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The Lexicographic Sum of Cohen-Macaulay and Shellable Ordered Sets

Jerrold R. Griggs*[1], Andrew R. Kustin*, Jeffrey A. Ross*[2] and Jürgen Stahl[3]

Department of Mathematics and Statistics, University of South Carolina, Columbia, SC 29208, USA

Abstract. We study the problem of determining when the lexicographic sum $\sum_{q \in Q} P_q$ of a family of posets $\{P_q | q \in Q\}$ over a poset Q is Cohen-Macaulay or shellable. Our main result, a characterization of when the lexicographic sum is Cohen-Macaulay, is proven using combinatorial methods introduced by Garsia. A similar characterization for when the lexicographic sum is CL(chainwise-lexicographically)-shellable, is derived using the recursive atom ordering method due to Björner and Wachs.

1. Introduction

The study of Cohen-Macaulay posets has produced a fruitful mingling of ideas from combinatorics, commutative algebra, and topology. The references listed at the end of this paper form only a fraction of the recent outpouring of results in this area.

In short, to a finite poset P can be associated a commutative ring $R[P]$, and P is Cohen-Macaulay in the ring-theoretic sense if and only if $R[P]$ is a Cohen-Macaulay ring. A simplicial complex $\Delta(P)$ can also be associated to P, and P is said to be Cohen-Macaulay in the topological sense if and only if the homology of $\Delta(P)$ satisfies certain conditions. The most important theorem in the area, due to Reisner [16], states that the ring-theoretic and topological definitions of Cohen-Macaulay posets are equivalent.

The interplay between the algebraic and topological interpretations figures prominently in applications of Cohen-Macaulay posets (and, more generally, simplicial complexes) such as, e.g., the solution of the upper bound conjecture concerning the number of faces of a spherical simplicial complex [17], and in obtaining information about the Möbius function of P [6].

Working from the ring-theoretic definition, Garsia [10] has derived a purely combinatorial and linear-algebraic characterization of Cohen-Macaulay posets.

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2 Current address: Bell Communications Research, 3D-638, Holmdel, NJ 07733, USA

3 Current address: F64 AG1, Technische Hochschule Darmstadt, Schlossgartenstr.7, 6100 Darmstadt, Federal Republic of Germany

In practice, showing that a poset or a class of posets satisfying certain conditions is Cohen-Macaulay from the definition or Garsia's characterization is often difficult. A topological property, shellability, has proven to be very useful. It is stronger than the Cohen-Macaulay property, but many interesting Cohen-Macaulay posets are shellable. It has an essentially combinatorial definition. Björner has introduced the notion of lexicographic shellability, which involves slightly stronger properties than shellability. Two properties of this type, EL-shellable and CL-shellable, are particularly natural, useful, and interesting combinatorially.

In dealing with posets having certain special properties one asks whether these properties are preserved under standard constructions. In this paper we investigate when the lexicographic sum $\sum_{q\in Q} P_q$ of a collection of posets P_q over a poset Q is Cohen-Macaulay, shellable, EL-shellable, or CL-shellabe. For Cohen-Macaulay and CL-shellable posets we can solve the problem completely, and for shellable and EL-shellable posets we obtain partial results.

The lexicographic sum $\sum_{q\in Q} P_q$ is more general than constructions studied previously in the area, so we shall cite connections to earlier results. In particular, special cases of our main result for Cohen-Macaulay posets have been obtained before by homological arguments and sometimes also by ring-theoretic arguments. We use only Garsia's combinatorial characterization of Cohen-Macaulay posets in hopes of making this area accessible to a wider audience. Although we succeeded in formulating and proving our main theorem on when the ordinal sum is Cohen-Macaulay, it must be conceded that the homological definition yields a shorter and less complicated proof.

The paper is organized as follows. After setting our poset terminology and surveying the Cohen-Macaulay definitions, we state our results for Cohen-Macaulay posets in Section 4. The proof is divided into several lemmas which are of independent interest and are proven in the next few sections. In Section 10 we survey the area of shellability and consider when the lexicographic sum of posets is shellable. The last two sections look at constructions of lexicographically shellable posets.

2. Poset Terminology

In a poset (partially ordered set) P, a *chain* (respectively, *antichain*) is a totally ordered (respectively, totally unordered) subset of P. A chain C has *length* $|C|$. We denote by $\mathscr{C} = \mathscr{C}(P)$ (respectively, $\mathscr{M} = \mathscr{M}(P)$) the set of all chains (respectively, maximal chains) of P.

All posets in this paper are assumed to be finite and *pure*, which means that all maximal chains have the same length, called the *rank*, $r = r(P)$. For $x \in P$, $r(x) = r_P(x)$ denotes the *rank* of x, which is the maximum length of the *chains in* P with maximum element x. The *rank set* P_i consists of the elements x with rank $r(x) = i$. If c is a chain, then $r(c) = \{r(x) | x \in c\}$ is the *rank set of* c.

The poset P is *graded* if it is pure and contains a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. We use $\hat{P}$ to denote P with a minimum element $\hat{0}$ and a maximum element $\hat{1}$ added to it.

The *ordinal sum* $P \oplus Q$ of disjoint posets P and Q has elements $P \cup Q$ ordered by $x < y$ if and only if x and y are both in P or both in Q with $x < y$ or $x \in P$ and

$y \in Q$. If $\mathscr{P} = \{P_q | q \in Q\}$ is a family of posets indexed by the elements of a poset Q, then the *lexicographic sum of* $\mathscr{P}$ *over* Q, denoted $\sum_{q \in Q} P_q$, is the poset obtained by replacing each element $q \in Q$ by the poset P_q. More precisely, $\sum_{q \in Q} P_q = \{(q, p) | q \in Q, p \in P_q\}$, ordered by $(q, p) < (q', p')$ if and only if $q < q'$ in Q or $q = q'$ in Q and $p < p'$ in P_q. If $P_q = P$ for all $q \in Q$, $\sum_{q \in Q} P_q$ is denoted by $\sum_Q P$ and called the *lexicographic product of* P *over* Q. Observe that $P \oplus Q$ is the lexicographic sum of P and Q over a two element chain.

An element x is *covered* by an element y of P, written $x \to y$, if $x < y$ and if $x < z \leq y$ only if $z = y$. A *closed interval* $[x, y]$ for $x \leq y$ in P contains all elements z such that $x \leq z \leq y$, ordered as in P. An *open interval* (x, y) contains all z such that $x < z < y$. The *restriction* of P to an element x, denoted $P|x$, contains all elements z of P which are related to x, that is, $z \leq x$ or $x \leq z$. The *extension* of P by an element x, denoted $P \propto x$, is P with a new element, call it x', which is just like x: $z < x'$ (respectively, $z > x'$) if and only if $z < x$ (respectively, $z > x$). The subposet of P in which x is *deleted* is denoted $P \backslash x$. The *dual poset* P^d of P has the same elements as P does, but with the ordering reversed.

We next describe the lexicographic order that we impose on the chains of a poset P of rank $r = r(P)$. For each integer $n \geq 1$, $[n]$ is the set $\{1, \ldots, n\}$. If $S, T \subseteq [r]$, write $S < T$ if either $|S| < |T|$ or $|S| = |T|$ and $\min(S \Delta T) \in S$. Label the elements of P by the integers $1, 2, \ldots, |P|$, starting with the elements of P_1, then the elements of P_2, etc. The chains $\mathscr{C}(P)$ are ordered as follows: For $c, c' \in \mathscr{C}(P)$, write $c < c'$ if either $r(c) < r(c')$ or $r(c) = r(c')$ and, looking at the labels of the elements, $\min(c \Delta c') \in c$.

A *simplicial complex* Δ is a family of subsets of a finite set, called *faces*, such that for any face $F \in \Delta$ and any $G \subseteq F$, $G \in \Delta$. To a pure poset P we associate a simplicial complex $\Delta(P)$, called the *order complex*, in this way: The vertices (0-dimensional faces) are the elements of P and in general the i-dimensional faces are the chains in P of length $i + 1$, including $\varnothing$ as a (-1)-dimensional face.

3. Cohen-Macaulay Posets

One may study a poset P topologically by looking at its order complex, $\Delta(P)$. This was done by Baclawski [1], who introduced the idea of Cohen-Macaulay posets. Without going into details here (cf. the survey [6]) it suffices to mention that one looks at the reduced simplicial homology groups of $\Delta(P)$. Throughout the paper let k denote any field. The poset P is *Cohen-Macaulay over* k if every open interval (x, y) in $\hat{P}$ is a *bouquet*, which means that its reduced simplicial homology $\tilde{H}_i((x, y), k)$ vanishes, except possibly at the top level, $i = r(y) - r(x) - 2$. Indeed this definition of Cohen-Macaulay can be extended to any simplicial complex Δ. More generally, k can be replaced by any commutative ring, for example, the integers.

Reisner [16] and Stanley [17] associated a commutative ring $R[P]$ with a poset P in the following way. To each element i in P associate an indeterminate x_i. With k being some field, let $R[P] = R/I$, where $R = k[x_1, \ldots, x_n]$ is the polynomial ring for the indeterminates from the n elements of P, and where $I = (\{x_i x_j | i, j$ unrelated in $P\})$ is the ideal generated by products from unrelated elements of P. Thus the surviving monomials in $R[P]$ are of the form $c x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$, where $c \in k$, each $a_j \in \mathbb{N}$, and $i_1 < i_2 < \cdots < i_k$ is a chain P. Note that different posets P may have the same ring $R[P]$.

The algebraic definition of Cohen-Macaulay posets says simply that P is *Cohen-Macaulay over k* if its ring $R[P]$ is Cohen-Macaulay. We mention one method for deciding if $R[P]$ is Cohen-Macaulay. See, for example, [4] for more details. Let θ_j be the *rank j polynomial* in $R[P]$ given by

$$\theta_j = \sum_{i \in P_j} x_i.$$

Then P is Cohen-Macaulay over k if and only if there exists some set $\{\beta_1, \ldots, \beta_2\} \subseteq R[P]$ such that for every $f \in R[P]$ there exist *unique* polynomials $p_1, \ldots, p_s$ with coefficients in k such that

$$f = \sum_{i=1}^{s} \beta_i p_i(\theta_1, \ldots, \theta_r),$$

where $r = r(P)$.

The algebraic definition of Cohen-Macaulay can be extended to any simplicial complex Δ by looking at its ring R_Δ in which the surviving monomials come from faces of Δ. Reisner's Theorem [16] asserts that for any poset P, or more generally, for any simplicial complex Δ, the topological and algebraic definitions of Cohen-Macaulay are equivalent. It is worth noting that the definition of Cohen-Macaulay can be applied to finite posets in general, whether or not they are pure, but it can be shown, see [6], that every Cohen-Macaulay poset is pure, which is why we only consider pure posets.

We will not work directly with either the topological or algebraic definitions. Instead, we rely on the combinatorial-linear algebraic characterizations contained in Garsia [10]. Form a matrix N in which the rows correspond to the chains $c_i \in \mathscr{C}(P)$ and the columns to the maximum chains $m_j \in \mathscr{M}(P)$. (Be sure to order the rows and columns of N as described in section 2.) The entry N_{ij} is 1 if $c_i \subseteq m_j$ and 0 otherwise. The matrix N has rank $|\mathscr{M}(P)|$ since its last $|\mathscr{M}(P)|$ rows form an identity matrix. Now form the set $\mathscr{B}$ of chains which correspond to rows of N that form the basis in which every row is linearly independent (over k) of the rows above it. Garsia has shown that P is Cohen-Macaulay over k if and only if this lexicographically least basis $\mathscr{B}$ satisfies the combinatorial condition:

$$|\{b \in \mathscr{B} \mid r(b) \subseteq S\}| = |\{c \in \mathscr{C} \mid r(c) = S\}|$$

for every $S \subseteq [r]$.

Although this and other related characterizations are inspired by visualizing the matrix N, it is more natural overall to work directly in the ring $R[P]$. We need some notation for this.

Fix $c = \{p_1, p_2, \ldots, p_n\} \subseteq P$. Let $x(c)$ be the element $x_{p_1} \cdots x_{p_n}$ of $R[P]$. Note that $x(c) = 0$ if c is not a chain. For any $S \subseteq [r]$, let $H_S = H_S^P$ be the vector subspace of $R[P]$ with basis $\{x(c) \mid c \in \mathscr{C}, r(c) = S\}$. Let $Lc = L^P c$ be the element of $H_{[r]}$ given by

$$Lc = x(c) \prod_{\substack{i \in [r] \\ i \notin r(c)}} \theta_i,$$

which is 0 unless c is a chain. It follows from the definition of $R[P]$ and θ_i that

$$Lc = \sum_{\substack{m \in M \\ c \subseteq m}} x(m).$$

In particular, for $m \in \mathscr{M}$, $Lm = x(m)$.

Let LH_S be the vector subspace of $H_{[r]}$ generated by $\{Lc|c\in\mathscr{C}, r(c)=S\}$. Since each θ_i is a nonzero divisor, multiplication by $\prod_{i\in[r]\setminus S}\theta_i$ gives an isomorphism from H_S to LH_S for all $S\subseteq[r]$. Consequently, $\dim_k LH_S = |\{c\in\mathscr{C}|r(c)=S\}|$. Let $\sum_{T\subsetneq S} LH_T$ denote the vector subspace of $H_{[r]}$ generated by the LH_T with $T\subsetneq S$, that is, by $\{Lc|c\in\mathscr{C}, r(c)\subsetneq S\}$. Similarly define $\sum_{T<S} LH_T$ to be the subspace generated by the LH_T with $T<S$ in the lexicographic ordering on $[r]$.

The conditions of the following lemma are very similar. We shall require each condition later on. It is to our advantage to see that they are equivalent once and for all.

Lemma 3.1. *Let $\mathscr{B}$ be a collection of chains from the poset P. The following conditions are equivalent*:

(3.1) *For all chains $c\in\mathscr{C}$, Lc can be written uniquely in the form*

$$\sum_{\substack{b\in\mathscr{B}\\ r(b)\subseteq r(c)}} \alpha_b Lb.$$

(3.2) a) *For all chains $c\in\mathscr{C}$, Lc can be written in the form*

$$\sum_{\substack{b\in\mathscr{B}\\ r(b)\subseteq r(c)}} \alpha_b Lb.$$

b) $\{Lb|b\in\mathscr{B}\}$ *is a basis for* $H_{[r]}$.

(3.3) a) $|\{c\in\mathscr{C}|r(c)=S\}| = |\{b\in\mathscr{B}|r(b)\subseteq S\}|$ *for all* $S\subseteq[r]$.
b) $\{Lb|b\in\mathscr{B}\}$ *is a basis for* $H_{[r]}$.

(3.4) a) $|\{c\in\mathscr{C}|r(c)=S\}| = |\{b\in\mathscr{B}|r(b)\subseteq S\}|$ *for all* $S\subseteq[r]$.
b) *For all* $m\in\mathscr{M}$, *Lm can be written in the form* $\sum_{b\in\mathscr{B}}\alpha_b Lb$.

The proof of the lemma is a simple exercise. From the lemma and Garsia [10, Section 3] we obtain the following characterizations of Cohen-Macaulay posets.

Theorem 3.1. *The following conditions are equivalent for a poset P*:

(3.5) *P is a Cohen-Macaulay over k.*

(3.6) *There is a collection of chains of P which satisfies the conditions of Lemma* 3.1.

(3.7) *If $\mathscr{B}$ is the lexicographically least set of chains of $\mathscr{B}$ with $\{Lb|b\in\mathscr{B}\}$ a basis for $H_{[r]}$, then $\mathscr{B}$ satisfies the conditions of Lemma* 3.1.

(3.8) *For every subset S of* $[r]$

$$LH_S \cap \sum_{T<S} LH_T \subseteq \sum_{T\subsetneq S} LH_T.$$

From this theorem it is straightforward to verify the following well-known elementary observations about Cohen-Macaulay posets:

(3.9) A chain is Cohen-Macaulay.

(3.10) An antichain is Cohen-Macaulay.

(3.11) The poset $\{1<3, 2<4\}$ is not Cohen-Macaulay.

(3.12) If the element $p\in P$ is comparable to every other element of P, then P is Cohen-Macaulay if and only if $P\setminus p$ is Cohen-Macaulay.

(3.13) The dual poset P^d is Cohen-Macaulay if and only if P is Cohen-Macaulay.

4. The Lexicographic Sum of Cohen-Macaulay Posets

We prove the following result about Cohen-Macaulay posets.

Theorem 4.1. *The poset $\sum_{q \in Q} P_q$ is Cohen-Macaulay over k if and only if all of the following conditions hold*:

(4.1) *Each poset P_q is Cohen-Macaulay over k,*

(4.2) *The poset Q is Cohen-Macaulay over k,*

(4.3) *If q and q' are distinct elements of Q having the same rank, then P_q and $P_{q'}$ are both antichains.*

An immediate consequence of this result is

Corollary 4.1. *The poset $\sum_Q P$ is Cohen-Macaulay over k if and only if one of the following conditions hold*:

(4.4) *Q is a chain and P is Cohen-Macaulay over k, or*

(4.5) *P is an antichain and Q is Cohen-Macaulay over k.*

Theorem 4.1 will be derived in Section 9 from a sequence of intermediate results, Lemmas 4.1–4.4 below, which are of interest in themselves.

Lemma 4.1. *The sum $P_1 \oplus P_2$ is Cohen-Macaulay over k if and only if P_1 and P_2 are both Cohen-Macaulay over k.*

Lemma 4.2. *If $\sum_{q \in Q} P_q$ is Cohen-Macaulay over k, and if q and q' are distinct elements having the same rank, then P_q and $P_{q'}$ are both antichains.*

Lemma 4.3. *If q belongs to the Cohen-Macaulay poset Q, then the restriction $Q|q$ is also Cohen-Macaulay.*

Lemma 4.4. *If $q \in Q$ then the extension $Q \propto q$ is Cohen-Macaulay over k if and only if Q is Cohen-Macaulay over k.*

A nice topological proof of Lemma 4.1 can be found in [19, 9.1]. Baclawski [2, 7.3] discovered Lemma 4.4 using the Leray spectral sequence. Indeed, after learning of our Theorem 4.1, Walker produced a purely topological proof.

Some further well-known properties of Cohen-Macaulay posets [6] can now be supplied with elementary proofs, given Theorem 3.1, once we establish the results above:

Corollary 4.2. *A closed interval $[x, y]$ in a Cohen-Macaulay poset is also Cohen-Macaulay.*

Proof. Let $[x, y]$ be an interval in the Cohen-Macaulay poset Q. Let $q_1 \to q_2 \to \cdots \to q_n \to x$ and $y \to q_{n+1} \to \cdots \to q_m$ be unrefinable chains in Q, where q_1 is a minimal element and q_m is a maximal element of Q. By Lemma 4.3 and observation 3.12, $Q|q_1|q_2|\ldots|q_m|x|y\backslash q_1\backslash q_2\ldots\backslash q_m = [x, y]$ is also Cohen-Macaulay. □

For a poset P and $T \subseteq [r]$, the *rank-selected subposet* P_T is the subposet of P consisting of $p \in P$ with $r(p) \in T$.

Corollary 4.3. If P is Cohen-Macaulay over k and $T \subseteq [r]$, then P_T is Cohen-Macaulay.

Proof. By Theorem 3.1 there exists a set $\mathscr{B}$ of chians of P satisfying (3.3). Let $\mathscr{B}' = \{b \in \mathscr{B} \mid r(b) \subseteq T\}$. For every $S \subseteq T$,

$$\begin{aligned} |\{c \in \mathscr{C}(P) \mid r(c) = S\}| &= |\{b \in \mathscr{B} \mid r(b) \subseteq S\}| \\ &= |\{b \in \mathscr{B}' \mid r(b) \subseteq S\}|. \end{aligned}$$

Moreover, $\{L^{P_T}b \mid b \in \mathscr{B}'\}$ are linearly independent since $L^P b = (\prod_{i \in [r] \setminus T} \theta_i) L^{P_T} b$ and $\prod_{i \in [r] \setminus T} \theta_i$ is a nonzero divisor. Since $|\{L^{P_T}b \mid b \in \mathscr{B}'\}| = |\mathscr{C}(P_T)|$, $\{L^{P_T}b \mid b \in \mathscr{B}'\}$ is the required basis, and it now follows from (3.3) and Theorem 3.1 that P_T is Cohen-Macaulay. □

Say P is *rank-connected* if each subposet of P consisting of two consecutive ranks has a connected Hasse diagram. Note that if $T \subseteq [r]$ with $|T| \geq 2$ and P is rank-connected, then P_T has a connected Hasse diagram. An antichain is vacuously rank-connected.

Corollary 4.4. *Every Cohen-Macaulay poset is rank-connected.*

Proof. We prove the contrapositive. Suppose there is a subset $T \subseteq [r]$ of two consecutive ranks with P_T disconnected. Then P_T can be written as a lexicographic sum of posets of rank 2 over a two point antichain, so it follows from Lemma 4.2 that P_T is not Cohen-Macaulay. Hence, by Corollary 4.3, P is not Cohen-Macaulay. □

In the following sections we prove the four lemmas above and then Theorem 4.1.

5. Proof of Lemma 4.1

We prove that the sum $P_1 \oplus P_2$ is Cohen-Macaulay over k if and only if P_1 and P_2 are both Cohen-Macaulay.

Let $P = P_1 \oplus P_2$, $r_i = r(P_i)$, $r = r(P)$, $\mathscr{C}_i = \mathscr{C}(P_i)$, $\mathscr{C} = \mathscr{C}(P)$, $\mathscr{M}_i = \mathscr{M}(P_i)$, and $\mathscr{M} = \mathscr{M}(P)$.

First assume that P_1 and P_2 are both Cohen-Macaulay. Find collections of chains $\mathscr{B}_i \subseteq \mathscr{C}_i$ which satisfy the conditions of Lemma 3.1. Let $\mathscr{B}$ be the set of chains $\{b_1 \cup b_2 \mid b_1 \in \mathscr{B}_1, b_2 \in \mathscr{B}_2\}$ of P. A typical subset of $[r]$ is

$$S = \{\alpha_1 < \cdots < \alpha_m < r_1 + \beta_1 < \cdots < r_1 + \beta_n\}$$

with $S_1 = \{\alpha_1, \ldots, \alpha_m\}$ a subset of $[r_1]$ and $S_2 = \{\beta_1, \ldots, \beta_n\}$ a subset of $[r_2]$. The number of chains $c \in \mathscr{C}$ with rank set S is

$$|\{c_1 \in \mathscr{C}_1 \mid r(c_1) = S_1\}| \cdot |\{c_2 \in \mathscr{C}_2 \mid r(c_2) = S_2\}|.$$

Similarly the number of chains $b \in \mathscr{B}$ with rank set contained in S is

$$|\{b_1 \in \mathscr{B}_1 \mid r(b_1) \subseteq S_1\}| \cdot |\{b_2 \in \mathscr{B}_2 \mid r(b_2) \subseteq S_2\}|.$$

Hence $\mathscr{B}$ satisfies condition (3.4a) of Lemma 3.1.

Let $c = c_1 \cup c_2$ be a typical chain of P with $c_i \in \mathscr{C}_i$. Each maximal chain of P containing c has the form $m_1 \cup m_2$ where m_i is a maximal chain of P_i containing c_i. The ring $R[P_i]$ is contained in $R[P]$; hence $L^{P_1}m_1 L^{P_2}m_2 = L^P(m_1 \cup m_2)$ and

$$L^P(c_1 \cup c_2) = \sum_{\substack{c_1 \subseteq m_1 \\ c_2 \subseteq m_2 \\ m_i \in \mathscr{M}_i}} L^{P_1}m_1 \cdot L^{P_2}m_2 = L^{P_1}c_1 \cdot L^{P_2}c_2.$$

If m is the maximal chain $m_1 \cup m_2$ of P, then by hypothesis

$$L^{P_i}m_i = \sum_{b_i \in \mathscr{B}_i} \alpha_{b_i} L^{P_i} b_i.$$

Thus

$$L^P m = L^{P_1}m_1 \cdot L^{P_2}m_2 = \sum_{\substack{b_1 \cup b_2 \in \mathscr{B} \\ b_1 \in \mathscr{B}_1, b_2 \in \mathscr{B}_2}} \alpha_{b_1}\alpha_{b_2} L^P(b_1 \cup b_2).$$

Therefore $\mathscr{B}$ satisfies (4) of Lemma 3.1 and P is Cohen-Macaulay.

To prove the converse, suppose that $P = P_1 \oplus P_2$ is Cohen-Macaulay. Let $S \subseteq [r_1]$ and let

$$\zeta \in L^{P_1}H_S \cap \sum_{\substack{T<S \\ T \subseteq [r_1]}} L^{P_1}H_T.$$

Multiply by $\theta = \prod_{i=r_1+1}^{r} \theta_i$ and allow the sum to be taken over all subsets of $[r]$ which are lexicographically less than S to get

$$\zeta\theta \in L^P H_S \cap \sum_{\substack{T<S \\ T \subseteq [r]}} L^P H_T.$$

Since P is Cohen-Macaulay we have by Theorem 3.1

$$\zeta\theta = \sum_{T \subsetneqq S} L^P H_T = \theta \sum_{T \subsetneqq S} L^{P_1} H_T.$$

The element θ of $R[P]$ is a non-zero divisor; hence $\zeta \in \sum_{T \subsetneqq S} L^{P_1}H_T$, and P_1 is Cohen-Macauley by Theorem 3.1.

By (3.13) P^d is Cohen-Macaulay since P is. Since $P^d = P_2^d \oplus P_1^d$, by the same argument as above, P_2^d is Cohen-Macaulay. Applying (3.13) again, P_2 is Cohen-Macaulay. □

6. Proof of Lemma 4.2

Now we prove the (new) result that if the lexicographic sum $\sum_{q \in Q} P_q$ is Cohen-Macaulay and if $q \neq q' \in Q$ and $r(q) = r(q')$, then both P_q and $P_{q'}$ are antichains.

For notational convenience, let $\mathscr{S}$ be the sum $\sum_{q \in Q} P_q$. Recall that a typical element of $\mathscr{S}$ is (q, x) with $q \in Q$ and $x \in P_q$. The maximal chains of $\mathscr{S}$ are of the form M: $(q_1, m_{q_1}), (q_2, m_{q_2}), \ldots, (q_n, m_{q_n})$ where $m = \{q_1, \ldots, q_n\}$ is a maximal chain of Q and for each i, m_{q_i} is a maximal chain of P_{q_i}. If M is the maximal chain of $\mathscr{S}$ described above and q is an element of Q, then we say $q \in M$ if q is equal to one of the q_i. Select any q in Q and any n, $1 \leq n \leq r(P_q)$. Observe that

$$\sum_{\substack{x \in P_q \\ r(x)=n}} L^{\mathscr{S}}(q, x) = \sum_{\substack{M \in \mathscr{M}(\mathscr{S}) \\ q \in M}} L^{\mathscr{S}} M$$

because every maximal chain of $\mathscr{S}$ containing q passes through exactly one of the points (q, x) as x varies over all the elements of P_q with a fixed rank.

We establish the Lemma by induction. Let $n \geq 1$. Suppose that if $q \in Q$ and $r(q) < n$, then either P_q is an antichain or q is the only element of Q of its rank. Let $q_1, \ldots, q_k$ be the distinct elements of Q having rank n. Assume that $k \geq 2$ and that $r(P_{q_1}) \geq 2$. By the induction hypothesis $r_{\mathscr{S}}(q_1, x_1) = r_{\mathscr{S}}(q_2, x_2)$ if x_i is a rank one element of P_{q_i}. It follows from the preceding display that

$$\sum_{\substack{x \in P_{q_1} \\ r(x)=2}} L^{\mathscr{S}}(q_1, x) = L^{\mathscr{S}}\varnothing - \sum_{i=2}^{k} \sum_{\substack{y \in P_{q_i} \\ r(y)=1}} L^{\mathscr{S}}(q_i, y),$$

which is an element of

$$L^{\mathscr{S}} H_S \cap \sum_{T<S} L^{\mathscr{S}} H_T$$

where $\mathscr{S}$ is the singleton consisting of the rank of (q_1, x), and where the element x of P_{q_1} has rank 2. This element is not in $\sum_{T \subsetneq S} L^{\mathscr{S}} H_T$, which is the k-vector space spanned by $L^{\mathscr{S}}\varnothing$. By (3.8) of Theorem 3.1, $\mathscr{S}$ is not Cohen-Macaulay. Thus, either $k = 1$ or $r(P_{q_i}) = 1$ for $1 \leq i \leq k$. □

7. Proof of Lemma 4.3

Next we show that if Q is Cohen-Macauley and $q \in Q$, then the restriction $Q|q$ is also Cohen-Macaulay. This new result can be proven algebraically by localizing the ring $R[P]$ at the element x_q, and it can be proven topologically, but we will present here an elementary combinatorial proof.

Label Q so that q is the lexicographically largest element of its rank. Let $\mathscr{B}$ be the lexicographically least collection of chains of Q with $\{Lb | b \in \mathscr{B}\}$ a basis for $H_{[r]}$. In this proof L means L^Q and $r = r(Q)$. Observe that if c is a chain of Q containing q, then Lc is equal to the following linear combination involving lexicographically smaller chains:

$$Lc = L(c \backslash q) - \sum_{\substack{y \in Q \\ y \neq q \\ r(y)=r(q)}} L((c \backslash q) \cup y).$$

Consequently, if $b \in \mathscr{B}$, then q is not an element of b.

The ring $R[Q|q]$ is a subring of $R[Q]$. If c is a chain, then the maximal chains of $Q|q$ which contain c are precisely the maximal chains of Q which contain c and q; consequently,

$$L^{Q|q} c = L(c \cup q).$$

Of course, both sides are 0 if $c \cup q$ is not a chain.

Let $\overline{\mathscr{B}}$ be the lexicographically least collection of chains of $Q|q$ such that $\{L(\bar{b} \cup q) | \bar{b} \in \overline{\mathscr{B}}\}$ is a basis for $H_{[r]}^{Q|q}$. By Theorem 3.1 and (3.2) it will suffice to show hat for each chain c in Q there are $\alpha_{\bar{b}} \in k$ so that

$$L(c \cup q) = \sum_{\substack{\bar{b} \in \overline{\mathscr{B}} \\ r(\bar{b}) \subseteq r(c)}} \alpha_{\bar{b}} L(\bar{b} \cup q). \tag{7.1}$$

Suppose that every chain of Q which is lexicographically smaller than c can be written in form (7.1). Observe that if $q \in c$, then $c \backslash q$ is lexicographically less than c; hence,

$$L(c \cup q) = L((c \backslash q) \cup q) = \sum_{\substack{\bar{b} \in \mathscr{B} \\ r(\bar{b}) \subseteq r(c \backslash q)}} \alpha_{\bar{b}} L(\bar{b} \cup q).$$

Since $r(c \backslash q)$ is contained in $r(c)$, this expression has the form of (7.1). Henceforth, let $S = r(c)$. We may assume that $r(q) \notin S$ and that $c \notin \mathscr{B}$. Thus

$$L(c \cup q) = \sum_{\substack{\bar{b} \in \mathscr{B} \\ \bar{b} < c}} \beta_{\bar{b}} L(\bar{b} \cup q).$$

If $\bar{b} < c$, then either $r(\bar{b}) < S$, or $r(\bar{b}) = S$ and $\bar{b} < c$. Let

$$\zeta = L(c \cup q) - \sum_{\substack{\bar{b} \in \mathscr{B} \\ \bar{b} < c \\ r(\bar{b}) = S}} \beta_{\bar{b}} L(\bar{b} \cup q) = \sum_{\substack{\bar{b} \in \mathscr{B} \\ r(\bar{b}) < S}} \beta_{\bar{b}} L(\bar{b} \cup q). \tag{7.2}$$

Since Q is Cohen-Macaulay, Lc' can be written uniquely in the form

$$\sum_{\substack{b \in \mathscr{B} \\ r(b) \subseteq r(c')}} \alpha_b Lb$$

for all chains c' of Q. If we write (7.2) in terms of Lb for $b \in \mathscr{B}$ we have

$$\zeta = \sum_{\substack{b \in \mathscr{B} \\ r(b) \subseteq S \cup r(q)}} \gamma_b Lb = \sum_{\substack{\bar{b} \in \mathscr{B} \\ r(\bar{b}) < S}} \sum_{\substack{b \in \mathscr{B} \\ r(b) \subseteq r(\bar{b}) \cup q}} \gamma_{b\bar{b}} Lb \tag{7.3}$$

Observe that if $r(b) = S$, then the right hand expression for ζ in (7.3) does not involve Lb, because $r(q) \notin S \subseteq r(\bar{b} \cup q)$ implies $S \subseteq r(\bar{b})$ which implies $S \leq r(\bar{b}) < S$, which is impossible. Since $\{Lb \mid b \in \mathscr{B}\}$ is a basis for $H_{[r]}$, this implies that in the left hand expression for ζ in (7.3), $\gamma_b = 0$ for all b with $r(b) = S$. Hence,

$$\begin{aligned} \zeta &= \sum_{\substack{b \in \mathscr{B} \\ r(b) \subseteq S \cup r(q) \\ r(b) \neq S}} \gamma_b Lb \\ &= \sum_{\substack{b \in \mathscr{B} \\ r(b) \subsetneq S}} \gamma_b Lb + \sum_{\substack{b \in \mathscr{B} \\ r(q) \in r(b) \subseteq S \cup r(q)}} \gamma_b Lb. \end{aligned} \tag{7.4}$$

Let $\sigma \colon R[Q] \to R[Q|q]$ be the ring map which sends all elements of Q not comparable to q to zero and all elements comparable to q to themselves. If b is a chain of Q, then $\sigma(Lb) = L(b \cup q)$. Since $\zeta \in R[Q|q]$ we have $\sigma(\zeta) = \zeta$. Apply σ to the last expression in (7.4) for ζ to get

$$\zeta = \sum_{\substack{b \in \mathscr{B} \\ r(b) \subsetneq S}} \gamma_b L(b \cup q) + \sum_{\substack{b \in \mathscr{B} \\ r(q) \in r(b) \subseteq S \cup r(q)}} \gamma_b L(b \cup q).$$

We have seen that if b is in $\mathscr{B}$, then q is not an element of b. Thus if $r(q) \in r(b)$, then $b \cup q$ is not a chain and $L(b \cup q) = 0$. Hence

$$\zeta = \sum_{\substack{b \in \mathscr{B} \\ r(b) \subsetneq S}} \gamma_b L(b \cup q),$$

and from (7.2)

$$L(c \cup q) = \sum_{\substack{\bar{b} \in \bar{\mathscr{B}} \\ \bar{b} < c \\ r(\bar{b}) = S}} \alpha_{\bar{b}} L(\bar{b} \cup q) + \sum_{\substack{b \in \mathscr{B} \\ r(b) \subsetneq S}} \gamma_b L(b \cup q). \qquad (7.5)$$

All of the $\bar{b}$ *and* b on the right side of (7.5) are lexicographically smaller than c. By induction each term $L(b \cup q)$ on the right hand side can be put in form (7.1). Thus $L(c \cup q)$ can be put in form (7.1) and $Q|q$ is Cohen-Macaulay.

8. Proof of Lemma 4.4

For the last of this series of lemmas, we are to show that if $q \in Q$, then the extension $Q \propto q$ is Cohen-Macaulay over k if and only if Q itself is Cohen-Macaulay.

Assume that Q is Cohen-Macaulay. Let $\bar{Q} = Q|q$, $\tilde{Q} = Q \propto q$, $L = L^Q$, $\bar{L} = L^{\bar{Q}}$, $\tilde{L} = L^{\tilde{Q}}$. and $r = r(Q) = r(\bar{Q}) = r(\tilde{Q})$. Let $\tilde{q}$ denote the extra copy of q in $\tilde{Q}$. Let $\mathscr{B}$ be a collection of chains of Q satisfying the conditions of Lemma 3.1. By Lemma 4.3, $\bar{Q}$ is also Cohen-Macaulay. Let $\bar{\mathscr{B}}$ be the lexicographically least collection of chains of $\bar{Q}$ with $\{\bar{L}\bar{b} | \bar{b} \in \bar{\mathscr{B}}\}$ a basis for $H^{\bar{Q}}_{[r]}$. If $\bar{c}$ is a chain of $\bar{Q}$ and $q \in \bar{c}$, then $\bar{c} \backslash q$ is lexicographically less than $\bar{c}$ and $\bar{L}(\bar{c} \backslash q) = \bar{L}\bar{c}$. It follows that if $\bar{b}$ is in $\bar{\mathscr{B}}$, then q is not an element of $\bar{b}$. Let $\tilde{\mathscr{B}}$ be the set of chains

$$\mathscr{B} \cup \{\bar{b} \cup \tilde{q} | \bar{b} \in \bar{\mathscr{B}}\}.$$

We will show that $\tilde{\mathscr{B}}$ satisfies (3.4) of Lemma 3.1.

Let $\mathscr{C} = \mathscr{C}(Q)$, $\tilde{\mathscr{C}} = \mathscr{C}(\tilde{Q})$, and $\bar{\mathscr{C}} = \mathscr{C}(\bar{Q})$; similarly, let $\mathscr{M} = \mathscr{M}(Q)$, $\tilde{\mathscr{M}} = \mathscr{M}(\tilde{Q})$, and $\bar{\mathscr{M}} = \mathscr{M}(\bar{Q})$. If S is a subset of $[r]$, then

$$\begin{aligned} |\{\tilde{c} \in \tilde{\mathscr{C}} | r(\tilde{c}) = S\}| &= |\{\tilde{c} \in \tilde{\mathscr{C}} | r(\tilde{c}) = S, \tilde{q} \notin \tilde{c}\}| + |\{\tilde{c} \in \tilde{\mathscr{C}} | r(\tilde{c}) = S, \tilde{q} \in \tilde{c}\}| \\ &= |\{c \in \mathscr{C} | r(c) = S\}| + \chi(r(q) \in S) |\{\bar{c} \in \bar{\mathscr{C}} | r(\bar{c}) = S\}| \end{aligned}$$

where

$$\chi(r(q) \in S) = \begin{cases} 1 \text{ if } r(q) \in S, \\ 0 \text{ otherwise.} \end{cases}$$

Similarly

$$\begin{aligned} |\{\tilde{b} \in \tilde{\mathscr{B}} | r(\tilde{b}) \in S\}| &= |\{\tilde{b} \in \tilde{\mathscr{B}} | r(\tilde{b}) \subseteq S, \tilde{q} \notin \tilde{b}\}| + |\{\tilde{b} \in \tilde{\mathscr{B}} | r(b) \subseteq S, \tilde{q} \in \tilde{b}\}| \\ &= |\{b \in \mathscr{B} | r(b) \subseteq S\}| + |\{\bar{b} \cup \tilde{q} | \bar{b} \in \bar{\mathscr{B}} \text{ and } r(\bar{b} \cup \tilde{q}) \subseteq S\}| \\ &= |\{b \in \mathscr{B} | r(b) \subseteq S\}| + \chi(r(q) \in S) |\{\bar{b} \in \bar{\mathscr{B}} | r(\bar{b}) \subseteq S\}|. \end{aligned}$$

Since $\mathscr{B}$ and $\bar{\mathscr{B}}$ both satisfy (3.4a) of Lemma 3.1, we conclude that $\tilde{B}$ also satisfies this condition.

In order to show that $\tilde{\mathscr{B}}$ satisfies (3.4b) we must carefully distinguish L, $\bar{L}$, and $\tilde{L}$. Let l be the rank of q. Let c, $\bar{c}$, $\tilde{c}$ be chains of Q, $\bar{Q}$, $\tilde{Q}$ respectively. Then

$$Lc = x(c) \prod_{i \in [r] \backslash r(c)} \theta_i, \quad \bar{L}\bar{c} = x(\bar{c}) \prod_{i \in [r] \backslash r(\bar{c})} \bar{\theta}_i, \quad \tilde{L}\tilde{c} = x(\tilde{c}) \prod_{i \in [r] \backslash r(c)} \tilde{\theta}_i$$

where

$$\theta_i = \sum_{\substack{y \in Q \\ r(y) = i}} x_y, \quad \bar{\theta}_i = \sum_{\substack{y \in \bar{Q} \\ r(y) = i}} x_y, \text{ and } \tilde{\theta}_i = \sum_{\substack{y \in \tilde{Q} \\ r(y) = i}} x_y.$$

It is clear that

$$\tilde{\theta}_i = \theta_i \qquad \text{if } i \neq l,$$
$$\tilde{\theta}_i = \theta_i + x(\tilde{q}) \qquad \text{if } i = l,$$
$$x(\tilde{q})\tilde{\theta}_i = x(\tilde{q})\bar{\theta}_i \qquad \text{if } i \neq l.$$

If $\tilde{m}$ is a maximal chain of $\tilde{Q}$, then there are two cases: either $\tilde{q} \in \tilde{m}$ or $\tilde{q} \notin \tilde{m}$.

Case 1. Suppose $\tilde{q} \in \tilde{m}$. Since $\tilde{m} \backslash \tilde{q}$ is a chain of $\bar{Q}$, we have

$$\bar{\theta}_l x(\tilde{m}\backslash\tilde{q}) = \bar{L}(\tilde{m}\backslash\tilde{q}) = \sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} \overline{Lb} = \sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} x(\bar{b}) \prod_{i \in [r] \backslash r(\bar{b})} \bar{\theta}_i.$$

Since $\bar{b} \in \bar{\mathscr{B}}$ implies that $q \notin \bar{b}$, which implies $l \notin r(\bar{b})$, we have that

$$\bar{\theta}_l x(\tilde{m}\backslash\tilde{q}) = \bar{\theta}_l \sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} x(\bar{b}) \prod_{i \in [r] \backslash r(\bar{b} \cup \tilde{q})} \bar{\theta}_i.$$

Cancel the non-zero divisor $\bar{\theta}_l$ and multiply by $x(\tilde{q})$ to see that

$$L(\tilde{m}) = x(\tilde{m}) = x(\tilde{q})x(\tilde{m}\backslash\tilde{q}) = \sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} x(\bar{b} \cup \tilde{q}) \prod_{i \in [r] \backslash r(\bar{b} \cup \tilde{q})} \bar{\theta}_i$$

Since $x(\tilde{q})\bar{\theta}_i = x(\tilde{q})\tilde{\theta}_i$ for $i \neq l$, the last expression is equal to

$$\sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} x(\bar{b} \cup \tilde{q}) \prod_{i \in [r] \backslash r(b \cup \tilde{q})} \tilde{\theta}_i = \sum_{\bar{b} \in \bar{\mathscr{B}}} \alpha_{\bar{b}} \tilde{L}(\bar{b} \cup \tilde{q}).$$

This is a linear combination of the $\tilde{L}\tilde{b}$, $\tilde{b} \in \tilde{\mathscr{B}}$.

Case 2. Suppose that $\tilde{q} \notin \tilde{m}$. Then

$$\tilde{L}\tilde{m} = L\tilde{m} = \sum_{b \in \mathscr{B}} \alpha_b Lb = \sum_{b \in \mathscr{B}} \alpha_b x(b) \prod_{r \in [r] \backslash r(b)} \theta_i$$

Since $\tilde{\theta}_i = \theta_i$ for $i \neq l$ and $\tilde{\theta}_l = \theta_l + x(\tilde{q})$, we have

$$\tilde{L}\tilde{m} = \sum_{b \in \mathscr{B}} \alpha_b \tilde{L}b - \sum_{\substack{m \in \mathscr{M} \\ \tilde{q} \in m}} \beta_m \tilde{L}m$$

for some $\beta_m \in k$. By Case 1 each $\tilde{L}m$ is a linear combination of the $\tilde{L}(\bar{b} \cup \tilde{q})$, and thus, of the $\tilde{L}\tilde{b}$.

Hence, for all maximal chains $\tilde{m}$ of $\tilde{Q}$,

$$\tilde{L}\tilde{m} = \sum_{\tilde{b} \in \tilde{\mathscr{B}}} \alpha_{\tilde{b}} \tilde{L}\tilde{b}.$$

Thus $\tilde{Q}$ is Cohen-Macaulay.

We now prove the converse. Assume $Q \propto q$ is Cohen-Macaulay. Let the elements of Q with the same rank as q be $q = q_1, \ldots, q_m$. If the characteristic of the field k is not 2, let $n = 2$ and $\tilde{Q} = Q \propto q_1 \propto \cdots \propto q_m$. If char $k = 2$, let $n = 3$ and $\tilde{Q} = Q \propto q_1 \propto \cdots \propto q_m \propto q_1 \propto \cdots \propto q_m$. By repeated applications of the direction of the Lemma which has been established we have $Q \propto q$ is Cohen-Macaulay implies $\tilde{Q}$ is Cohen-Macaulay.

As always let $\mathscr{C}$ be the chains of Q, $\tilde{\mathscr{C}}$ the chains of $\tilde{Q}$, $L = L^Q$, $\tilde{L} = L^{\tilde{Q}}$, and $r = r(Q) = r(\tilde{Q})$. Let $f\colon \tilde{Q} \to Q$ be the poset map which is the identity on Q and which send each duplication of q_i to q_i. Extend f the natural way to obtain maps $f\colon \tilde{\mathscr{C}} \to \mathscr{C}$ and $f\colon R[\tilde{Q}] \to R[Q]$.

Claim 1. *If* $\sum_{c\in\mathscr{C}} \alpha_c Lc = 0$, *then* $\sum_{\tilde{c}\in\tilde{\mathscr{C}}} \alpha_{f(\tilde{c})} \tilde{L}\tilde{c} = 0$.

Proof. The sum $\sum_{c\in\mathscr{C}} \alpha_c Lc = 0$ means that if m is a maximal chain of Q and $c_1, \ldots, c_N$ are the subchains of m, then $\sum_{i=1}^{N} \alpha_{c_i} = 0$. If $\tilde{m}$ is a maximal chain of $\tilde{Q}$ and $\tilde{c}_1, \ldots, \tilde{c}_N$ are the subchains of $\tilde{m}$, then $f(\tilde{m})$ is a maximal chain of Q and $f(\tilde{c}_1), \ldots, f(\tilde{c}_N)$ are the subchains of $f(\tilde{m})$. Hence $\sum_{i=1}^{N} \alpha_{f(\tilde{c}_i)} = 0$, and $\sum_{\tilde{c}\in\tilde{\mathscr{C}}} \alpha_{f(\tilde{c})} \tilde{L}\tilde{c} = 0$.

Claim 2. *If* $\zeta = \sum_{c\in\mathscr{C}} \alpha_c Lc$, *and* $\tilde{\zeta} = \sum_{\tilde{c}\in\tilde{\mathscr{C}}} \alpha_{f(\tilde{c})} \tilde{L}\tilde{c}$, *then* $f(\tilde{\zeta}) = n\zeta$ *where* $n = 2$ *or* 3 *or defined earlier.*

Proof. Let l be the rank of the elements $q_1, \ldots, q_m$. If $l \in r(c)$, then there are n chains $\tilde{c} \in \tilde{\mathscr{C}}$ with $f(\tilde{c}) = c$, but for each of these $f(\tilde{L}\tilde{c}) = Lc$. If $l \notin r(c)$, then there is only one chain $\tilde{c} \in \tilde{\mathscr{C}}$ with $f(\tilde{c}) = c$, but $f(\tilde{L}\tilde{c}) = nLc$. It follows that $f(\tilde{\zeta}) = n\zeta$.

We use (3.4) of Theorem 3.1 to show that Q is Cohen-Macaulay. Let S be a subset of $[r]$ and let $\zeta = \sum_{r(c)=S} \alpha_c Lc = \sum_{r(c)<S} \alpha_c Lc$ be an element of $LH_S \cap \sum_{T<S} LH_T$. Let $\tilde{\zeta}$ be $\sum_{r(\tilde{c})=S} \alpha_{f(\tilde{c})} \tilde{L}\tilde{c}$. It follows from Claim 1 that $\tilde{\zeta}$ is also equal to $\sum_{r(\tilde{c})<S} \alpha_{f(\tilde{c})} Lc$, thus $\tilde{\zeta}$ is in $\tilde{L}H_S^{\tilde{Q}} \cap \sum_{T<S} \tilde{L}H_T^{\tilde{Q}}$. Since $\tilde{Q}$ is Cohen-Macaulay, $\tilde{\zeta}$ is in $\sum_{T\subsetneq S} \tilde{L}H_T^{\tilde{Q}}$ by Theorem 3.1. Thus for some $\beta_{\tilde{c}}$, $\tilde{\zeta} = \sum_{r(c)\subsetneq S} \beta_{\tilde{c}} Lc$. Now consider $f(\tilde{\zeta}) = \sum_{r(\tilde{c})\subsetneq S} \beta_{\tilde{c}} f(\tilde{L}\tilde{c})$. By analysis in the proof of Claim 2 it follows that $f(\tilde{\zeta}) = \sum_{r(c)\subsetneq S} \beta_c Lc$, where

$$\beta_c = \begin{cases} \sum_{f(\tilde{c})=c} \beta_{\tilde{c}}, \text{ if } l \in r(c) \\ n\beta_{\tilde{c}}, \text{ where } \tilde{c} = f^{-1}(c), \text{ if } l \notin r(c). \end{cases}$$

Thus $f(\tilde{\zeta}) \in \sum_{T\subsetneq S} LH_T$. By Claim 2, $f(\tilde{\zeta}) = n\zeta$, and, by design, n is a unit; therefore ζ itself belongs to $\sum_{T\subsetneq S} LH_T$, and Q is Cohen-Macaulay. □

9. Proof of Theorem 4.1

We now combine the lemmas above to prove our main theorem characterizing when the lexicographic sum $\sum_{q\in Q} P_q$ is Cohen-Macaulay over k.

The poset Q is the sum $Q_1 \oplus Q_2 \oplus \cdots \oplus Q_N$ where, for all i, Q_i is the union of consecutive ranks of Q, and Q_i is either a singleton or else each rank of Q_i has more than one element. Thus $\sum_{q\in Q} P_q = \bigoplus_{i=1}^{N} \sum_{q\in Q_i} P_q$.

If $\sum_{q\in Q} P_q$ is Cohen-Macaulay, then by Lemma 4.1 each $\sum_{q\in Q_i} P_q$ is Cohen-Macaulay. If Q_i is the singleton $\{q_i\}$, then $\sum_{q\in Q_i} P_q$ is P_q and P_q is Cohen-Macaulay. If each rank of Q_i has more than one element, then by Lemma 4.2, P_q is an antichain for each $q \in Q_i$. Thus (4.1) and (4.3) of the Theorem hold. By repeated application of Lemma 4.4 any Q_i which is not a singleton is Cohen-Macaulay, thus by Lemma 4.1, $Q = Q_1 \oplus \cdots \oplus Q_N$ is Cohen-Macaulay, which is (4.2).

Conversely, if Conditions (4.1) through (4.3) hold, then each Q_i is Cohen-Macaulay by Lemma 4.1, hence by Lemma 4.4 each $\sum_{q\in Q_i} P_q$ is Cohen-Macaulay. Thus $\sum_{q\in Q} P_q$ is Cohen-Macaulay by Lemma 4.1. □

Note that Lemma 4.3 is not used directly in the proof above. However, it is used to prove Lemma 4.4, which is needed to prove the theorem. It is also crucial in the proof we gave to Corollary 4.2.

10. Shellable Posets

We now turn our attention to the notion of shellability, which originated in the study of polyhedra. A finite simplicial complex Δ is *pure d-dimensional* if its maximal faces, called *facets*, each contain $d + 1$ vertices. Such a simplicial complex is *shellable* if its facets can be ordered $F_1, F_2, \ldots, F_t$ in such a way that for all $i < j$ there exists $k < j$ such that $F_i \cap F_j \subseteq F_k \cap F_j$ and $|F_k \cap F_j| = d$. Such an ordering of the facets of Δ is called a *shelling*.

McMullen [15] used shellability to prove Motzkin's upper bound conjecture for convex polytopes Δ. The conjecture gives the maximum possible number $f_i(n, d)$ of i-dimensional faces of any d-dimensional convex polytope with n vertices. Klee extended the upper bound conjecture to arbitrary manifolds. The case in which the geometric realization $|\Delta|$ of Δ is a sphere was proven by Stanley [17]. Shellability could not be used because there exist triangulations of spheres which are not shellable. Instead, Stanley used the Cohen-Macaulay property, which holds for Δ if $|\Delta|$ is a sphere.

For further information on shellability see [5, 7]. Björner [5, p.183] notes that shellable complexes are Cohen-Macaulay but the converse is not true in general. Indeed, although Cohen-Macaulayness is preserved under homeomorphisms, shellability is not, e.g., consider the nonshellable sphere mentioned above.

Shellability carries down to order theory directly by saying a poset P is *shellable* if its order complex $\Delta(P)$ is shellable. Most interesting examples of Cohen-Macaulay posets are actually shellable, e.g., finite distributive, semimodular, and supersolvable lattices. So the interest in shellability for order theorists is that one can show combinatorially that a poset is Cohen-Macaulay (and therefore has nice algebraic and combinatorial interpretations) by giving a shelling order of its maximal chains.

Generalizing a technique of Stanley for labelling the edges of the Hasse diagram of certain nice lattices, Björner [5] introduced a condition on such a labelling for any finite pure poset P, now called EL-shellability, which implies shellability. A slightly more general version of this concept, CL-shellability, was formulated by Björner and Wachs in [7] and further studied in [8]. Thus EL-shellability implies CL-shellability, which in turn implies shellability. It has been shown that shellability need not imply CL-shellability, by Vince and Wachs [18] and by Walker [20]. However, it remains an open problem to find a CL-shellable poset which is not EL-shellable. In practice the notion of CL-shellability has proven to be particularly valuable because contructing a CL-shellability labelling of the maximal chains of a poset is typically the easiest way to show that a poset is Cohen-Macaulay. Refer to the survey [6] for more about these properties. In subsequent sections we define EL- and CL-shellability and consider their ordinal sums.

Now we consider constructions of shellable posets. Using the shelling order, Björner [5, 4.4] proved the analogue of our Lemma 4.1:

Theorem 10.1. *The poset $P \oplus Q$ is shellable if and only if P and Q are shellable.*

Since shellable posets are Cohen-Macaulay [5], the analogue of Lemma 4.2 for shellable posets is immediate from Lemma 4.2. The analogue of Lemma 4.3 is easy to verify:

Theorem 10.2. *If Q is shellable and if $q \in Q$, then $Q|q$ is shellable.*

If Q is shellable and $q \in Q$, one can also construct a shelling of the extension $Q \propto q$, as follows: In a given shelling of the maximum chains of Q, each time a chain appears which contains q, call it c, insert immediately after it the chain $c' = c \cup \{q'\} \setminus \{q\}$, where q' is the new copy of q. This proves one direction of the analogue of Lemma 4.4:

Theorem 10.3. *If Q is shellable and if $q \in Q$, then $Q \propto q$ is shellable.*

The converse of this theorem, which is the analogue of the other direction of Lemma 4.4, seems equally natural, but we have not succeeded in proving it:

Conjecture. If $q \in Q$ and if $Q \propto q$ is shellable, then Q is shellable.

This conjecture is the last piece of the puzzle needed to prove the analogue of our main ordinal sum theorem. Thus it is equivalent to this

Conjecture. Theorem 4.1 remains true when "Cohen-Macaulay" is replaced by "shellable".

The shellable analogue of Corollary 4.1 for $\sum_Q P$ would follow from this conjecture. The analogue of Corollary 4.2, which says that an interval in a shellable poset is shellable, was noted in [5, 4.2] and follows from the analogue of Lemma 4.3, Björner proved the analogue of Corollary 4.3 in [5, 4.1], that rank-selected subposets P_T of a shellable poset P are shellable. That every shellable poset is rank-connected, the analogue of Corollary 4.4, is well-known and immediate from the definition of shellable.

11. EL-Shellable Posets

We now present the definition of EL-shellable posets, as introduced in [5], and study some of their properties. "EL" stands for "edgewise-lexicographically". Assume P is graded (not merely pure). We label each edge $x \to y$ in the Hasse diagram of P by an integer $\lambda(x \to y)$. An edge-labelling λ induces a labelling of each unrefinable chain $c = (x_0 \to x_1 \to \cdots \to x_k)$ in P by $\lambda(c) = (\lambda(x_0 \to x_1), \ldots, \lambda(x_{k-1} \to x_k))$. We may then order the maximal chains in an interval $[x, y]$ of P lexicographically according to the corresponding labels: We write $c <_L c'$ if $\lambda(c)$ lexicographically precedes $\lambda(c')$. A chain $c = (x_0 \to x_1 \to \cdots \to x_k)$ is *increasing* if $\lambda(x_0 \to x_1) \leq \lambda(x_1 \to x_2) \leq \cdots \leq \lambda(x_{k-1} \to x_k)$. P is said to be *EL-shellable* if it admits a labelling λ of its edges such that for every interval $[x, y]$ in P,

(11.1) There is a unique increasing maximal chain c in $[x, y]$, and

(11.2) $c <_L c'$ for all other maximal chains c' in $[x, y]$.

Such a labelling λ is called an *EL-labelling*. All distributive, semimodular and supersolvable lattices are EL-shellable Posets.

Less is known about how constructions are preserved for EL-shellability than for the other properties. As noted by Björner [5, 4.4] it is easy to prove the analogue of one direction of our Lemma 4.1. Recall that $\hat{P}$ is P with $\hat{0}$ and $\hat{1}$ added.

Theorem 11.1. *If $\hat{P}$, $\hat{Q}$ are EL-shellable, then $\hat{P} \oplus \hat{Q}$ is EL-shellable.*

The analogue of the other direction of Lemma 4.1, the converse of the theorem above, remains open. The analogue of Lemma 4.2 follows directly from 4.2 itself. The analogue of Lemma 4.3 is again easy to verify:

Theorem 11.2. *If q is an element of the EL-shellable poset Q, then $Q|q$ is EL-shellable.*

Both directions of the analogue of Lemma 4.4, concerning whether the extension $Q \propto q$ is EL-shellable, are open. One needs to prove both directions of this analogue as well as the missing direction of the analogue of Lemma 4.1 to prove the analogue of the main result for ordinal sums, Theorem 4.1. For now, we will merely make this weaker conjecture, which generalizes Theorem 11.1 above:

Conjecture. If each poset $\hat{P}_q$, $q \in Q$, is EL-shellable, if $\hat{Q}$ is EL-shellable, and if P_q and $P_{q'}$ are both antichains whenever q, $q' \in Q$ have the same rank then $\widehat{\sum_{q \in Q} P_q}$ is EL-shellable.

The analogue of Corollary 4.2, which follows from Theorem 11.2 above, has previously been verified by Björner [5, 4.2]: An interval in an EL-shellable poset is EL-shellable. However, the analogue of Corollary 4.3 remains open: Is the rank-selected subposet $\hat{P}_T$ of an EL-shellable poset $\hat{P}$ necessarily EL-shellable? This is frustrating inasmuch as Corollary 4.3 guarantees that $\hat{P}_T$ is at least Cohen-Macaulay. Finally, we note that EL-shellable posets are rank-connected by Corollary 4.4.

12. CL-Shellable Posets

Björner and Wachs [7] introduced a slightly more general definition of lexicographic shellability than EL-shellability, called CL-shellability, "CL" for "chainwise-lexicographic". As we noted earlier, this notion is known to be strictly stronger than shellability itself. By design it is weaker than EL-shellability, and probably strictly weaker. Although there are some classes of posets that have been shown to be CL-shellable for which no EL-labelling has been found, e.g., Bruhat order, no one has yet proven that there exist any CL-shellable posets which are not EL-shellable [6].

The idea of CL-shellability is to label the chains themselves, rather than the edges, and to put a condition on the labelling which is sufficiently strong to preserve the argument that the lexicographic order on the maximal chains of P is a shelling, so that P is shellable (and, hence, Cohen-Macaulay).

Let P be a graded poset of rank $d + 2$. We consider labellings λ of the maximal chains $m = (\hat{0} = x_0 \to x_1 \to \cdots \to x_{d+1} = \hat{1})$ of P where each such chain receives a label $\lambda(m) = (\lambda_1(m), \lambda_2(m), \ldots, \lambda_{d+1}(m))$, each $\lambda_i(m)$ an integer, such that whenever two maximal chains m and m' coincide in their first k edges, then $\lambda_i(m) = \lambda_i(m')$, for $1 \leq i \leq k$. If $[x, y]$ is an interval and c is an unrefinable chain from $\hat{0}$ to x, then the pair $(c, [x, y])$ is called a *rooted interval.* If b is a maximal chain in $[x, y]$, it makes sense to define the rooted label $\lambda^c(b) = (\lambda_{r(x)+1}(m), \ldots, \lambda_{r(y)}(m))$, where m is any maximal chain in P containing c and b, since this label does not depend on m. We now lexicographically order the maximal chains b in the rooted interval $(c, [x, y])$ by $b <_L b'$ is $\lambda^c(b)$ lexicographically precedes $\lambda^c(b')$. The chain b is *increasing* if

$\lambda_{r(x)+1}(m) \leq \cdots \leq \lambda_{r(y)}(m)$. Then we can define P to be *CL-shellable* if for every rooted interval $(c, [x, y])$,

(12.1) there is a unique increasing maximal chain b in $(c, [x, y])$, and

(12.2) $b <_L b'$ for all other maximal chains b' in $(c, [x, y])$.
Such a labelling λ is called a *CL-labelling*.

An EL-labelling of the edges of a poset induces a CL-labelling of the maximal chains automatically, but not conversely: A given edge $x \to y$ receives different labels, depending on the root taken from 0 to x, so no edge-labelling is induced by a CL-labelling.

For CL-shellable posets all analogues of the results above for Cohen-Macaulay posets work. In particular, we shall show this:

Theorem 12.1. *The poset $\widehat{\sum_{q\in Q} P_q}$ is CL-shellable if and only if all of the following conditions hold:*

(12.1) *Each poset $\widehat{P_q}$ is CL-shellable,*

(12.2) *The poset $\hat{Q}$ is CL-shellable, and*

(12.3) *If q and q' are distinct elements of Q having the same rank, then P_q and $P_{q'}$ are both antichains.*

Corollary 12.1. The poset $\widehat{\sum_Q P}$ is CL-shellable if and only if one of the following conditions hold:

(12.4) Q is a chain and $\hat{P}$ is CL-shellable, or

(12.5) P is an antichain and $\hat{Q}$ is CL-shellable.

The proof of Theorem 12.1 follows from analogues of Lemmas 4.1, 4.2, 4.3, and 4.4 just as Theorem 4.1 followed from those lemmas. The analogue of Lemma 4.1, which states that $\widehat{P \oplus Q}$ is CL-shellable if and only if $\hat{P}$ and $\hat{Q}$ are CL-shellable was discovered by Björner and Wachs [8, 8.6]. The analogue of Lemma 4.2 is a special case of Lemma 4.2 itself since CL-shellable posets are Cohen-Macaulay. Next we prove the analogue of Lemma 4.3.

Lemma 12.1. *If Q is CL-shellable and $q \in Q$, then $Q|q$ is CL-shellable.*

Proof. Take a CL-labelling of Q and add a very large number to the labels corresponding to edges in $Q|q$ above q: Specifically, let N be an integer greater than $\lambda_i(m)$ for all i and all maximal chains m in $Q|q$. Let $\lambda'(m) = (\lambda'_1(m), \ldots, \lambda'_{d+1}(m))$, where

$$\lambda'_i(m) = \begin{cases} \lambda_i(m) & \text{if } i \leq r(q) \\ \lambda_i(m) + N & \text{if } i > r(q) \end{cases}$$

Then it is easily verified that λ' is a CL-labelling of $Q|q$. □

Before discussing the analogue of Lemma 4.4, we consider the analogues of the other results, Corollaries 4.2, 4.3, and 4.4 The analogue of Corollary 4.2, that every interval in a CL-shellable poset is CL-shellable, is immediate from the definition.

The analogue of Corollary 4.3 that states that the rank-selected subposet P_T of a CL-shellable poset $\hat{P}$ is CL-shellable was proven by Björner and Wachs [8, 8.1]. The analogue of Corollary 4.4 that states that CL-shellable posets are rank-connected is a special case of Corollary 4.4 itself.

It remains to prove the analogue of Lemma 4.4 that if $q \in Q$, then $Q \propto q$ is CL-shellable if and only if $\hat{Q}$ is too. This is Lemma 12.2 below. As we noted with EL-shellability, it is not clear how to prove this from the definition of lexicographic labellings. What enables us to establish the result in this case, unlike before, is a powerful tool for proving results about CL-shellability due to Björner and Wachs.

Recall that the elements of a graded poset which cover $\hat{0}$ are called *atoms*. A graded poset P of rank $d+2$ is said to admit a *recursive atom ordering* (RAO) if $d=0$ or if $d>0$ and there is an ordering $a_1, a_2, \ldots, a_t$ of the atoms of P which satisfies:

(12.3) For all $j = 1, 2, \ldots, t$, $[a_j, \hat{1}]$ admits a RAO in which the atoms of $[a_j, \hat{1}]$ that come first in the ordering are those that cover some a_i where $i < j$.

(12.4) For all $i < j$, if $a_i, a_j < y$ then there exists $k < j$ and an element z such that $a_k, a_j \rightarrow z \leq y$.

Note that a RAO does not order the elements of P of every rank. For example, the same elements of rank 2 may be ordered differently in the atom orderings of different intervals $[a_j, \hat{1}]$.

Although RAO is harder to define and to grasp than CL-shellability, it is more useful for induction proofs. The key result is this.

Theorem 12.2. [8, 3.2] *A graded poset P admits an RAO if and only if P is CL-shellable.*

We can now prove our Lemma.

Lemma 12.2 *If $q \in Q$, then $\widehat{Q \propto q}$ is CL-shellable if and only if $\hat{Q}$ is CL-shellable.*

Proof. First suppose $\widehat{Q \propto q}$ has a RAO. If q is not an atom of $\hat{Q}$, then $\hat{Q}$ has a RAO by induction on the rank of Q. So suppose q is an atom of $\hat{Q}$. Then take the RAO for $\widehat{Q \propto q}$ and delete the copy of q which comes later in the atom ordering. It can be checked that this leaves a RAO for $\hat{Q}$.

Conversely, suppose $\hat{Q}$ has a RAO and $q \in Q$. By induction on the rank of Q, it can be shown that $\widehat{Q \propto q}$ admits a RAO if q is not an atom of $\hat{Q}$. If q is an atom of $\hat{Q}$, simply insert the new copy of q immediately after the original copy of q in the RAO of $\hat{Q}$. It is straightforward to check that this gives a RAO of $\widehat{Q \propto q}$. □

Theorem 12.1 now follows from the results above just as Theorem 4.1 following from Lemmas 4.1–4.4.

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