an immediate consequence of (b) and 3.9.(e).

Through out the rest of the proof $n=1$.

(a) Take $n=1$ and let
$$F: \quad 0 \to F_4 \to F_3 \to F_2 \to F_1 \to P$$
be the minimal homogeneous resolution of R by free P-modules. According to Corollary 5.8, the ideal
$$(x^d, y^d, z^d, w^d) : (x+y+z+w),$$
is minimally generated by x^d, y^d, z^d, w^d, together with d additional generators of degree $2d-2$. It follows that
$$F_1 = P(-d)^4 \oplus P(-(2d-2))^d.$$
The socle of R has degree $4(d-1)-1$; hence, $F_4 = P(-(4d-1))$ by 3.9.(e). Use the self-duality of F; see (40); to compute
$$F_3 = P(-(3d-1))^4 \oplus P(-(2d+1))^d.$$
Rank is additive on exact sequences and the rank of R as a P-module is zero. It follows that F_2 has rank $2d+6$ as a P-module. Use Hilbert functions to determine the degrees of the generators of F_2. It is shown in Proposition 4.2 that
$$H_R(k) = \begin{cases} \;\; H_P(k+\varepsilon) - 4H_P(k+\varepsilon-d) + 6H_P(k+\varepsilon-2d) \\ -4H_P(k+\varepsilon-3d) + H_P(k+\varepsilon-4d), \end{cases}$$
where
$$\varepsilon = \begin{cases} 0, & \text{if } k < 2d-2, \text{ and} \\ 1, & \text{if } 2d-2 \le k. \end{cases}$$
Hilbert functions are additive on exact sequences; hence
$$H_{F_2}(k) = H_R(k) - H_P(k) + H_{F_1}(k) + H_{F_3}(k) - H_{F_4}(k) = \begin{cases} 0, & \text{if } k \le 2d-2, \text{ and} \\ d+3, & \text{if } k = 2d-1. \end{cases}$$
Thus, $P(-(2d-1))^{d+3}$ is a summand of F_2. The resolution F is self-dual; hence $P(-2d)^{d+3}$ is also a summand of F_2. The rank of F_2 is $2d+6$. We conclude that $F_2 = P(-(2d-1))^{d+3} \oplus P(-2d)^{d+3}$. Assertion (a) has been established.

One can easily produce a resolution of A by free P-modules using linkage (see 3.14); however some care has to be taken in order to convert this resolution into a minimal resolution. We prefer to first resolve A by free $\overline{P}$ modules and then lift the resolution by free $\overline{P}$ modules to a resolution by free P-modules. That is, we first prove (d) and then we deduce (b).

(d) Consider the $\mathbf{k}[x,y,z]$-algebra isomorphism
$$\overline{P} = \frac{\mathbf{k}[x,y,z,w]}{(x+y+z+w)} \to Q = \mathbf{k}[x,y,z]$$
which sends

(56) $$\text{the class of } w \text{ to } -(x+y+z).$$

Observe that the map (56) sends A isomorphically onto $A' = Q/(x^d, y^d, z^d, (x+y+z)^d)$. We know from Proposition 5.7 and the Buchsbaum-Eisenbud Theorem (see for example, Lemma 5.4) that the minimal homogeneous resolution of
$$\frac{Q}{(x^d, y^d, z^d) : (x+y+z)^d},$$
by free Q-modules, has the form of the bottom complex of

(57) $$\begin{tikzcd} 0 \arrow[r] & Q(-3d) \arrow[r] \arrow[d] & Q(-2d)^3 \arrow[r] \arrow[d] & Q(-d)^3 \arrow[r] \arrow[d, "\alpha_1"] & Q \arrow[d, "="] \\ 0 \arrow[r] & Q(-2d) \arrow[r] & Q(-(d+1))^d \oplus Q(-d)^{\chi} \arrow[r] & Q(-(d-1))^d \oplus Q(-d)^{\chi} \arrow[r] & Q, \end{tikzcd}$$

where

$$\chi = \begin{cases} 1, & \text{if } d \text{ is even, and} \\ 0, & \text{if } d \text{ is odd.} \end{cases}$$

It follows from the theory of linkage (see 3.14) that a resolution of

$$\frac{Q}{(x^d, y^d, z^d, (x+y+z)^d)} = A',$$

by free Q-modules, is obtained by taking the mapping cone of the dual of (57). It is important to recall from Proposition 5.7 that if χ is positive, then z^d is a minimal generator of

$$(x^d, y^d, z^d) : (x+y+z)^d;$$

and therefore, one copy of $Q(-d) \xrightarrow{=} Q(-d)$ is a direct summand of α_1 in (57); and the corresponding summand $Q(-2d)^\chi \to Q(-2d)^\chi$ splits from the shifted mapping cone of the dual of (57). At any rate, one obtains a homogeneous resolution of A' by free Q-modules of the form

$$0 \to Q(-(2d+1))^d \to \begin{matrix} Q(-2d)^3 \\ \oplus \\ Q(-(2d-1))^d \end{matrix} \to Q(-d)^4 \to Q.$$

A quick look at the twists in the resolution guarantees that the resolution is minimal. This completes the proof of (d).

(b) One may interpret (d) to say that $\frac{\mathbf{k}[x,y,z,w]}{(x^d,y^d,z^d,(x+y+z)^d)}$ has a minimal homogeneous resolution, by free P-modules, of the form

$$L: \quad 0 \longrightarrow P(-(2d+1))^d \longrightarrow P(-2d)^3 \oplus P(-(2d-1))^d \longrightarrow P(-d)^4 \longrightarrow P\ .$$

Observe that

$$0 \to \frac{\mathbf{k}[x,y,z,w]}{(x^d,y^d,z^d,(x+y+z)^d)} \xrightarrow{x+y+z+w} \frac{\mathbf{k}[x,y,z,w]}{(x^d,y^d,z^d,(x+y+z)^d)} \to A \to 0$$

is an exact sequence of P-modules; and therefore, the mapping cone of $L[-1] \xrightarrow{x+y+z+w} L$ is minimal homogeneous resolution of A. □

9. THE GENERATORS OF $(x^d, y^d, z^d) : (x+y+z)^d$ AND $(x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}$.

We exhibit explicit elements in the ideals

$$(x^d, y^d, z^d) : (x+y+z)^d \quad \text{and} \quad (x^d, y^{d+1}, z^d) : (x+y+z)^{d+1}.$$

We used these explicit elements in Proposition 5.7 to determine the graded Betti numbers of these ideals when the field has characteristic zero. (Indeed, in Proposition 5.7 we record a minimal generating set for these ideals.) These ideals define compressed codimension three Gorenstein quotient rings with odd socle degree. Compressed codimension three Gorenstein quotient rings with even socle degree are well-understood, see Lemma 5.4 or [6], in particular, they have linear resolutions. However, no theory provides the graded Betti numbers of compressed codimension three Gorenstein quotient rings with odd socle degree. In order to produce the graded Betti numbers, we were forced to produce a minimal generating set.

Proposition 9.1. *Let R be the ring $\mathbf{Z}[x,y,z]$. Let j, d, ε, and σ be integers with*

$$1 \le j \le d, \quad 0 \le \varepsilon \le 1, \quad \textit{and} \quad j-1 \le \sigma \le d-1.$$

Let

$$F_{\sigma,j,\varepsilon} = \sum_{k=0}^{j-1} (-1)^{\sigma+k} \binom{d-1-k}{d-j} \binom{d-1-\sigma+k}{k} \binom{\sigma+\varepsilon}{j-1+\varepsilon} x^k z^{\sigma-k} \tag{58}$$

and

$$f_{j,\varepsilon}=\sum_{s=j-1}^{d-1}F_{s,j,\varepsilon}y^{d-1-s}. \tag{59}$$

Then $f_{j,\varepsilon}\in(x^d,y^{d+\varepsilon},z^d):(x+y+z)^{d+\varepsilon}$.

The proof of Proposition 9.1 uses the following identity twice.

Lemma 9.2. [9, Lemma 2.24] *Let a, z, p, w, and c be integers with* $0\le a$. *Then*

$$\sum_{m\in\mathbf{Z}}(-1)^m\binom{a}{m}\binom{m+z}{p}\binom{m+w}{c}=(-1)^a\sum_{\ell\in\mathbf{Z}}\binom{z}{p-\ell}\binom{a}{\ell}\binom{w+\ell}{c-a+\ell}.$$

In Lemma 9.2 the binomial coefficient $\binom{a}{b}$ makes sense for all pairs of integers (a,b), with

$$\binom{a}{b}=\begin{cases}\dfrac{a(a-1)\cdots(a-b+1)}{b!}, & \text{if } 0<b,\\ 1, & \text{if } 0=b, \text{ and}\\ 0, & \text{if } b<0.\end{cases}$$

In particular,

$$\tbinom{a}{b}=(-1)^b\tbinom{b-a-1}{b}, \tag{60}$$

holds for all integers a and b. The other fact about binomial coefficients that we use often is the familiar fact that

$$\text{if } 0\le a<b, \text{ then } \quad \binom{a}{b}=0. \tag{61}$$

The proof of Proposition 9.1. There are three steps.
Step 1. We expand the product $(x+y+z)^{d+\varepsilon}f_{j,\varepsilon}$ into the form

$$(x+y+z)^{d+\varepsilon}f_{j,\varepsilon}=\sum_{A+B+C=2d-1+\varepsilon}X_{ABC}x^Ay^Bz^C,$$

where, for each 3-tuple of exponents (A,B,C), there is a finite set of ordered pairs S_{ABC} such that

$$X_{ABC}=\sum_{(k,s)\in S_{ABC}}(-1)^{s+k}Y_{jd\varepsilon ABCsk}$$

with

$$Y_{jd\varepsilon ABCsk}=\binom{d+\varepsilon}{B+s+1-d}\binom{2d-B-s-1+\varepsilon}{A-k}\binom{d-1-k}{d-j}\binom{d-1-s+k}{d-1-s}\binom{s+\varepsilon}{j-1+\varepsilon}. \tag{62}$$

Step 2. We prove that if A, B, and C satisfy

$$0\le A,C\le d-1,\quad 0\le B\le d-1+\varepsilon,\quad \text{and}\quad A+B+C=2d-1+\varepsilon, \tag{63}$$

then

$$X_{ABC}=\sum_{s\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}(-1)^{s+k}Y_{jd\varepsilon ABCsk}.$$

Step 3. For A,B,C satisfying (63), we apply Lemma 9.2 to

$$\sum_{k\in\mathbf{Z}}(-1)^k\binom{2d-B-s-1}{A-k}\binom{d-1-k}{d-j}\binom{d-1-s+k}{d-1-s};$$

then we apply Lemma 9.2 to $\sum_{s\in Z}(-1)^s$ of the three remaining terms which involve s. We conclude $X_{ABC}=0$.

We carry out Step 1. Write

$$(x+y+z)^{d+\varepsilon} = \sum_{\substack{a+b+c=d+\varepsilon\\ 0\le a,b,c}} \binom{d+\varepsilon}{a,b,c} x^a y^b z^c.$$

It follows that $(x+y+z)^{d+\varepsilon} f_{j,\varepsilon}$ is equal to

$$\begin{aligned}
&\sum_{\substack{a+b+c=d+\varepsilon\\ 0\le a,b,c}} \sum_{s=j-1}^{d-1} \sum_{k=0}^{j-1} (-1)^{s+k} \tbinom{d+\varepsilon}{a,b,c} \tbinom{d-1-k}{d-j} \tbinom{d-1-s+k}{k} \tbinom{s+\varepsilon}{j-1+\varepsilon} x^{k+a} y^{d-1-s+b} z^{s-k+c}\\
&= \sum_{\substack{A+B+C=2d-1+\varepsilon\\ 0\le A,B,C}} X_{ABC} x^A y^B z^C,
\end{aligned}$$

where X_{ABC} is equal to

$$\sum_{s=j-1}^{d-1} \sum_{k=0}^{j-1} (-1)^{s+k} \binom{d+\varepsilon}{a,b,c} \binom{d-1-k}{d-j} \binom{d-1-s+k}{k} \binom{s+\varepsilon}{j-1+\varepsilon} Q_{dskabcABC},$$

for

$$\begin{aligned}
Q_{dskabcABC} &= \chi(k+a=A)\chi(d-1-s+b=B)\chi(s-k+c=C)\chi(0\le a)\chi(0\le b)\chi(0\le c)\\
&= \begin{cases} 1, & \text{if } k+a=A,\ d-1-s+b=B,\ s-k+c=C, 0\le a,\ 0\le b, \text{ and } 0\le c \text{ and}\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

In general, we use the convention that if "S" is a statement then

$$\chi(S) = \begin{cases} 1, & \text{if } S \text{ is true, and}\\ 0, & \text{if } S \text{ is false.}\end{cases}$$

At this point the binomial coefficients all have the form $\binom{a}{b}$ with $0\le b\le a$; so we may write

$$\binom{d+\varepsilon}{a,b,c} = \binom{d+\varepsilon}{b}\binom{d-b+\varepsilon}{a} \quad \text{and} \quad \binom{d-1-s+k}{k} = \binom{d-1-s+k}{d-1-s},$$

without penalty. Replace a with $A-k$; b with $B-d+1+s$; and c with $C-s+k$. Observe that

$$X_{ABC} = \sum_{s=j-1}^{d-1} \sum_{k=0}^{j-1} (-1)^{s+k} Y_{jd\varepsilon ABCsk} \chi(0\le A-k)\chi(0\le B+s+1-d)\chi(0\le C+k-s).$$

In other words,

$$X_{ABC} = \sum_{k=0}^{\min\{A,j-1\}} \sum_{s=\max\{j-1,d-1-B\}}^{\min\{d-1,k+C\}} (-1)^{s+k} Y_{jd\varepsilon ABCsk}.$$

This completes Step 1.

We carry out Step 2. Assume (63). We prove that

$$X_{ABC} = \sum_{s\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} (-1)^{s+k} Y_{jd\varepsilon ABCsk}.$$

It suffices to show that if $Y_{jd\varepsilon ABCsk} \neq 0$, then

(a) $d-1-B\le s$
(b) $j-1\le s$,
(c) $s\le C+k$,
(d) $s\le d-1$,
(e) $k\le A$,
(f) $k\le j-1$, and
(g) $0\le k$.

The hypothesis that $\binom{d+\varepsilon}{B+s+1-d} \neq 0$ implies $0 \le B+s+1-d$; thus $d-1-B \le s$, which is (a). Add ε to both sides of (a) and use $B \le d-1+\varepsilon$ from (63) in order to see that

$$0 \le d-1+\varepsilon-B \le s+\varepsilon. \tag{64}$$

The hypothesis that $\binom{s+\varepsilon}{j-1+\varepsilon} \neq 0$, together with (64) and (61), guarantees that

$$j-1+\varepsilon \le s+\varepsilon;$$

hence, $j-1 \le s$, which is (b). The sum $d+\varepsilon$ is positive; hence, the hypothesis

$$\binom{d+\varepsilon}{B+s+1-d} \neq 0,$$

together with (61), implies $B+s+1-d \le d+\varepsilon$, which, in turn, implies

$$0 \le 2d-B-s-1+\varepsilon. \tag{65}$$

The hypothesis $\binom{2d-B-s-1+\varepsilon}{A-k} \neq 0$, together with (65) and (61) implies

$$A-k \le 2d-B-s-1+\varepsilon,$$

which implies $s \le (2d-1-A-B+\varepsilon)+k$. Recall from (63) that $2d-1-A-B+\varepsilon = C$. We have established $s \le C+k$, which is (c). It also follows, by way of (63), that

$$s-k \le C \le d-1;$$

hence

$$0 \le d-1-s+k. \tag{66}$$

The hypothesis $\binom{d-1-s+k}{d-1-s} \neq 0$ implies that $0 \le d-1-s$; thus $s \le d-1$, which is (d). The hypothesis $\binom{2d-B-s-1+\varepsilon}{A-k} \neq 0$ implies $0 \le A-k$; thus $k \le A$, which is (e). Recall from (63) that $A \le d-1$. It follows that $k \le d-1$; hence

$$0 \le d-1-k. \tag{67}$$

The hypothesis $\binom{d-1-k}{d-j} \neq 0$, together with (67) and (61), ensures that $d-j \le d-1-k$. It follows that $k \le j-1$ and this is (f). The hypothesis $\binom{d-1-s+k}{d-1-s} \neq 0$, together with (66) and (61), implies

$$d-1-s \le d-1-s+k.$$

Thus, $0 \le k$, and this is (g). This completes Step 2.

We carry out Step 3. The parameters satisfy

$$0 \le A, C \le d-1, \quad 0 \le B \le d-1+\varepsilon, \quad A+B+C = 2d-1+\varepsilon, \quad \text{and} \quad 1 \le j \le d.$$

We prove that X_{ABC}, which is equal to

$$\sum_{s\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}(-1)^{s+k}\binom{d+\varepsilon}{B+s+1-d}\binom{2d-B-s-1+\varepsilon}{A-k}\binom{d-1-k}{d-j}\binom{d-1-s+k}{d-1-s}\binom{s+\varepsilon}{j-1+\varepsilon},$$

is equal to zero. Apply (60) to write

$$\binom{d-1-s+k}{d-1-s} = (-1)^{d-1-s}\binom{-k-1}{d-1-s}.$$

Let $K = A-k$. Observe that X_{ABC} is equal to

$$\sum_s (-1)^{A+d+1}\binom{d+\varepsilon}{B+s+1-d}\left[\sum_K (-1)^K\binom{2d-B-s-1+\varepsilon}{K}\binom{K+d-1-A}{d-j}\binom{K-A-1}{d-1-s}\right]\binom{s+\varepsilon}{j-1+\varepsilon}.$$

The sum $d+\varepsilon$ is non-negative; consequently the binomial coefficient $\binom{d+\varepsilon}{B+s+1-d}$ is zero unless

$$B+s+1-d \le d+\varepsilon.$$

Thus, X_{ABC} is also equal to

$$\sum_{s\le 2d-B-1+\varepsilon}(-1)^{A+d+1}\tbinom{d+\varepsilon}{B+s+1-d}\left[\sum_K(-1)^K\tbinom{2d-B-s-1+\varepsilon}{K}\tbinom{K+d-1-A}{d-j}\tbinom{K-A-1}{d-1-s}\right]\tbinom{s+\varepsilon}{j-1+\varepsilon}.$$

Apply Lemma 9.2 with

$$a=2d-B-s-1+\varepsilon,\quad z=d-1-A,\quad p=d-j,\quad w=-A-1,\quad \text{and}\quad c=d-1-s.$$

Notice that $0\le a$. Conclude that X_{ABC} is equal to

$$\sum_{s\le 2d-B-1+\varepsilon}(-1)^{A+B+s+d+\varepsilon}\tbinom{d+\varepsilon}{B+s+1-d}\left[\sum_{\ell\in\mathbf{Z}}\tbinom{d-1-A}{d-j-\ell}\tbinom{2d-B-s-1+\varepsilon}{\ell}\tbinom{-A-1+\ell}{-d+B+\ell-\varepsilon}\right]\tbinom{s+\varepsilon}{j-1+\varepsilon}.$$

The binomial coefficient $\binom{d+\varepsilon}{B+s+1-d}$ continues to be zero unless $s\le 2d-B-1+\varepsilon$; thus, we may remove the bound $s\le 2d-B-1+\varepsilon$ from the summation sign without changing the value of the sum. Apply (60) to write

$$\binom{2d-B-s-1+\varepsilon}{\ell}=(-1)^\ell\binom{\ell-2d+B+s-\varepsilon}{\ell}.$$

Gather the factors which involve s and observe that X_{ABC} is equal to

$$\sum_{\ell\in\mathbf{Z}}(-1)^{A+B+d+\ell+\varepsilon}\tbinom{d-1-A}{d-j-\ell}\left[\sum_{s\in\mathbf{Z}}(-1)^s\tbinom{d+\varepsilon}{B+s+1-d}\tbinom{\ell-2d+B+s-\varepsilon}{\ell}\tbinom{s+\varepsilon}{j-1+\varepsilon}\right]\tbinom{-A-1+\ell}{-d+B+\ell-\varepsilon}.$$

Let $S=B+s+1-d$ to see that X_{ABC} is equal to

$$\sum_{\ell\in\mathbf{Z}}(-1)^{A+1+\ell+\varepsilon}\tbinom{d-1-A}{d-j-\ell}\left[\sum_{S\in\mathbf{Z}}(-1)^S\tbinom{d+\varepsilon}{S}\tbinom{S+\ell-d-1-\varepsilon}{\ell}\tbinom{S+d-B-1+\varepsilon}{j-1+\varepsilon}\right]\tbinom{-A-1+\ell}{-d+B+\ell-\varepsilon}.$$

Use Lemma 9.2 with

$$a=d+\varepsilon,\quad z=\ell-d-1-\varepsilon,\quad p=\ell,\quad w=d-B-1+\varepsilon,\quad \text{and}\quad c=j-1+\varepsilon$$

to see that X_{ABC} is equal to

$$\sum_{\ell\in\mathbf{Z}}(-1)^{A+d+1+\ell}\tbinom{d-1-A}{d-j-\ell}\left[\sum_{L\in\mathbf{Z}}\tbinom{\ell-d-1-\varepsilon}{\ell-L}\tbinom{d+\varepsilon}{L}\tbinom{d-B-1+L+\varepsilon}{j-1-d+L}\right]\tbinom{-A-1+\ell}{-d+B+\ell-\varepsilon}.$$

Every non-zero term in the above sum satisfies

$$0\le d-j-\ell,\quad 0\le \ell-L,\quad \text{and}\quad 0\le j-1-d+L.$$

Thus, every non-zero term satisfies

$$d+1-j\le L\le \ell\le d-j.$$

There are no ordered pairs (ℓ,L) which satisfy these constraints; thus, $X_{ABC}=0$, and the proof is complete. $\square$

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