Mathematics Related to Spinning Tops

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CHAPTER 1

Introduction.

This thesis is concerned with some topics related to the mechanics of moving rigid bodies, with the main goal being a description of the motion of a spinning top.

The motion of a rigid body in space can be described by giving two Euclidean coordinate systems k and K, where k is a fixed coordinate system, by which we mean that k is thought of as being "attached" to the background space. In particular, k is an inertial coordinate system. The second coordinate system, K, is "attached" to the rigid body and, therefore, is moving with respect to the fixed coordinate system k. This motion will be described as a map $B = B_t$, so that for each time t, we have a mapping $B_t \colon K \to k$ that preserves distances and orientation. Such mappings are usually called rigid motions. For each t, the map B_t relates the coordinates in the frame K of the moving body to the frame k of space.

In Chapter 2, formulas relating motion in the moving coordinate system K are related to those in the fixed coordinate system. In an inertial coordinate system, such as k, Newton's second law relating the force $\mathbf{f}(t)$ on a particle $\mathbf{q}(t)$ of mass m and acceleration $\ddot{\mathbf{q}}$ takes its usual form $\mathbf{f} = m\ddot{\mathbf{q}}$. However, if the same particle has its coordinates $\mathbf{Q}(t)$ given in the moving coordinate system K, then as K is not inertial, Newton's law can not be written in the straightforward manner $\mathbf{F} = m\ddot{\mathbf{Q}}$. Thus, the main result of Chapter 2 is the corrected version of Newton's second law (Theorem 2.2.1) in a rotating coordinate system. The extra terms that result can be understood by pretending that K is an inertial frame with three extra forces acting on the particle $\mathbf{Q}$. These are the *inertial force of rotation* , the *Coriolis force*, and the *centrifugal force*. Further, a frame attached to the earth can be viewed as a moving coordinate system due to the rotation of the earth. In §2.2.1 we show how the Coriolis force affects the path of a falling stone and the motion of a long pendulum in the latitude of Columbia, South Carolina.

In Chapter 3, we study the motion of a rigid body rotating about a stationary point that has no external forces acting on it. Among other things this models the motion of a body, such as an asteroid or spaceship with its engines offand sufficiently far away from surrounding astronomical objects, moving without any forces acting on it. This is provided we choose the coordinate system k so that the center of mass of the object in question is at the origin of k. For objects in a gravitational field, it models the motion of a top that has as its stationary point the center of gravity of the top. The main result here is the beautiful theorem of Poinsot which states that an ellipsoid (the *ellipsoid of inertia*) centered at the stationary point of the body can be attached rigidly so that the motion of this ellipsoid rolls without slipping along a fixed plane.

Finally, Chapter 4 analyzes the equations of motion for a rotational symmetric top (Lagrange's top). Unlike Chapter 3, where the equations of motion were deduced directly from Newton's second law, in this chapter, the equations of motion are derived from a variational principle, Hamilton's principle, which we assume without proof.

The presentation in this thesis follows that of Arnold [**1**], except that we do not use Neother's theorem to deduce the conservation laws such as conservation of energy and conservation of angular momentum. While Neother's theorem does give a unified treatment of these facts, it is non-trivial both to state and to apply. Also, without putting it in larger context, it does not shed any light on the geometry or the physics. Therefore, in the presentation of the conservation laws in Chapter 3, we have followed the presentation in [**4**]. In Chapter 4, the conservation laws are derived by direct calculation, which amounts to the proofs of Neother's theorem in the cases we need.

CHAPTER 2

Moving Coordinates.

2.1. Coordinate Systems.

A ***rigid body*** is a body whose particles do not move relative to one another. In order to describe the motion of a rigid body, we begin by defining the two coordinate systems that are used to describe the motion of a rigid body. The first one will be the usual coordinate system for space and the other will be "attached" to the rigid body. We will call these two, three dimensional oriented Euclidean spaces k and K, respectively. It is generally convenient to have fixed Euclidean coordinates on k and K. Here, $\mathbf{q}$ will describe the Cartesian radius vector of a point relative to an inertial coordinate system k which is the coordinate system in space. The notation $\mathbf{Q}$ will be used for the Cartesian radius vector of a point relative to a moving coordinate system K.

A ***motion of*** K ***relative to*** k is a mapping smoothly dependent on t,

$$D_t \colon K \to k$$

which preserves the metric and orientation. Here, t is to be thought of as time. So, for a fixed time t, the mapping D_t gives the relationship between the coordinates K on the moving body and the coordinates k in space. In general, a map D_t that preserves the metric and orientation is a translation C_t followed by a rotation B_t about a line. (Figure 2.1.) Proofs of this can be found in [**3**] and [**8**].

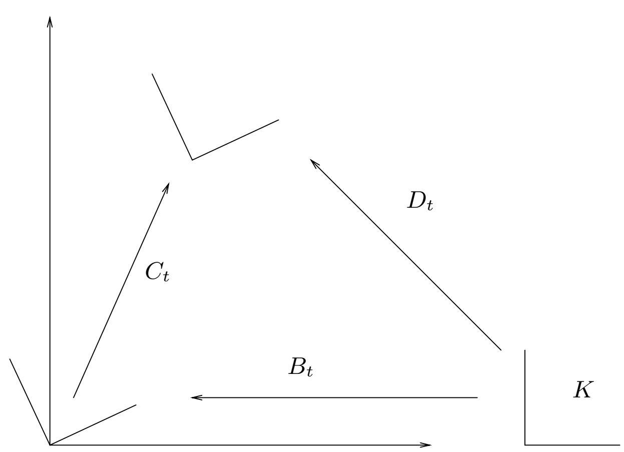

FIGURE 2.1. The rigid motion D_t is a translation C_t followed by a rotation B_t.

The motion B_t in Figure 2.1 is called a ***rotation*** since it takes the origin of K to the origin of k. Therefore, B_t is a linear operator. The motion C_t is a ***translation*** as it does not involve any rotation.

$$C_t\mathbf{Q} = \mathbf{Q} + \mathbf{r}(t)$$

Since $\mathbf{q}$ is the position of a particle relative to a fixed coordinate system, we have that

$$\dot{\mathbf{q}} = \text{velocity of particle,} \tag{2.1}$$

$$\ddot{\mathbf{q}} = \text{acceleration of particle.} \tag{2.2}$$

In what follows, we will assume that there is a point of the moving body that stays fixed during the motion. If this point is O then it is natural to take O to be the origin of both the fixed coordinate system k and the moving coordinate system K. In this case the motion D_t will leave the origin fixed and thus the translational part, C_t, will vanish. Therefore, we will be assuming that the motion of K relative to k is a smooth map $t \mapsto B(t)$ of $\mathbb{R}$ into $SO(3)$, the group of rotations about the origin O. (Concretely, $SO(3)$ is the group of all 3×3 orthogonal matrices.)

We can think of $B(t)$ as giving a moving (rotating) coordinate system. Letting $\mathbf{Q}$ be the position coordinates of the particle with respect to this moving coordinate system allows $\mathbf{q}$ and $\mathbf{Q}$ to be related by

$$\mathbf{q} = B\mathbf{Q} = B(t)\mathbf{Q}. \tag{2.3}$$

Also, if $\mathbf{q} = \mathbf{q}(t)$ and $\mathbf{Q} = \mathbf{Q}(t)$ depend on time (so that they can be thought of as the motion of a particle), then (2.3) shows that knowing $\mathbf{q}(t)$ and $B(t)$ determines $\mathbf{Q}(t)$ and likewise knowing $\mathbf{Q}(t)$ and $B(t)$ determines $\mathbf{q}(t)$. Thus, we can work in whichever coordinate system, k or K, in which it is easier to compute. Note, however, that the first and second derivatives $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ represent the actual velocity and acceleration of the particle as the coordinate system k is inertial, while the derivative $\dot{\mathbf{Q}}$ and $\ddot{\mathbf{Q}}$ do not have obvious physical or geometric meaning since this coordinate system is moving and in general also accelerating.

Our immediate goal is to relate the derivatives $\dot{\mathbf{Q}}$ and $\ddot{\mathbf{Q}}$ to $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ and interpret the results geometrically. We start by differentiating both sides of equation (2.3).

$$\begin{aligned}\dot{\mathbf{q}} &= \dot{B}\mathbf{Q} + B\dot{\mathbf{Q}} \\ &= \dot{B}B^{-1}B\mathbf{Q} + B\dot{\mathbf{Q}} \\ &= \dot{B}B^{-1}\mathbf{q} + B\dot{\mathbf{Q}} \qquad (2.4)\end{aligned}$$

LEMMA 2.1.1. *The matrix $\dot{B}B^{-1}$ is skew symmetric.*

PROOF. Since $B \in SO(3)$ (which is the group of all 3×3 orthogonal matrices), we have that $BB^t = I$ or $B^t = B^{-1}$ by definition. Differentiating both sides of $BB^t = I$ with respect to t, we get

$$\dot{B}B^t + B\dot{B}^t = 0$$

Substituting $B^t = B^{-1}$, we get

$$\dot{B}B^{-1} + (\dot{B}B^{-1})^t = 0.$$

So

$$\dot{B}B^{-1} = -(\dot{B}B^{-1})^t.$$

Therefore, $\dot{B}B^{-1}$ is skew symmetric. $\square$

THEOREM 2.1.2. *Let $\mathbf{q} = B\mathbf{Q}$ as above. Then there is a map $t \mapsto \boldsymbol{\omega}(t) \in \mathbb{R}^3$ so that*

$$\dot{\mathbf{q}} = \boldsymbol{\omega}(t) \times \mathbf{q} + B\dot{\mathbf{Q}}.$$

*(Here $\times$ is the usual vector cross product from vector analysis.) The vector $\boldsymbol{\omega}(t)$ is the **instantaneous angular velocity**, which for brevity sake will be referred to as the **angular velocity** in the sequel. It is an eigenvector of $\dot{B}B^{-1}$ with eigenvalue 0.*

PROOF. By equation (2.4), we know,

$$\dot{\mathbf{q}} = \dot{B}B^{-1}\mathbf{q} + B\dot{\mathbf{Q}}.$$

So we must show $\dot{B}B^{-1}\mathbf{q} = \boldsymbol{\omega}(t) \times \mathbf{q}$ for some $\boldsymbol{\omega}(t)$. Since $S = \dot{B}B^{-1}$ is skew symmetric by the previous lemma, it is of the form

$$S = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

for some w_1, w_2, w_3. Letting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{R}^3$ and setting

$$\boldsymbol{\omega} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3,$$
$$\mathbf{q} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3,$$

then

$$\boldsymbol{\omega} \times \mathbf{q} = (w_2q_3 - w_3q_2)\mathbf{e}_1 - (w_1q_3 - w_3q_1)\mathbf{e}_2 + (w_1q_2 - w_2q_1)\mathbf{e}_3 = S\mathbf{q}.$$

as required. Direct calculation shows $S\boldsymbol{\omega} = 0$ so that $\boldsymbol{\omega}$ is an eigenvector of S with eigenvalue 0. $\square$

The following notation will be used in the description of the motion:

$$\mathbf{v} = \dot{\mathbf{q}} \in k = \text{absolute velocity},$$
$$\mathbf{v}' = B\dot{\mathbf{Q}} \in k = \text{relative velocity},$$
$$\mathbf{v}_n = \dot{B}\mathbf{Q} = \boldsymbol{\omega} \times \mathbf{q} \in k = \text{transferred velocity of rotation}.$$

Then, the previous theorem showed

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_n.$$

So, $\mathbf{v}$ is the velocity of a particle of the rigid body in a stationary coordinate system, k. The vector $\mathbf{v}'$ is the velocity of the particle in the stationary plane relative to the moving coordinate system, K. The vector $\mathbf{v}_n$, the transferred velocity, is the velocity of the particle when the particle is at rest with respect to the moving coordinate system K, while K is only rotating (i.e., no translation takes place). Therefore, the absolute velocity of the particle is the sum of the particle's relative velocity and its transferred velocity of rotation.

Now, consider again our basic relation, $\mathbf{q} = B\mathbf{Q}$, which relates coordinates in the fixed and moving coordinate systems. We let

$$\boldsymbol{\omega} = B\boldsymbol{\Omega}, \tag{2.5}$$

where $\boldsymbol{\Omega}$ is ***angular velocity in the moving coordinate system,*** K, and $\boldsymbol{\omega}$ is ***angular velocity in the fixed frame,*** k. We also assume that $\mathbf{q}$ moves by Newton's Second Law:

$$m\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}})$$

where $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}})$ is the force acting on the particle $\mathbf{q}$ and m is the mass. Then there is enough information to find the acceleration of a particle relative to K.

PROPOSITION 2.1.3. *The acceleration of a particle relative to a moving frame can be found by the following equation.*

$$\ddot{\mathbf{q}} = B(\ddot{\mathbf{Q}} + 2(\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) + (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q})) + (\dot{\boldsymbol{\Omega}} \times \mathbf{Q}))$$

PROOF. Differentiating $\mathbf{q} = B\mathbf{Q}$, we get

$$\dot{\mathbf{q}} = \dot{B}\mathbf{Q} + B\dot{\mathbf{Q}}.$$

From the proof of the previous theorem, we know $\dot{B}\mathbf{Q} = \boldsymbol{\omega} \times \mathbf{q}$. Thus,

$$\dot{B}\mathbf{Q} = \boldsymbol{\omega} \times \mathbf{q} = B\boldsymbol{\Omega} \times B\mathbf{Q} = B(\boldsymbol{\Omega} \times \mathbf{Q})$$

So

$$\dot{\mathbf{q}} = B(\boldsymbol{\Omega} \times \mathbf{Q} + \dot{\mathbf{Q}}).$$

Differentiating $\dot{\mathbf{q}}$ gives the following:

$$\begin{aligned}
\ddot{\mathbf{q}} &= \dot{B}(\boldsymbol{\Omega} \times \mathbf{Q} + \dot{\mathbf{Q}}) + B[(\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) + (\dot{\boldsymbol{\Omega}} \times \mathbf{Q}) + \ddot{\mathbf{Q}}] \\
&= B[(\boldsymbol{\Omega} \times ((\boldsymbol{\Omega} \times \mathbf{Q}) + \dot{\mathbf{Q}}) + (\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) + (\dot{\boldsymbol{\Omega}} \times \mathbf{Q}) + \ddot{\mathbf{Q}}] \\
&= B[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q}) + 2(\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) + (\dot{\boldsymbol{\Omega}} \times \mathbf{Q}) + \ddot{\mathbf{Q}}].
\end{aligned}$$

In this calculation, at the second equality, we have used that for any vector $\mathbf{u}$, $\dot{B}B^{-1}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$. This implies that for any $\mathbf{v}$, we have $\dot{B}\mathbf{v} = \dot{B}B^{-1}B\mathbf{v} = \boldsymbol{\omega} \times B\mathbf{v} = B(B^{-1}\boldsymbol{\omega} \times \mathbf{v}) = B(\boldsymbol{\Omega} \times \mathbf{v})$. We also used $\mathbf{v} = (\boldsymbol{\Omega} \times \mathbf{Q} + \dot{\mathbf{Q}})$. □

2.2. Acting Forces.

The next result shows that in a moving coordinate system Newton's second law looks as if there are three extra forces acting on a particle. These forces will be called the ***acting forces***.

THEOREM 2.2.1. *Motion in a rotating coordinate system takes place as if three inertial forces act on each point* $\mathbf{Q}$ *of mass* m.

1. ***The inertial force of rotation:***

$$m\dot{\boldsymbol{\Omega}} \times \mathbf{Q}.$$

2. ***The Coriolis force:***

$$2m(\mathbf{\Omega} \times \dot{\mathbf{Q}}).$$

3. ***The centrifugal force:***

$$m(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})).$$

Thus,

$$m\ddot{\mathbf{Q}} = \mathbf{F} - (m\dot{\mathbf{\Omega}} \times \mathbf{Q}) - (2m\mathbf{\Omega} \times \dot{\mathbf{Q}}) - [m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})] \tag{2.6}$$

where $\mathbf{F}$ *is defined by*

$$B\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}) = \mathbf{f}(B\mathbf{Q}, B\dot{\mathbf{Q}}).$$

PROOF. From the previous proposition, we have

$$\ddot{\mathbf{q}} = B\ddot{\mathbf{Q}} + 2B(\mathbf{\Omega} \times \dot{\mathbf{Q}}) + B(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})) + B(\dot{\mathbf{\Omega}} \times \mathbf{Q})$$

$$\ddot{\mathbf{q}} - 2B(\mathbf{\Omega} \times \dot{\mathbf{Q}}) - B(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})) - B(\dot{\mathbf{\Omega}} \times \mathbf{Q}) = B\ddot{\mathbf{Q}}.$$

Multiplying thru by the mass m, we get

$$m\ddot{\mathbf{q}} - 2Bm(\mathbf{\Omega} \times \dot{\mathbf{Q}}) - Bm(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})) - Bm(\dot{\mathbf{\Omega}} \times \mathbf{Q}) = Bm\ddot{\mathbf{Q}}.$$

Now, realizing that

$$m\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}(B\mathbf{Q}, B\dot{\mathbf{Q}}) = B\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}),$$

we find

$$B\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}) - 2Bm(\mathbf{\Omega} \times \dot{\mathbf{Q}}) - Bm(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})) - Bm(\dot{\mathbf{\Omega}} \times \mathbf{Q}) = Bm\ddot{\mathbf{Q}}.$$

Since B is an orthogonal matrix, it is nonsingular. Therefore, it can be canceled off of both sides of this equation, leaving the desired equality:

$$\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}) - 2m(\mathbf{\Omega} \times \dot{\mathbf{Q}}) - m(\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Q})) - m(\dot{\mathbf{\Omega}} \times \mathbf{Q}) = m\ddot{\mathbf{Q}}.$$

□

2.2.1. Examples. The frame of the earth is not an inertial frame due to the rotation of the earth about its axis and its orbiting the sun. We now give a couple of examples showing how the acting forces of Theorem 2.2.1 affect objects near the earth's surface. When studying effects of relatively short duration, say only lasting a few days at most, the main contribution of the motion of the earth is its rotation about its axis. Thus, for such events we can assume that the motion is uniform rotation about the axis that runs through the North and South poles. Then, the angular velocity vector is constant and of the form $\boldsymbol{\omega} = |\boldsymbol{\omega}|\mathbf{e}$, where $\mathbf{e}$ is the unit vector parallel to the axis of the earth. The rate of rotation is 2π radians every 24 hours (86,400 seconds). Therefore, $|\boldsymbol{\omega}| = 2\pi/(86400) \approx .00007272$ radians/second. $\boldsymbol{\omega}$ and $\mathbf{\Omega}$ are related by $\mathbf{\Omega} = B^{-1}\boldsymbol{\omega}$. As B is the rotation about $\mathbf{e}$, it follows that $B\mathbf{e} = B^{-1}\mathbf{e} = \mathbf{e}$ and, therefore,

$$\mathbf{\Omega} = B|\boldsymbol{\omega}|\mathbf{e} = |\boldsymbol{\omega}|\mathbf{e} \approx .00007272\mathbf{e}.$$

This implies that

$$|\mathbf{\Omega}| \approx .00007272,$$

an approximation that will be used in our examples. Also, as $\mathbf{e}$ is a constant vector, we have $\dot{\boldsymbol{\Omega}} = 0$. Therefore, in Theorem 2.2.1 the inertial force of rotation vanishes and the equation for $m\ddot{\boldsymbol{\Omega}}$ in the theorem simplifies to

$$m\ddot{\mathbf{Q}} = \mathbf{F} - 2m(\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) - [m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q})] \tag{2.7}$$

2.2.1.1. *The Coriolis force and horizontal displacement of a falling body.* We can denote $m\ddot{\mathbf{Q}}$ as $\mathbf{F}_{\text{eff}}$, or the effective force. When the particle is stationary in the moving coordinate system, the centrifugal force, $m(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q}))$, is the only added term in the effective force. This is due to the fact that $\dot{\boldsymbol{\Omega}} = 0$ and $\dot{\mathbf{Q}} = 0$ in this case.

However, if the particle is moving relative to the moving coordinate system, K, the Coriolis force, $2m(\boldsymbol{\Omega} \times \dot{\mathbf{Q}})$, is also added. In effect, the Coriolis force is just an apparent force visible only due to the movement of a chosen frame. Therefore, if any object is dropped onto the earth, it will deviate from the vertical of a chosen fixed coordinate system due to the Coriolis force.

For example, consider a stone of mass m that is dropped with zero initial velocity, and falls a vertical distance of h meters. Then choose a frame $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ attached to the earth at the initial point of the stone so that $\mathbf{e}_z$ points upward and $\mathbf{e}_x$ and $\mathbf{e}_y$ are parallel to the earth's surface. As the acceleration due to gravity at the earth's surface is $g = 9.8\text{m}/\text{sec}^2$, the force of gravity on the stone is $m\mathbf{g}$ where

$$\mathbf{F} = m\mathbf{g} = -gm\mathbf{e}_z = -9.8m\mathbf{e}_z.$$

Substituting this into equation (2.7)and dividing by m gives

$$\ddot{\mathbf{Q}} = \mathbf{g} - 2(\boldsymbol{\Omega} \times \dot{\mathbf{Q}}) - (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q})).$$

But as the size of $\boldsymbol{\Omega}$ is small, the quadratic term, $(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{Q}))$, will be very small in relation to the term $(\boldsymbol{\Omega} \times \dot{\mathbf{Q}})$. So, we further simplify by dropping the quadratic term. Then the equation for the falling stone reduces to

$$\ddot{\mathbf{Q}} = \mathbf{g} + 2(\dot{\mathbf{Q}} \times \boldsymbol{\Omega}) \tag{2.8}$$

We then split $\mathbf{Q}$ as

$$\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2 \tag{2.9}$$

where, as in Figure 2.2, $\mathbf{Q}_1$ is the vertical displacement and $\mathbf{Q}_2$ is the horizontal displacement.

Then

$$\mathbf{Q}_1 = \mathbf{Q}_1(0) + \mathbf{g}t^2/2$$

where $\mathbf{g}t^2/2$ is the distance traveled after time t. Differentiating $\mathbf{Q}_1$ twice with respect to t, we get

$$\ddot{\mathbf{Q}}_1 = \mathbf{g}.$$

From equation (2.9),

$$\begin{aligned} \mathbf{Q}_2 &= \mathbf{Q} - \mathbf{Q}_1 \\ \ddot{\mathbf{Q}}_2 &= \ddot{\mathbf{Q}} - \ddot{\mathbf{Q}}_1 = \mathbf{g} + 2(\dot{\mathbf{Q}} \times \boldsymbol{\Omega}) - \mathbf{g} \\ \ddot{\mathbf{Q}}_2 &= 2(\dot{\mathbf{Q}} \times \boldsymbol{\Omega}). \qquad (2.10) \end{aligned}$$

Integrating equation (2.8), we get

$$\dot{\mathbf{Q}} = 2(\mathbf{Q} \times \mathbf{\Omega}) + \mathbf{g}t.$$

Then substituting this into equation (2.10), we get

$$\begin{aligned} \ddot{\mathbf{Q}}_2 &= 2((\mathbf{g}t + 2(\mathbf{Q} \times \mathbf{\Omega})) \times \mathbf{\Omega}) \\ &= 2(\mathbf{g}t \times \mathbf{\Omega}) + O(\mathbf{\Omega}^2). \end{aligned}$$

Integrating $\ddot{\mathbf{Q}}_2$ twice, we find

$$\mathbf{Q}_2 \approx \frac{t^3}{3}(\mathbf{g} \times \mathbf{\Omega}) \approx \frac{2t}{3}(\mathbf{h} \times \mathbf{\Omega}),$$

where $\mathbf{h} = \mathbf{h}(t) = \mathbf{g}t^2/2$ is the vertical displacement after t seconds. The magnitude of this is

$$|\mathbf{Q}_2(t)| \approx \frac{2t}{3}|\mathbf{h} \times \mathbf{\Omega}| = \frac{2t}{3}|\mathbf{h}||\mathbf{\Omega}| \cos\lambda, \tag{2.11}$$

where λ is the latitude of the stone. It takes the stone a time of $t_h = \sqrt{(2h)/g}$ to fall h meters. Using this in (2.11), we find that the horizontal displacement in falling h meters is about

$$|\mathbf{Q}_2(t_h)| \approx \frac{(2h)^{3/2}}{3\sqrt{g}}|\mathbf{\Omega}| \cos\lambda \approx .00002190(h)^{3/2} \cos\lambda.$$

Given that the latitude of Columbia is $\lambda = 34°$, we can generate the following table giving the horizontal displacements for some different heights (all distance are in meters).

Vertical drop	100	250	500	1000	5000	10000
Horizontal displacement	.018	.071	.202	.574	6.419	18.156

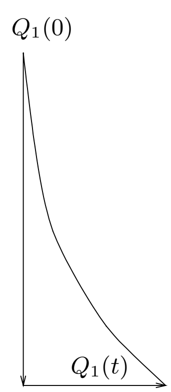

FIGURE 2.2. The Coriolis force causes a falling object to deviate from the vertical. In the latitude of Columbia and considering a drop of 250 meters, the horizontal displacement is a little over 7cm.

2.2.1.2. *Foucault's Pendulum.* Another example of the Coriolis force is when one considers the small oscillations of a pendulum. We will consider a long pendulum undergoing small oscillations. Then, if $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ constitute a frame as in the last example, and if we ignore the Coriolis force, we can make the usual approximations to get the equations of motion. (This involves $\sin\theta \approx \theta$ which is very good for small θ. For example, see [**7**, pp. 216–217].) Then

$$\begin{aligned}\ddot{x} &= -\omega^2 x,\\ \ddot{y} &= -\omega^2 y,\\ \ddot{z} &= 0,\end{aligned}$$

where ω is a constant depending on the length of the pendulum. (We use the traditional notation of ω for this constant and remark that it should not be confused with the angular velocity $\boldsymbol{\omega}$.) The solutions to these equations give simple harmonic motion in the x-y plane. We now consider what happens when the Coriolis force is also taken into account. The horizontal component of the Coriolis force is easily computed to be

$$2m\dot{y}\Omega_z\mathbf{e}_x - 2m\dot{x}\Omega_z\mathbf{e}_y.$$

The equations of motion for the pendulum with the Coriolis force added in are

$$\ddot{x} = -\omega^2 x + 2\dot{y}\Omega_z \tag{2.12}$$

$$\ddot{y} = -\omega^2 y - 2\dot{x}\Omega_z, \tag{2.13}$$

where $\Omega_z = |\mathbf{\Omega}|\sin\lambda$ and λ is the latitude. Set $u = x + iy$, where, as usual, $i = \sqrt{-1}$. Then

$$\begin{aligned}\dot{u} &= \dot{x} + i\dot{y}\\ \ddot{u} &= \ddot{x} + i\ddot{y}.\end{aligned}$$

By substituting these into the equations of motion, we get

$$\begin{aligned}\ddot{u} &= -\omega^2 x + 2\dot{y}\Omega_z + i(-\omega^2 y - 2\dot{x}\Omega_z)\\ \ddot{u} - i\ddot{y} &= -\omega^2 x + 2\dot{y}\Omega_z.\end{aligned}$$

Substituting (2.13) for $\ddot{y}$, we find

$$\begin{aligned}\ddot{u} - i(-\omega^2 y - 2\dot{x}\Omega_z) &= -\omega^2 x + 2\dot{y}\Omega_z\\ \ddot{u} - i(-\omega^2 y - 2(\dot{u} - i\dot{y})\Omega_z) &= -\omega^2 x + 2\dot{y}\Omega_z\\ \ddot{u} + \omega^2 yi + 2(\dot{u} - i\dot{y})\Omega_z i &= -\omega^2 x + 2\dot{y}\Omega_z\\ \ddot{u} + \omega^2 yi + 2\dot{u}\Omega_z i + 2\dot{y}\Omega_z &= -\omega^2 x + 2\dot{y}\Omega_z\\ \ddot{u} + \omega^2 yi + 2\dot{u}\Omega_z i + \omega^2 x &= 0\\ \ddot{u} + 2\dot{u}\Omega_z i + \omega^2(x + yi) &= 0.\end{aligned}$$

Substituting the relationship $u = x + iy$, equations (2.12) and (2.13) become the following single complex equation:

$$\ddot{u} + 2\Omega_z \dot{u} i + \boldsymbol{\omega}^2 u = 0 \tag{2.14}$$

We begin to solve this complex equation in the following manner: Let

$$u = e^{\lambda t} \tag{2.15}$$

Substituting this into (2.14) implies λ satisfies

$$\lambda^2 + 2i\Omega_z \lambda + \omega^2 = 0.$$

Therefore,

$$\lambda = -i\Omega_z \pm i\sqrt{\Omega_z^2 + \omega^2}.$$

Since Ω_z is relative to the earth and ω is relative to space, $\Omega_z^2 \ll \omega^2$. Thus,

$$\sqrt{\Omega_z^2 + \omega^2} = \omega + O(\Omega_z^2),$$

from which we can conclude

$$\lambda \approx -i\Omega_z \pm i\omega.$$

By substituting this into (2.15), we find that

$$\begin{aligned} u &= e^{(-i\Omega_z \pm i\omega)t} \\ &= e^{-i\Omega_z t \pm i\omega t} \\ &= e^{-i\Omega_z t} e^{i\omega t}, e^{-i\Omega_z t} e^{-i\omega t}. \end{aligned} \tag{2.16}$$

So, the general solution to (2.14) is

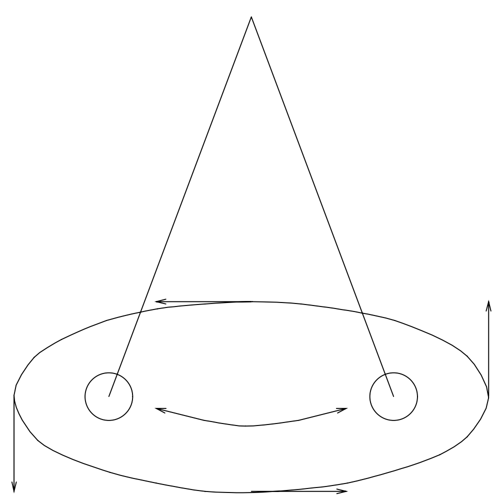

FIGURE 2.3. A pendulum rotates about its axis with angular velocity $-\Omega_z$. In the latitude of Columbia, one revolution about the central axis takes a little less than 1 day and 19 hours.

$$\begin{aligned} u &= e^{-i\Omega_z t}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) \\ &= (\cos(-\Omega_z t) + i\sin(-\Omega_z t))(c_1 \cos(\omega t) + c_2 \sin(\omega t)). \end{aligned}$$

Since $u = x(t) + iy(t)$,

$$x(t) = \cos(-\Omega_z t)(c_1 \cos(\omega t) + c_2 \sin(\omega t)) \tag{2.17}$$

$$y(t) = \sin(-\Omega_z t)(c_1 \cos(\omega t) + c_2 \sin(\omega t). \tag{2.18}$$

Then for $\Omega_z = 0$, equations (2.17) and (2.18) reduce to usual equations for the the harmonic oscillations of a spherical pendulum where ω is the angular frequency. Therefore, in a moving frame where $\Omega_z \neq 0$, the effect of the Coriolis force is that, when viewed from above, the pendulum will rotate with an angular velocity of $-\Omega_z$. Recalling that $\Omega_z = |\mathbf{\Omega}| \sin \lambda$ and that $|\mathbf{\Omega}|$ corresponds to one rotation per day, it follows that the plane of the pendulum will make one complete revolution each $1/\sin\lambda = \csc\lambda$ days. This shows latitude can be determined by the observation of a pendulum as Ω_z determines the latitude. This is the principal behind Foucault's Pendulum. In the latitude of Columbia SC, the period of the rotation about the vertical axis is $1/\sin(34°) \approx 1.788$ days (1 day 18 hours and 55 minutes). See Figure 2.3.

CHAPTER 3

Motion of a Rigid Body with no External Forces.

In this chapter we give a detailed description of the motion of a rigid body around a stationary point O with no outside forces acting on it. This can be thought of as a spinning top without any external forces acting on it. This is achieved by showing that the motion of the body is the same as the motion of an ellipsoid (the ellipsoid of inertia) centered at O and defined in terms of the moments of inertia of the body. It is then shown by Poinsot's Theorem that this ellipsoid moves so that it rolls without slipping on a plane perpendicular to the angular momentum $\mathbf{m}$. The description of the ellipsoid of inertia and the kinetic energy is in terms of a symmetric linear operator A, the inertia operator, which is also introduced and studied.

We recall some notation. We have a fixed inertial coordinate system k centered at the fixed point O of the body. The coordinates in k are thought of as giving the position in space. In this coordinate system we have the quantities

$$\begin{aligned}
\mathbf{q} &= \text{the radius vector of a point in space,}\\
\mathbf{v} &= \dot{\mathbf{q}} \text{ is the velocity vector of a point in space,}\\
\boldsymbol{\omega} &= \text{the angular velocity vector of a point in space and}\\
\mathbf{m} &= \text{the angular momentum vector of a point in space.}
\end{aligned}$$

To be more specific, as the body has the point O fixed, the motion is rotational. Therefore, the only contribution to the velocity is the angular velocity. Also, by Theorem 2.1.2 and using that $\dot{\mathbf{Q}} = 0$ for a point at rest with respect to the moving frame K, the velocity vector $\mathbf{v}$ of a point can be computed directly from the angular velocity $\boldsymbol{\omega}$ and the position $\mathbf{q}$ by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{q}. \tag{3.1}$$

By definition, the angular momentum $\mathbf{m}$ of a point with mass m is the cross product of the position and the momentum $m\mathbf{v}$. That is,

$$\mathbf{m} = \mathbf{q} \times m\mathbf{v} = \mathbf{q} \times m(\boldsymbol{\omega} \times \mathbf{q}). \tag{3.2}$$

Attached to the body there is a moving coordinate system K, also centered at O, that gives the position in the frame of the body. Therefore, a particle with constant coordinates in K is at rest with respect to the body. We will say that such a particle or point is attached to the body. In the coordinate system K, we have defined the quantities

$$\begin{aligned}
\mathbf{Q} &= \text{the radius vector of the body,}\\
\mathbf{V} &= \text{the velocity vector in the body,}\\
\mathbf{\Omega} &= \text{the angular velocity in the body and}\\
\mathbf{M} &= \text{the angular momentum in the body.}
\end{aligned}$$

The coordinates k and K are related by a time dependent map $B = B_t \colon K \to k$ which, for all t, is an element of the group $SO(3)$. Therefore, each B_t preserves the inner products and vector cross products on k and K. The various quantities defined in these coordinate systems are related by having a vector in K carried over to k by B:

$$\begin{aligned}
\mathbf{q} &= B\mathbf{Q},\\
\mathbf{v} &= B\mathbf{V},\\
\boldsymbol{\omega} &= B\mathbf{\Omega} \text{ and}\\
\mathbf{m} &= B\mathbf{M}.
\end{aligned}$$

After the last chapter, it might seem natural to use the equations of motion in the frame of the body as given in Theorem 2.2.1 to try to get information about the motion of the body. However, this does not lead to much useful information. The difference is that here we are looking for B_t as a function of t which is basically the same as looking for $\mathbf{q}(t) = B_t\mathbf{Q}$ as $\mathbf{Q}$ is constant, while in the last chapter the motion B_t was known by looking for $\mathbf{Q}(t)$ as a function of t.

3.1. The Inertia Operator.

Using formula (3.2) for the angular momentum $\mathbf{m}$ and the relationship between $\mathbf{m}$ and the angular momentum $\mathbf{M}$ in the body given by the rotational operator B, we have for a point of mass m attached to the body,

$$\mathbf{M} = B^{-1}\mathbf{m} = mB^{-1}(\mathbf{q}\times(\boldsymbol{\omega}\times\mathbf{q})) = m(B^{-1}\mathbf{q}\times(B^{-1}\boldsymbol{\omega}\times B^{-1}\mathbf{q})) = m(\mathbf{Q}\times(\mathbf{\Omega}\times\mathbf{Q})).$$

Therefore, if we introduce a linear operator $A\colon K \to K$ by

$$A\mathbf{X} = m\mathbf{Q}\times(\mathbf{X}\times\mathbf{Q}), \tag{3.3}$$

then $\mathbf{\Omega}$ and $\mathbf{M}$ are related by

$$A\mathbf{\Omega} = \mathbf{M}. \tag{3.4}$$

(Note that $A = A_{\mathbf{Q}}$ depends on $\mathbf{Q}$.) It is important to realize that as $\dot{\mathbf{Q}} = 0$, that $\dot{A} = 0$. Therefore, A is constant with respect to time. The linear operator A is the ***inertia operator*** of the point $\mathbf{Q}$.

Lemma 3.1.1. *The inertia operator satisfies the identities*

$$\begin{aligned}
A\mathbf{X}\cdot\mathbf{Y} &= m(\mathbf{X}\times\mathbf{Q})\cdot(\mathbf{Y}\times\mathbf{Q}),\\
A\mathbf{X}\cdot\mathbf{X} &= m(\mathbf{X}\times\mathbf{Q})^2.
\end{aligned}$$

In particular, A is symmetric (that is, $A\mathbf{X} \cdot \mathbf{Y} = \mathbf{X} \cdot A\mathbf{Y}$) and positive semidefinite (that is, $A\mathbf{X} \cdot \mathbf{X} \geq 0$ for all $\mathbf{X}$).

PROOF. Consider any $\mathbf{X}$ and $\mathbf{Y}$ in K. Because of equation (3.3),

$$A\mathbf{X} \cdot \mathbf{Y} = m[(\mathbf{Q} \times (\mathbf{X} \times \mathbf{Q})) \cdot \mathbf{Y}].$$

Since for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the identity

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

holds, we get

$$\begin{aligned} A\mathbf{X} \cdot \mathbf{Y} &= m[\mathbf{Q} \times (\mathbf{X} \times \mathbf{Q}) \cdot \mathbf{Y}] \\ &= m[(\mathbf{Y} \times \mathbf{Q}) \cdot (\mathbf{X} \times \mathbf{Q})] \\ &= m(\mathbf{X} \times \mathbf{Q}) \cdot (\mathbf{Y} \times \mathbf{Q}). \end{aligned}$$

(The symmetry of the inner product was used at the last step.) This proves the first of the required identities. The second follows by setting $\mathbf{Y} = \mathbf{X}$. The symmetry of A follows as $m(\mathbf{X} \times \mathbf{Q}) \cdot (\mathbf{Y} \times \mathbf{Q})$ is symmetric with respect to $\mathbf{X}$ and $\mathbf{Y}$. That A is positive semi-definite follows from the identity for $A\mathbf{X} \cdot \mathbf{X}$. $\square$

3.2. The Kinetic Energy, T, and its Conservation.

Recall that the kinetic energy of a point of mass m moving with velocity $\mathbf{v}$ is $T = \frac{1}{2}m\mathbf{v}^2$. Using that B is an orthogonal linear map, so that it preserves inner products, we have that $\mathbf{v}^2 = (B\mathbf{V})^2 = \mathbf{V}^2$. Therefore, in the coordinates of the body, the kinetic energy of a point with mass m is

$$T = \frac{1}{2}m\mathbf{V}^2.$$

PROPOSITION 3.2.1. *The kinetic energy of a point of mass m on the body is a quadratic form with respect to the vector of angular velocity $\mathbf{\Omega}$, namely*

$$\begin{aligned} T &= \frac{1}{2}(A\mathbf{\Omega} \cdot \mathbf{\Omega}) \\ &= \frac{1}{2}(\mathbf{M} \cdot \mathbf{\Omega}). \end{aligned}$$

PROOF. From equation (3.1), and using $\mathbf{v} = B\mathbf{V}$, we have $\mathbf{V} = B^{-1}(\boldsymbol{\omega} \times \mathbf{q}) = \mathbf{\Omega} \times \mathbf{Q}$. Therefore, using one of the identities of the previous lemma

$$\begin{aligned} T &= \frac{1}{2}m\mathbf{V}^2 \\ &= \frac{1}{2}m(\mathbf{\Omega} \times \mathbf{Q})^2 \\ &= \frac{1}{2}A\mathbf{\Omega} \cdot \mathbf{\Omega} \\ &= \frac{1}{2}\mathbf{M} \cdot \mathbf{\Omega}, \end{aligned}$$

where, at the last step, we have used that $A\mathbf{\Omega} = \mathbf{M}$. $\square$

The derivative of kinetic energy is interpreted as the rate of change of work done in moving the particle. This is important as it leads to the conservation of kinetic energy for a body with no outside forces acting on it. Letting $\mathbf{f}$ be the total force acting on $\mathbf{q}$, the kinetic energy of the point is $T = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\dot{\mathbf{q}}^2$. Its derivative is

$$\frac{dT}{dt} = m\dot{\mathbf{q}} \cdot \ddot{\mathbf{q}} = \dot{\mathbf{q}} \cdot \mathbf{f},$$

where we have used Newton's second law $m\ddot{\mathbf{q}} = \mathbf{f}$. Therefore, if dt is a small change in time, the corresponding change in kinetic energy between times t and $t + dt$ is

$$dT = \frac{dT}{dt}dt = \dot{\mathbf{q}}dt \cdot \mathbf{f} = d\mathbf{q} \cdot \mathbf{f},$$

where $d\mathbf{q} = \dot{\mathbf{q}}dt$ is the displacement (change in position) between times t and $t + dt$. However, the dot product of the force and displacement is the work done in doing the displacement. This is the definition of work. So, dT is the work done in a time interval of length dt and then, $\frac{dT}{dt}$ is the rate of change of work. Therefore, we have shown

Proposition 3.2.2. *If $W = W(t)$ is the work done be a force $\mathbf{f}$ acting on particle $\mathbf{q} = \mathbf{q}(t)$ between the time t and time $t = 0$, then W is the derivative of the kinetic energy $T = T(t)$. That is*

$$\dot{T} = W.$$

So far we have only considered the equations resulting from the motion of one point on the body. We now wish to consider the motion of the entire body. We will simply assume that the body is made up of a finite set of points $\mathbf{Q}_1, \ldots, \mathbf{Q}_N$. So, to model a physical body, we can take the points $\mathbf{Q}_i$ to be the molecules of the body and thus get a very accurate description of the body. Fortunately, as we will see shortly, the formulas describing the motion are not any more complicated for a large number of points N than for a small number. So, choosing one point per molecule, or even one point per atom, is reasonable and does not make calculation a problem.

Letting m_i be the mass of the point $\mathbf{Q}_i$, and $\mathbf{q}_i = B\mathbf{Q}_i$, then each point has its own velocity, $\mathbf{v}_i = \dot{\mathbf{q}}_i$, and so on for all the quantities we have defined so far. In particular, each point $\mathbf{Q}_i$ will have its angular momentum $\mathbf{M}_i$ in the body and its own inertia operator A_i. Then the angular momentum of the entire body is defined as the sum

$$\mathbf{M} = \sum_i \mathbf{M}_i.$$

To avoid the trivial case, we make the assumption that there is not a line through O that contains all the points $\mathbf{Q}_i$.

Theorem 3.2.3. *The angular momentum $\mathbf{M}$ of a rigid body with respect to a stationary point O depends linearly on the angular velocity $\mathbf{\Omega}$. That is, there exists a linear operator $A\colon K \mapsto K$ (the* ***inertia operator of the body****) such that $A\mathbf{\Omega} = \mathbf{M}$. The operator A is symmetric and positive definite (that is, $A\mathbf{X}\cdot\mathbf{X} > 0$ for all $\mathbf{X} \neq 0$).*

Also, the kinetic energy of a body is a quadratic form with respect to the angular velocity $\mathbf{\Omega}$*:*

$$T = \frac{1}{2}A\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\mathbf{M}\cdot\mathbf{\Omega}.$$

PROOF. Because of (3.4) we have $A_i\mathbf{\Omega} = \mathbf{M}_i$. (Note that $\mathbf{\Omega}$ is defined in terms of B and, thus, is independent of any particular point of the body we are at. So, $\mathbf{\Omega}$ is independent of i.) Thus

$$\mathbf{M} = \sum_i \mathbf{M}_i = \sum_i A_i\mathbf{\Omega} = A\mathbf{\Omega},$$

where $A = \sum_i A_i$. Then, by Lemma 3.1.1, each A_i is symmetric, and so the operator A is symmetric. Again using Lemma 3.1.1

$$A\mathbf{X}\cdot\mathbf{X} = \sum_i A_i\mathbf{X}\cdot\mathbf{X} = \sum_i m_i(\mathbf{X}\times\mathbf{Q}_i)^2 \geq 0.$$

If there is an $\mathbf{X} \neq 0$ with $A\mathbf{X}\cdot\mathbf{X} = 0$, then, as each of the terms in the sum for $A\mathbf{X}\cdot\mathbf{X}$ is nonnegative, we have $(\mathbf{X}\times\mathbf{Q}_i)^2 = 0$ for all i. Thus $\mathbf{X}\times\mathbf{Q}_i = 0$ for all i. However, $\mathbf{X}\times\mathbf{Q}_i = 0$ implies that $\mathbf{Q}_i = \lambda_i\mathbf{X}$ for some scalar λ_i. So this implies that all of the points $\mathbf{Q}_i$ lie on the line through O and $\mathbf{X}$, contrary to one of our assumptions. Thus, $\mathbf{X} \neq 0$ implies $A\mathbf{X}\cdot\mathbf{X} > 0$ and A is positive definite as claimed.

The kinetic energy of $\mathbf{Q}_i$ is $T_i = \frac{1}{2}A_i\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\mathbf{M}_i\cdot\mathbf{\Omega}$. Therefore, using that A is the sum of the A_i, the total kinetic energy is

$$T = \sum_i T_i = \frac{1}{2}\sum_i A_i\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}A\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\mathbf{M}\cdot\mathbf{\Omega}, \tag{3.5}$$

where we have also used $A\mathbf{\Omega} = \mathbf{M}$. □

Finally, we give what may be the most important property of kinetic energy, which is that it is preserved under the motion of a rigid body. To do this we need to make a hypothesis concerning what forces are acting on the particles of a rigid body. Let $\mathbf{f}_i$ be the sum of all the forces acting on the i-th particle $\mathbf{q}_i$ of our body. Then kinetic energy of this particle is T_i, and by Proposition 3.2.2 the rate of change of T_i is

$$\dot{T}_i = W_i,$$

where W_i is the work done by the force $\mathbf{f}_i$ between the times 0 and t. But if the body is rigid and there are no outside forces acting on it, then no work is being done. Thus, for a rigid body we make the following assumption:

ASSUMPTION 3.2.4. *If* $\mathbf{q}_1, \ldots, \mathbf{q}_N$ *are the particles of a rigid body moving without any external forces on it, then the total work* $W = \sum_i W_i$ *done by the forces* $\mathbf{f}_i$ *acting on these particle vanishes. That is,*

$$W = \sum_i W_i = 0. \tag{3.6}$$

This is sometimed referred to as D'Alembert's principle [**5**, p. 18] and leads to

THEOREM 3.2.5 (Conservation of Kinetic Energy). *If $\mathbf{q}_1, \ldots, \mathbf{q}_N$ are the particles of a rigid body moving without any external forces acting on it, then the total kinetic energy $T = \sum_i T_i$ is constant. That is,*

$$\dot{T} = 0.$$

PROOF. We have from Proposition 3.2.2 that $\dot{T}_i = W_i$. Therefore, using equation (3.6) we have

$$\dot{T} = \sum_i \dot{T}_i = \sum_i W_i = 0,$$

which completes the proof. □

3.3. Principal Axes.

Since A is a symmetric linear map, by the principle axis theorem for selfadjoint linear maps (often also called the spectral), the space K has an oriented orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of eigenvectors of A. (This is in most linear algebra texts, for example [**6**, thm. 5 p. 193].) As A is positive definite, the eigenvalues of A are all positive. If we call the eigenvalues I_1, I_2, I_3, then

$$A\mathbf{e}_i = I_i\mathbf{e}_i$$

for $i = 1, 2, 3$. Writing $\mathbf{\Omega}$ and $\mathbf{M}$ in this basis, we have for scalars Ω_i and M_i that

$$\begin{aligned}\mathbf{\Omega} &= \Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2 + \Omega_3\mathbf{e}_3,\\ \mathbf{M} &= M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3.\end{aligned}$$

Then the relation $\mathbf{M} = A\mathbf{\Omega}$ implies

$$\begin{aligned}A(\Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2 + \Omega_3\mathbf{e}_3) &= M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3\\ &= I_1\Omega_1\mathbf{e}_1 + I_2\Omega_2\mathbf{e}_2 + I_3\Omega_3\mathbf{e}_3,\end{aligned}$$

which in turn implies

$$M_i = I_i\Omega_i \tag{3.7}$$

for $i = 1, 2, 3$. As the kinetic energy given by Theorem 3.2.3 is $T = \frac{1}{2}\mathbf{M}\cdot\mathbf{\Omega}$, we have

$$T = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2).$$

DEFINITION 3.3.1. The ***principal axes*** of the body at point O are these axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We give another interpretation of the kinetic energy by writing the vector $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = |\mathbf{\Omega}|\mathbf{e}$$

where $\mathbf{e}$ is a unit vector. The line through O in the direction of $\mathbf{e}$ is the ***axis of rotation***. (Note that $\mathbf{\Omega} = \mathbf{\Omega}(t)$ is time dependent so the axis of rotation changes with respect to time.)

The instantaneous motion of the body is a rotation with angular velocity $|\mathbf{\Omega}|$ around the axis $\mathbf{e}$. If r_i is the distance of the point $\mathbf{Q}_i$ from the axis of $\mathbf{e}$ (which is

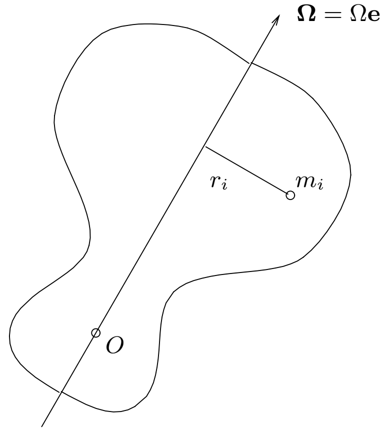

FIGURE 3.1. If the i-th point of the rigid body has distance r_i from the axis of rotation, then its speed is $v_i = |\mathbf{\Omega}|r_i$, where $\mathbf{\Omega} = |\mathbf{\Omega}|\mathbf{e}$ is the angular velocity. Thus, if the mass of the point is m_i, then its kinetic energy is $T_i = \frac{1}{2}m_i v_i^2 = \mathbf{\Omega}^2 \frac{1}{2} m_i r_i^2$.

the axis that is a line through the fixed point O and in the direction $\mathbf{\Omega}$), then the speed (i.e., length of the velocity vector) of $\mathbf{Q}_i$ is $v_i = |\mathbf{\Omega}|r_i$. Therefore, the kinetic energy of $\mathbf{Q}_i$ is

$$T_i = \frac{1}{2}m_i v_i^2 = \mathbf{\Omega}^2 \frac{1}{2} m_i r_i^2. \tag{3.8}$$

See Figure 3.1.

THEOREM 3.3.2. *For a rotation of a rigid body fixed at a point O with angular velocity*

$$\mathbf{\Omega} = \Omega\mathbf{e},\ (\Omega = |\mathbf{\Omega}|)$$

around the $\mathbf{e}$ *axis, the kinetic energy is equal to*

$$T = \frac{1}{2} I_e \Omega^2,$$

where

$$I_e = \sum_i m_i r_i^2$$

and Ω_i is the distance of the i-th point to the $\mathbf{e}$ *axis (Figure 3.1).*

PROOF. Summing the formula(3.8) over i, we get the desired result:

$$T = \sum_i T_i = \sum_i \mathbf{\Omega}^2 \frac{1}{2} m_i r_i^2 = \frac{1}{2} I_{\mathbf{e}} \mathbf{\Omega}^2.$$

□

DEFINITION 3.3.3. I_e is called the moment of ***inertia of the body with respect to the*** $\mathbf{e}$ ***axis***.

Comparing our formulas for the kinetic energy we have

$$T = \frac{1}{2}A\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} = \frac{1}{2}I_{\mathbf{e}}\boldsymbol{\Omega}^2.$$

If we choose $\boldsymbol{\Omega} = \mathbf{e}_1$, then $\mathbf{e} = \mathbf{e}_1$ and $\boldsymbol{\Omega}^2 = 1$. Using this in the formula for T gives

$$\frac{1}{2}I_1 = \frac{1}{2}A\mathbf{e}_1 \cdot \mathbf{e}_1 = \frac{1}{2}I_{\mathbf{e}_1},$$

so that the moment of inertia about $\mathbf{e}_1$ is I_1. Doing similar calculations for $\boldsymbol{\Omega} = \mathbf{e}_2$ and $\boldsymbol{\Omega} = \mathbf{e}_3$ leads to

Corollary 3.3.4. *The eigenvalues I_1, I_2, I_3 of the inertia operator A are the moments of inertia of the body with respect to the principal axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.*

3.4. Torque, Conservation of Angular Momentum, and Euler's Equations.

So far we have not given a precise definition of what it means for a rigid body with a fixed point to move without external forces acting on it. In the inertial coordinate system, if $\mathbf{q}_i$ is the position vector of the i-th point of the system, then let $\mathbf{f}_i$ be the force on $\mathbf{q}_i$. Then by Newton's second law

$$m_i\ddot{\mathbf{q}}_i = \mathbf{f}_i. \tag{3.9}$$

Consider the vector

$$\boldsymbol{\tau}_i = \mathbf{q}_i \times \mathbf{f}_i$$

which is defined to be the ***torque of the point*** $\mathbf{q}_i$ ***about*** O. The units on $\boldsymbol{\tau}_i$ are displacement ($\mathbf{q}_i$ is a vector) times force. So, $\boldsymbol{\tau}_i$ has the units of work. If the force $\mathbf{f}_i$ is parallel to $\mathbf{q}_i$ (that is, $\mathbf{f}_i = \lambda\mathbf{q}_i$ for some scalar λ), then $\boldsymbol{\tau}_i = \mathbf{q}_i \times \mathbf{f}_i = 0$ as $\mathbf{q}_i \times \mathbf{q}_i = 0$. Thus, for a force that is radial with respect to O, the contribution to the torque is zero. However, if the force $\mathbf{f}_i$ is orthogonal to $\mathbf{q}_i$, then $|\boldsymbol{\tau}_i| = |\mathbf{q}_i \times \mathbf{f}_i| = |\mathbf{q}_i||\mathbf{f}_i|$. More generally, if $\mathbf{f}_i$ is decomposed as

$$\mathbf{f}_i = \mathbf{f}_i^{\perp} + \mathbf{f}_i^{\top}$$

where $\mathbf{f}_i^{\perp}$ is orthogonal to $\mathbf{q}_i$ and $\mathbf{f}_i^{\top}$ is parallel to $\mathbf{q}_i$, then

$$\boldsymbol{\tau}_i = \mathbf{q}_i \times (\mathbf{f}_i^{\perp} + \mathbf{f}_i^{\top}) = \mathbf{q}_i \times \mathbf{f}_i^{\perp}.$$

Therefore, only the part of the force perpendicular to the line through O and the point $\mathbf{q}_i$ contributes to the torque about the point O. Thus, the torque can be thought of as the work done by the force $\mathbf{f}_i$ in rotating the point $\mathbf{q}_i$ about the point O. The following proposition gives an important relationship between the angular momentum $\mathbf{m}_i$ of $\mathbf{q}_i$ and its torque.

Proposition 3.4.1. *The angular momentum $\mathbf{m}_i$ and the torque $\boldsymbol{\tau}_i$ about the fixed point O of the body are related by*

$$\dot{\mathbf{m}}_i = \boldsymbol{\tau}_i.$$

That is, the torque is the rate of change of angular momentum.

PROOF. The definition of the angular momentum is $\mathbf{m}_i = m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i$. Therefore, by the product rule and Newton's second law $m_i \ddot{\mathbf{q}}_i = \mathbf{f}_i$, we have

$$\dot{\mathbf{m}}_i = m_i \dot{\mathbf{q}}_i \times \dot{\mathbf{q}}_i + m_i \mathbf{q}_i \times \ddot{\mathbf{q}}_i = 0 + \mathbf{q}_i \times \mathbf{f}_i = \boldsymbol{\tau}_i$$

as required. □

If a body has no external forces acting on it, then no work is done (as work is displacement times force). This motivates the following assumption:

ASSUMPTION 3.4.2. *If the rigid body composed of points $\mathbf{q}_1, \ldots, \mathbf{q}_N$ moves about the point O without external forces, then the total torque vanishes. That is*

$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = 0.$$

REMARK 3.4.3. If $\mathbf{F}_{ij}$ is the force that the particle $\mathbf{q}_i$ exerts on the particle $\mathbf{q}_j$, then Assumption 3.4.2 can be shown to hold if $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ (which is Newton's third law) and $\mathbf{F}_{ij}$ is parallel to the vector $\mathbf{q}_i - \mathbf{g}_j$. However Assumption 3.4.2 holds under more general conditions. □

The following is basic to the study of rotational motion.

THEOREM 3.4.4 (Conservation of Angular Momentum). *If the rigid body moves about the point O without external forces, then the total angular momentum*

$$\mathbf{m} = \sum_i \mathbf{m}_i$$

is constant.

PROOF. From the last proposition, we have that $\dot{\mathbf{m}}_i = \boldsymbol{\tau}_i$. Therefore,

$$\dot{\mathbf{m}} = \sum_i \dot{\mathbf{m}}_i = \sum_i \boldsymbol{\tau}_i = \boldsymbol{\tau} = 0$$

as $\boldsymbol{\tau} = 0$. By assumption, if the body moves without external forces about O, the total torque vanishes. However, $\dot{\mathbf{m}} = 0$ implies $\mathbf{m}$ is constant. □

We now rewrite the equation $\dot{\mathbf{m}} = 0$ in the moving coordinate system. Recall that the angular momentum $\mathbf{M}$ in the body is related to the usual angular momentum by $\mathbf{m} = B\mathbf{M}$.

THEOREM 3.4.5. *If a rigid body moves without any external forces, then the angular momentum $\mathbf{M}$ in the body satisfies the first order differential equation*

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\Omega}. \tag{3.10}$$

This is known as the ***Euler equation****.*

Proof. Differentiating the equation $\mathbf{m} = B\mathbf{M}$ and using $\dot{\mathbf{m}}$=0, we find

$$\begin{aligned} 0 &= \dot{\mathbf{m}} \\ &= B\dot{\mathbf{M}} + \dot{B}\mathbf{M} \\ &= B\dot{\mathbf{M}} + \dot{B}B^{-1}\mathbf{m}. \end{aligned}$$

From the proof of Theorem 2.1.2, we know that for any vector $\mathbf{v}$, $\dot{B}B^{-1}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$, and so $\dot{B}B^{-1}\mathbf{m} = \boldsymbol{\omega} \times \mathbf{m}$. Substituting this information into our equation for $0 = \dot{\mathbf{m}}$, we find

$$\begin{aligned} 0 &= B\dot{\mathbf{M}} + (\boldsymbol{\omega} \times \mathbf{m}) \\ &= B\dot{\mathbf{M}} + (B\boldsymbol{\Omega} \times B\mathbf{M}) \\ &= B\dot{\mathbf{M}} + B(\boldsymbol{\Omega} \times \mathbf{M}) \\ &= B(\dot{\mathbf{M}} + (\boldsymbol{\Omega} \times \mathbf{M})). \end{aligned}$$

Cancelling B out of this equation gives $\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\Omega}$ as required. □

3.5. Quadratic First Integrals of Euler's Equations.

Consider again the relation $\mathbf{M} = A\boldsymbol{\Omega}$. Remember in the notation of Section 3.3

$$\begin{aligned} \boldsymbol{\Omega} &= \Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2 + \Omega_3\mathbf{e}_3 \\ \mathbf{M} &= M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3. \end{aligned}$$

Then,

$$\begin{aligned} \dot{\mathbf{M}} &= A\dot{\Omega} \\ &= \dot{\Omega}_1 A\mathbf{e}_1 + \dot{\Omega}_2 A\mathbf{e}_2 + \dot{\Omega}_3 A\mathbf{e}_3 \\ &= \dot{\Omega}_1 I_1\mathbf{e}_1 + \dot{\Omega}_2 I_2\mathbf{e}_2 + \dot{\Omega}_3 I_3\mathbf{e}_3. \end{aligned}$$

Recall that A is constant or $\dot{A} = 0$, so we know the eigenvalues I_i of A are also constant. By Euler's Equation (3.10)

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\Omega} \\ &= (M_2\Omega_3 - M_3\Omega_2)\mathbf{e}_1 - (M_1\Omega_3 - M_3\Omega_1)\mathbf{e}_2 + (M_1\Omega_2 - M_2\Omega_1)\mathbf{e}_3. \end{aligned}$$

Comparing like components, we have

$$I_1\dot{\Omega}_1 = M_2\Omega_3 - M_3\Omega_2 \tag{3.11}$$

$$I_2\dot{\Omega}_2 = M_3\Omega_1 - M_1\Omega_3 \tag{3.12}$$

$$I_3\dot{\Omega}_3 = M_1\Omega_2 - M_2\Omega_1. \tag{3.13}$$

Since $M_i = I_i \Omega_i$ (from equation (3.7)),

$$\begin{aligned} I_1 \dot{\Omega}_1 &= I_2 \Omega_2 \Omega_3 - I_3 \Omega_3 \Omega_2 \\ &= (I_2 - I_3)\Omega_2 \Omega_3 \\ I_2 \dot{\Omega}_2 &= I_3 \Omega_3 \Omega_1 - I_1 \Omega_1 \Omega_3 \\ &= (I_3 - I_1)\Omega_1 \Omega_3 \\ I_3 \dot{\Omega}_3 &= I_1 \Omega_1 \Omega_2 - I_2 \Omega_2 \Omega_1 \\ &= (I_1 - I_2)\Omega_1 \Omega_2. \end{aligned}$$

Substituting the following information, $M_i = I_i \Omega_i$, $\dot{M}_i = I_i \dot{\Omega}_i$ and $\dot{M}_i / I_i = \dot{\Omega}_i$ into equations (3.11), (3.12), and (3.13), we find

$$\begin{aligned} I_1 \frac{\dot{M}_1}{I_1} &= \frac{M_2 M_3}{I_3} - \frac{M_3 M_2}{I_2} \\ I_2 \frac{\dot{M}_2}{I_2} &= \frac{M_1 M_3}{I_1} - \frac{M_3 M_1}{I_3} \\ I_3 \frac{\dot{M}_3}{I_3} &= \frac{M_2 M_1}{I_2} - \frac{M_2 M_1}{I_1}. \end{aligned}$$

Using this, Euler's equation becomes a system of differential equations:

$$\frac{dM_1}{dt} = a_1 M_2 M_3 \tag{3.14}$$

$$\frac{dM_2}{dt} = a_2 M_1 M_3 \tag{3.15}$$

$$\frac{dM_3}{dt} = a_3 M_1 M_2 \tag{3.16}$$

where a_1, a_2 and a_3 are the constants given by

$$\begin{aligned} a_1 &= \frac{I_2 - I_3}{I_2 I_3} \\ a_2 &= \frac{I_3 - I_1}{I_3 I_1} \\ a_3 &= \frac{I_1 - I_2}{I_1 I_2}. \end{aligned}$$

PROPOSITION 3.5.1. *The Euler Equations (3.14), (3.15), and (3.16) have two quadratic first integrals:*

$$2E = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}, \tag{3.17}$$

$$\mathbf{M}^2 = M_1^2 + M_2^2 + M_3^2. \tag{3.18}$$

(By first integral we mean that these equations are constant along solution curves to the Euler equation.)

PROOF. For this to be true, we must show that the first derivatives $\dot{E}$ and $\dot{M}$ are zero. For equation (3.17),

$$\dot{E} = \frac{M_1 \dot{M}_1}{I_1} + \frac{M_2 \dot{M}_2}{I_2} + \frac{M_3 \dot{M}_3}{I_3}.$$

Substituting Euler's equations (equations (3.14), (3.15), and (3.16)), we find

$$\begin{aligned}\dot{E} &= \frac{M_1}{I_1}(a_1 M_2 M_3) + \frac{M_2}{I_2}(a_2 M_3 M_1) + \frac{M_3}{I_3}(a_3 M_1 M_2) \\ &= M_1 M_2 M_3 \Big(\frac{a_1}{I_1} + \frac{a_2}{I_2} + \frac{a_3}{I_3} \Big).\end{aligned}$$

Substituting for a_1, a_2, and a_3, we find that

$$\begin{aligned}\dot{E} &= M_1 M_2 M_3 \Big(\frac{I_2 - I_3}{I_1 I_2 I_3} + \frac{I_3 - I_1}{I_1 I_2 I_3} + \frac{I_1 - I_2}{I_1 I_2 I_3} \Big) \\ &= 0.\end{aligned}$$

For equation (3.18), we find the derivative to be

$$\begin{aligned}\mathbf{M} \cdot \dot{\mathbf{M}} &= M_1 \dot{M}_1 + M_2 \dot{M}_2 + M_3 \dot{M}_3 \\ &= M_1(a_1 M_2 M_3) + M_2(a_2 M_3 M_1) + M_3(a_3 M_1 M_2) \\ &= M_1 M_2 M_3 (a_1 + a_2 + a_3) \\ &= M_1 M_2 M_3 \Big(\frac{I_2 - I_3}{I_2 I_3} + \frac{I_3 - I_1}{I_1 I_3} + \frac{I_1 - I_2}{I_1 I_2} \Big) \\ &= M_1 M_2 M_3 \Big(\frac{I_1 I_2 - I_1 I_3 + I_2 I_3 - I_2 I_1 + I_3 I_1 - I_2 I_3}{I_1 I_2 I_3} \Big) \\ &= 0.\end{aligned}$$

□

Geometrically this implies that in the space with coordinates M_1, M_2, and M_3, the vector $\mathbf{M}$ moves so that it lies on the intersection of a sphere of radius $|\mathbf{M}|$ and an Ellipsoid with axis of length $\sqrt{2EI_1}$, $\sqrt{2EI_2}$, and $\sqrt{2EI_3}$. (Equation (3.17) is an ellipsoid as I_1, I_2 and I_3 are positive.)

Therefore, $\mathbf{M}$ lies in the intersection of an ellipsoid and a sphere in a three dimensional space with coordinates M_1, M_2, M_3. To get a feel for the geometry, we assume that $I_1 > I_2 > I_3$. Also, we hold the energy fixed and vary the magnitude $|\mathbf{M}|$ of the angular momentum. This keeps the ellipsoid defined by (3.17) fixed and varies the sphere defined by (3.18) . Since

$$\frac{M_1^2}{2EI_1} + \frac{M_2^2}{2EI_2} + \frac{M_3^2}{2EI_3} = 1,$$

the semi-axes of the ellipsoid will be $\sqrt{2EI_1} > \sqrt{2EI_2} > \sqrt{2EI_3}$. Therefore (see Figure 3.2), we can make the following observations:

1. If the radius of the sphere $|\mathbf{M}| < \sqrt{2EI_3}$ or $|\mathbf{M}| > \sqrt{2EI_1}$, then there is no intersection and thus no corresponding motion.

2. If $|\mathbf{M}| = \sqrt{2EI_3}$ or $|\mathbf{M}| = \sqrt{2EI_1}$, then the intersection is two points.
3. If $\sqrt{2EI_3} < |\mathbf{M}| < \sqrt{2EI_2}$, there will be two closed curves around the ends of the smallest semi-axis.
4. If $\sqrt{2EI_2} < |\mathbf{M}| < \sqrt{2EI_1}$, then we get two closed curves close to the ends of the largest semi-axis.
5. If $|\mathbf{M}| = \sqrt{2EI_2}$, then the intersection consists of two circles that intersect at the ends of the middle semi-axis.

3.6. Poinsot's Description of the Motion.

DEFINITION 3.6.1. The ellipsoid

$$E = \{\mathbf{X} : A\mathbf{X} \cdot \mathbf{X} = 1\} \subset K$$

is called the ***inertia ellipsoid of the body*** at the point O, where A is the inertia operator and K is the moving frame of the body.

As E is defined in K, it will be moving in the inertial frame k. We will give a description of how E moves in k. Also, as it is attached to the body this gives a description of how the body moves. Using the result $T = \frac{1}{2}(A\mathbf{\Omega}, \mathbf{\Omega})$ of Theorem 3.2.3 and the definition of the inertia ellipsoid of a body, the equation of the inertia ellipsoid in terms of the principal axes $\mathbf{e}_i$ has the form

$$I_1 X_1^2 + I_2 X_2^2 + I_3 X_3^2 = 1,$$

where $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$. This implies that the semi-axes of the inertia ellipsoid are directed along the principal axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of the moments of inertia of the body.

Poinsot describes the motion of this inertia ellipsoid in order to describe the motion of the body. The following theorem is Poinsot's description of the motion of this body.

THEOREM 3.6.2. *The inertia ellipsoid rolls without slipping along a stationary plane perpendicular to the angular momentum vector* $\mathbf{m}$. *(Recall that the angular momentum is constant.)*

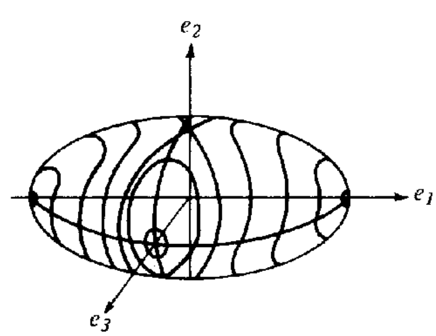

FIGURE 3.2. The curves on the ellipsoid defined by (3.17) are the intersections with the spheres defined by (3.18).

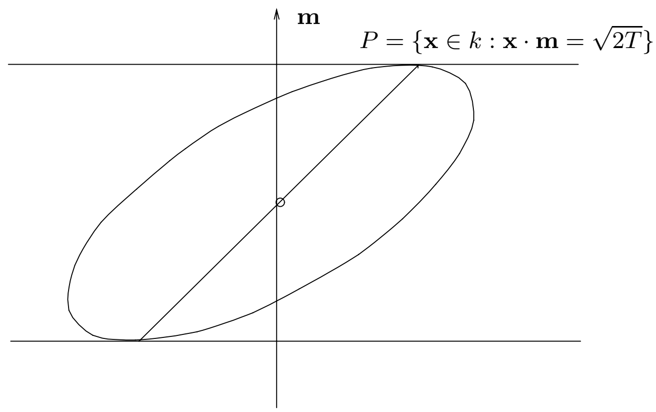

FIGURE 3.3. The ellipsoid of inertia rolls without slipping on the plane $P = \{\mathbf{x} \in k : \mathbf{x} \cdot \mathbf{m} = \sqrt{2T}\}$ orthogonal to the angular momentum $\mathbf{m}$.

PROOF. We first find the normal vector to E at the point $\mathbf{X} \in E$. As E is a level set of the function $\mathbf{X} \mapsto A\mathbf{X} \cdot \mathbf{X}$, the gradient to this function will be a normal. This gradient is

$$\nabla A\mathbf{X} \cdot \mathbf{X} = 2A\mathbf{X}.$$

Letting $c > 0$ be the scalar so that $c\mathbf{\Omega} \in E$ and using Proposition 3.2.1, we see

$$c = 1/\sqrt{2T}$$

where T is the kinetic energy. We also know by Theorem 3.2.5 that the kinetic energy T, and thus, c is constant also. So, we have (using that $A\mathbf{\Omega} = \mathbf{M}$ and the form of the gradient)

$$\mathbf{M} = A\mathbf{\Omega} \text{ is a normal to } E \text{ at } c\mathbf{\Omega}.$$

In the inertial coordinate system k, this normal corresponds to the vector $\mathbf{m} = B\mathbf{M}$ and therefore, in this frame, the point in E with a vector in the direction of $\mathbf{m}$ is the point corresponding to $c\mathbf{\Omega}$, which is the point $c\boldsymbol{\omega}$. Therefore, E is tangent to the plane

$$P = \{\mathbf{x} \in k : \mathbf{x} \cdot \mathbf{m} = c\boldsymbol{\omega} \cdot \mathbf{m}\}$$

at the point $c\mathbf{m}$. We now need to show that $c\boldsymbol{\omega} \cdot \mathbf{m}$ is constant so that this plane does not vary with respect to time t. From Proposition 3.2.1, we have $\mathbf{\Omega} \cdot \mathbf{M} = 2T$. Using that B is orthogonal and thus preserving inner products, we have

$$c\boldsymbol{\omega} \cdot \mathbf{m} = cB^{-1}\boldsymbol{\omega} \cdot B^{-1}\mathbf{m} = c\mathbf{\Omega} \cdot \mathbf{M} = c2T = \frac{2T}{\sqrt{2T}} = \sqrt{2T}.$$

As the kinetic energy is constant, this show that if $\beta = c\boldsymbol{\omega} \cdot \mathbf{m} = \sqrt{2T}$, then E is tangent to $P = \{\mathbf{x} \in k : \mathbf{x} \cdot \mathbf{m} = \beta\}$ at the point $c\boldsymbol{\omega}$.

Let $\gamma(t)$ be the point of contact between E and P at time t so that $t \mapsto \gamma(t)$ is a curve in P. Then $\gamma(t)$ is the intersection of P with the instantaneous axis of rotation. As points on the instantaneous axis of rotation have no instantaneous rotational velocity viewed from either the body or from P, the movement is due to the motion of the instantaneous axis of rotation and not due to the rotation of E. So, the only motion of $\gamma(t)$ with respect to the ellipsoid E is the motion of the instantaneous axis of rotation. Therefore, the velocity vector of $\gamma(t)$ looks the same from either E or the place P. Thus, E rolls without slipping. $\square$

COROLLARY 3.6.3. *Under initial conditions close to a stationary rotation around the large (or small) axis of inertia, the angular velocity* $\boldsymbol{\omega}$ *always remains close to its initial position.*

PROOF. Since the inertial ellipsoid rolls without slipping around the large (or small) axis of inertia, $\boldsymbol{\omega}$ (by proof of the previous theorem) is normal to the ellipsoid as well as collinear with the stationary angular momentum $\mathbf{m}$. Therefore, $\boldsymbol{\omega}$ stays close to its initial position. $\square$

It is possible to be more precise about shape of the curve on the inertial ellipsoid that rolls along the plane P. Letting $\widetilde{\Omega}_i = \Omega_i/\sqrt{2E}$, where E is the energy as given in equation (3.17), and using $M_i = \Omega_i I_i$, equations (3.17) and (3.18) become

$$1 = I_1\widetilde{\Omega}_1^2 + I_2\widetilde{\Omega}_2^2 + I_3\widetilde{\Omega}_3^2, \tag{3.19}$$

$$\frac{\mathbf{M}^2}{2E} = I_1^2\widetilde{\Omega}_1^2 + I_2^2\widetilde{\Omega}_2^2 + I_3^2\widetilde{\Omega}_3^2. \tag{3.20}$$

The first of these says that $\widetilde{\boldsymbol{\Omega}} = \widetilde{\Omega}_1\mathbf{e}_1 + \widetilde{\Omega}_2\mathbf{e}_2 + \widetilde{\Omega}_3\mathbf{e}_3 = \boldsymbol{\Omega}/\sqrt{2E}$ moves on the inertial ellipsoid. So the curve of contact between the plane P and the inertial ellipsoid is the intersection of the two quadratic surfaces defined in $\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3$ space by (3.19) and (3.20).

CHAPTER 4

Lagrange's Top.

Consider a rigid body fixed at a stationary point O and subject to the action of the gravitational force $m\mathbf{g}$. In this chapter we consider the special case of a symmetric rigid body (referred to as a top here), which is a rigid body fixed at a stationary point O whose inertia ellipsoid at O is an ellipsoid of revolution, and whose center of gravity lies on the axis of symmetry $\mathbf{e}_3$. This particular rigid body, shown in the following figure, is often called ***Lagrange's Top*** as he was the first to give a complete analysis of its motion. See Figure 4.1

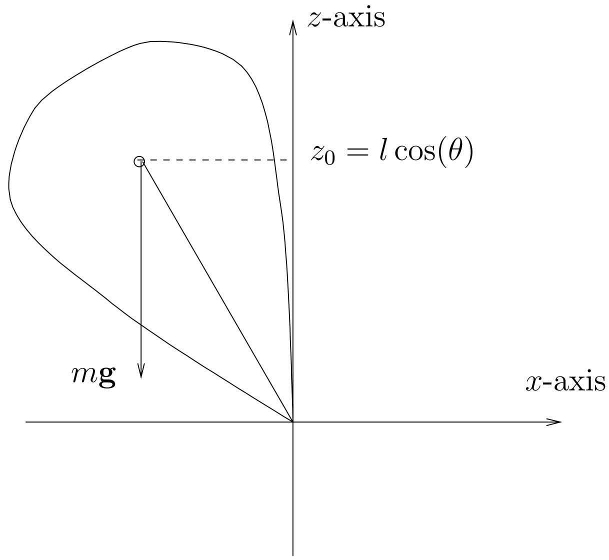

FIGURE 4.1. A rotational symmetric top with fixed point O at the origin. The height z_0 of the center of gravity above the point O is $z_0 = l\cos\theta$, where l is the distance between O and the center of gravity, and θ is the angle with the vertical axis.

Unlike our treatment of the motion of a body without any external forces, where we just used Newton's second law directly, we will use the formalism of the calculus of variations to treat the top under the influence of gravity. We will derive the equations of motion by using Hamilton's principle of least action. Recall that if U is the potential energy and T is the kinetic energy of a conservative dynamical system, then Hamilton's principle says that the equations of motion are the extremals for the variational problem with Lagrangian $L = T - U$. (The total energy $E = T + U$

is conserved.) So our first task is to get formulas for U and T in some convenient coordinate system.

4.1. Euler Angles.

We now introduce the ***Euler Angles*** as a method of describing the position of the top (that is the moving coordinate system) relative to the fixed coordinate system.

The following notation will be used: $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors of the stationary coordinate system at the stationary point O and $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the unit vectors of the moving coordinate system connected to the body and directed along the principal axes at O. Also, $I_1 = I_2 \neq I_3$ are the moments of inertia of the body at O.

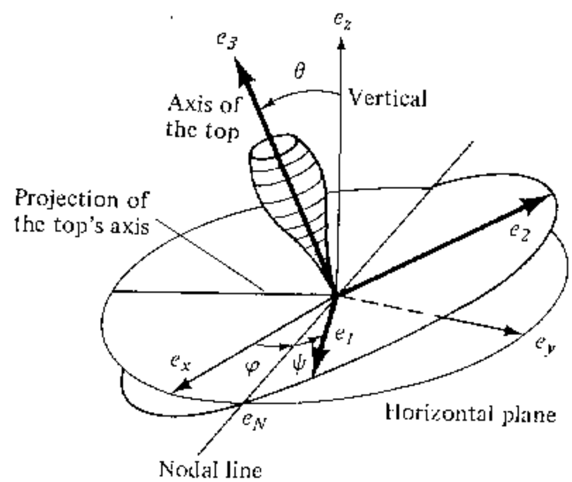

FIGURE 4.2. The geometric meaning of the Euler angles.

In order to carry the stationary frame into the moving frame, three rotations must be performed:

1. Rotate $\mathbf{e}_x$ and $\mathbf{e}_y$ about $\mathbf{e}_z$ by an angle ϕ. So, $\mathbf{e}_z$ is held stationary and $\mathbf{e}_x$ goes to a unit vector $\mathbf{e}_N$ which is in the direction of $\mathbf{e}_z \times \mathbf{e}_3$.
2. Rotate $\mathbf{e}_z$ and $\mathbf{e}_y$ about $\mathbf{e}_N$ an angle θ. So, $\mathbf{e}_z$ goes to $\mathbf{e}_3$ and $\mathbf{e}_N$ stays fixed.
3. Rotate $\mathbf{e}_N$ and $\mathbf{e}_y$ about $\mathbf{e}_3$ an angle ψ. So, $\mathbf{e}_N$ goes to $\mathbf{e}_1$ and $\mathbf{e}_3$ stays fixed.

Therefore, by the three rotations, $\mathbf{e}_x$ has gone to $\mathbf{e}_1$, $\mathbf{e}_z$ to $\mathbf{e}_3$, and $\mathbf{e}_y$ goes to $\mathbf{e}_2$ thru the angles ϕ, ψ, and θ. We can state this formally as a theorem.

THEOREM 4.1.1. *To every triple of numbers ϕ, θ, and ψ, the construction above associates a rotation of three-dimensional space, $B(\phi, \theta, \psi) \in SO(3)$, taking the frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ into the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.*

4.2. Calculation of the Lagrangian Function.

The Lagrangian function will be expressed in terms of the Euler angles and their derivatives. We first describe the potential energy by recalling that the Euler angle

θ is the angle between the upward vertical direction and the direction of the line connecting the fixed point O to the center of gravity of the top. Letting l be the distance between the center of gravity and the the fixed point O, the potential energy U of the top is proportional to the height of the center of gravity. (Figure 4.1.)

The potential energy of Lagrange's top is given by

$$\begin{aligned} U &= mgz_0 \\ &= mgl\cos\theta, \end{aligned}$$

where $m = \sum_i m_i$ is the total mass of the system and $g = |\mathbf{g}|$ is the magnitude of acceleration due to gravity.

The following lemma will help calculate the kinetic energy in terms of the Euler angles.

LEMMA 4.2.1. *The angular velocity of a top is expressed in terms of the derivatives of the Euler angles by the formula*

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_1 + (\dot{\phi}\sin\theta)\mathbf{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3$$

if $\phi = \psi = 0$.

PROOF. We will do the calculation assuming that $\phi = \psi = 0$. The general case can be done by a similar, but substantially messier calculation without this assumption. However, they lead to the same conclusion and so we continue with our assumption.

Consider the velocity of a point of the top occupying the position $\mathbf{r}$ at time t. At a change in time dt,

$$\mathbf{r}(t+dt) = B(\phi + d\phi, \theta + d\theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi)\mathbf{r}(t)$$

where

$$\begin{aligned} d\phi &= \dot{\phi}dt, \\ d\theta &= \dot{\theta}dt, \\ d\psi &= \dot{\psi}dt, \\ \mathbf{r} &= B(\phi, \theta, \psi)B^{-1}(\phi, \theta, \psi)\mathbf{r}. \end{aligned}$$

So,

$$\begin{aligned}
&\mathbf{r}(t+dt) - \mathbf{r}(t) \\
&\qquad = [B(\phi + d\phi, \theta + d\theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi) - B(\phi, \theta, \psi)B^{-1}(\phi, \theta, \psi)]\mathbf{r} \\
&\qquad = [B(\phi + d\phi, \theta + d\theta, \psi + d\psi) - B(\phi, \theta, \psi)]B^{-1}(\phi, \theta, \psi)\mathbf{r} \\
&\qquad = [B(\phi + d\phi, \theta, \psi) - B(\phi, \theta, \psi) + B(\phi, \theta + d\theta, \psi) \\
&\qquad\qquad - B(\phi, \theta, \psi) + B(\phi, \theta, \psi + d\psi) - B(\phi, \theta, \psi)]B^{-1}(\phi, \theta, \psi)\mathbf{r} \\
&\qquad = [B(\phi + d\phi, \theta, \psi)B^{-1}(\phi, \theta, \psi)\mathbf{r} - \mathbf{r}] \\
&\qquad\qquad + [B(\phi, \theta + d\theta, \psi)B^{-1}(\phi, \theta, \psi)\mathbf{r} - \mathbf{r}] \\
&\qquad\qquad + [B(\phi, \theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi)\mathbf{r} - \mathbf{r}] \\
&\qquad = (\boldsymbol{\omega}_\phi \times \mathbf{r}dt) + (\boldsymbol{\omega}_\theta \times \mathbf{r}dt) + (\boldsymbol{\omega}_\psi \times \mathbf{r}dt)
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial B}{\partial \phi}B^{-1}\mathbf{r} &\equiv \boldsymbol{\omega}_\phi \times \mathbf{r} \\
\frac{\partial B}{\partial \theta}B^{-1}\mathbf{r} &\equiv \boldsymbol{\omega}_\theta \times \mathbf{r} \\
\frac{\partial B}{\partial \psi}B^{-1}\mathbf{r} &\equiv \boldsymbol{\omega}_\psi \times \mathbf{r}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbf{r}(t + dt) - \mathbf{r}(t) &= (\boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi) \times \mathbf{r} \\
&= \boldsymbol{\omega} \times \mathbf{r}
\end{aligned}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$ is the angular velocity of the body.

If $\phi = \psi = 0$, then $B(\phi + d\phi, \theta, \psi)B^{-1}(\phi, \theta, \psi)$ is simply a rotation around the axis $\mathbf{e}_z$. So,

$$\boldsymbol{\omega}_\phi = \dot{\phi}\mathbf{e}_z.$$

Since for the rotation, $B(\phi, \theta + d\theta, \psi)B^{-1}(\phi, \theta, \psi)$, $\phi = 0$, $\mathbf{e}_x = \mathbf{e}_N$, and $\psi = 0$, we have $\mathbf{e}_x = \mathbf{e}_1$. So, $\mathbf{e}_N = \mathbf{e}_1$. Therefore, this rotation is simply a rotation $d\theta$ about the $\mathbf{e}_1$ axis. Thus,

$$\boldsymbol{\omega}_\theta = \dot{\phi}\mathbf{e}_1.$$

The rotation $B(\phi, \theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi)$ is just the rotation $d\psi$ about the $\mathbf{e}_3$ axis. So,

$$\boldsymbol{\omega}_\psi = \dot{\psi}\mathbf{e}_3.$$

Since

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi,$$

we have that

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{e}_z + \dot{\theta}\mathbf{e}_1 + \dot{\psi}\mathbf{e}_3. \tag{4.1}$$

Also since $\phi = \psi = 0$, we have

$$\mathbf{e}_z = \mathbf{e}_3 \cos\theta + \mathbf{e}_2 \sin\theta.$$

Substitution into equation (4.1) gives our conclusion:

$$\begin{aligned}\boldsymbol{\omega} &= \dot{\phi}(\mathbf{e}_3 \cos\theta + \mathbf{e}_2 \sin\theta) + \dot{\theta}\mathbf{e}_1 + \dot{\psi}\mathbf{e}_3 \\ &= \dot{\theta}\mathbf{e}_1 + (\dot{\phi}\sin\theta)\mathbf{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3\end{aligned}$$

□

From this lemma, we can conclude that the components of the angular velocity along the principal axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are

$$\begin{aligned}\omega_1 &= \dot{\theta} \\ \omega_2 &= \dot{\phi}\sin\theta \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta.\end{aligned}$$

Since we know that the kinetic energy T is

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

for $\phi = \psi = 0$, we can substitute for the angular velocities to get

$$T = \frac{1}{2}(I_1\dot{\theta}^2 + I_2\dot{\phi}^2\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2).$$

Since $I_1 = I_2$, we get

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2. \tag{4.2}$$

We now summarize our calculations for the potential and kinetic energy with the following proposition:

Proposition 4.2.2. *The Lagrangian, which is the difference of the kinetic and potential energy of the top, is given by*

$$\begin{aligned}L &\equiv T - U \\ &= \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta. \end{aligned} \tag{4.3}$$

Hamilton's Principle states that for a mechanical system described by generalized coordinates $y_1, \dots, y_n$ with Lagrangian given by

$$\begin{aligned}L &= (t, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n) \\ &\equiv T - U,\end{aligned}$$

the motion of the system from time t_0 to t_1 is such that the functional

$$J(y_1, \dots, y_n) = \int_{t_0}^{t_1} L(t, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n)dt \tag{4.4}$$

is stationary for the functions $y_1(t), \dots, y_n(t)$ which describe the actual time evolution of the system.

We can restate this principle in the following manner. Regard the set of coordinates $y_1, \ldots, y_n$ as coordinates in an n dimensional space. Then the equations

$$y_i(t) = y_i(t),\ i = 1, \ldots, n,\ t_o \leq t \leq t_1$$

can be regarded as parametric equations of a curve or path C in the space that joins two states $S_o : (y_1(t_0), \ldots, y_n(t_0))$ and $S_1 : (y_1(t_1), \ldots, y_n(t_1))$. Hamilton's principle than states that among all paths in the space connecting the initial state S_0 to the final state S_1, the actual motion will take place along the path that affords an extreme value to integral (4.4). Also, since the motion of the system from t_0 to t_1 is stationary for the functions $y_i = y_i(t), i = 1, \ldots, n$, it follows from the calculus of variations ([**1**, pp. 59–60] or [**2**, pp. 187–188]) that $y_i(t)$ must satisfy the Euler equations

$$L_{y_i} - \frac{d}{dt} L_{\dot{y}_i} = 0, \quad i = 1, \ldots, n.$$

We will just assume this result, and the reader can view it as an efficient method of writing down Newton's equations of motion for complicated systems.

4.3. Investigation of the Motion.

The Euler Lagrange equations for our body are

$$\frac{d}{dt} L_{\dot{\phi}} - L_{\phi} = 0 \tag{4.5}$$

$$\frac{d}{dt} L_{\dot{\theta}} - L_{\theta} = 0 \tag{4.6}$$

$$\frac{d}{dt} L_{\dot{\psi}} - L_{\psi} = 0. \tag{4.7}$$

Also note that $L_\phi = L_\psi = 0$. So the Euler Lagrange equations (4.5) and (4.7) imply that $L_{\dot{\phi}}$ and $L_{\dot{\psi}}$ are constant. This gives two first integrals for the motion.

Proposition 4.3.1. *There are constants M_z and M_3 so that*

$$\begin{aligned}
L_{\dot{\phi}} &= I_1 \dot{\phi} \sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\
&= \dot{\phi}(I_1 \sin^2\theta + I_3\cos^2\theta) + \dot{\psi} I_3 \cos\theta \\
&= M_z \qquad (4.8)\\
L_{\dot{\psi}} &= I_3(\dot{\psi} + \dot{\phi}\cos\theta) \\
&= \dot{\phi} I_3 \cos\theta + \dot{\psi} I_3 \\
&= M_3. \qquad (4.9)
\end{aligned}$$

We now show that the total energy E is constant which gives a third first integral.

Theorem 4.3.2 (Conservation of Energy). *Equations (4.5), (4.6), and (4.7) imply that the total energy $E = T + U$ is constant.*

PROOF. Using the form of the Lagrangian as given by (4.3)

$$\begin{aligned} L_{\dot\phi} &= T_{\dot\phi} = I_1\dot\phi\sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta \\ L_{\dot\theta} &= T_{\dot\theta} = I_1\dot\theta \\ L_{\dot\psi} &= T_{\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta), \end{aligned}$$

and using the form of the kinetic energy given by (4.2)

$$\begin{aligned} &\dot\phi L_{\dot\phi} + \dot\theta L_{\dot\theta} + \dot\psi L_{\dot\psi} \\ &\quad = \dot\phi(I_1\dot\phi\sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta) + \dot\theta(I_1\dot\theta) + \dot\psi I_3(\dot\psi + \dot\phi\cos\theta) \\ &\quad = I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + I_3(\dot\psi + \dot\phi\cos\theta)^2. \\ &\quad = 2T, \end{aligned}$$

we can rewrite the energy as

$$E = T + U = 2T - (T - U) = \dot\phi L_{\dot\phi} + \dot\theta L_{\dot\theta} + \dot\psi L_{\dot\psi} - L.$$

Using this formula for E, we have

$$\begin{aligned} \frac{dE}{dt} =& \ddot\phi L_{\dot\phi} + \dot\phi\frac{d}{dt}L_{\dot\phi} + \ddot\theta L_{\dot\theta} + \dot\theta\frac{d}{dt}L_{\dot\theta} + \ddot\psi L_{\dot\psi} + \dot\psi\frac{d}{dt}L_{\dot\psi} \\ & - \ddot\phi L_{\dot\phi} - \dot\phi L_\phi - \ddot\theta L_{\dot\theta} - \dot\theta L_\theta - \ddot\psi L_{\dot\psi} - \dot\psi L_\psi \\ =& \dot\phi\left(\frac{d}{dt}L_{\dot\phi} - L_\phi\right) + \dot\theta\left(\frac{d}{dt}L_{\dot\theta} - L_\theta\right) + \dot\psi\left(\frac{d}{dt}L_{\dot\psi} - L_\psi\right) \\ =& 0, \end{aligned}$$

where at the last step we have used the Euler Lagrange equations (4.5), (4.6), and (4.7). This completes the proof that E is constant. □

THEOREM 4.3.3. *The inclination θ of the axis of the top to the vertical changes with time in the same way as in the one-dimensional system with energy*

$$E_{\text{bar}} = \frac{I_1}{2}\dot\theta^2 + U_{\text{eff}}(\theta),$$

where the effective potential energy is given by

$$U_{\text{eff}} = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta.$$

(We will show that E_{bar} is constant as a function of time t. A stronger statement is true: The extremal curves $t \mapsto \theta(t)$ for this one dimensional problem are the same curves as for the three dimensional problem. As this is not needed in what follows, we omit the proof.)

PROOF. The total energy of the system is given by

$$\begin{aligned} E &= T + U \\ &= \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 + mgl\cos\theta. \end{aligned} \tag{4.10}$$

Now we shall express $\dot{\phi}$ and $\dot{\psi}$ in terms of M_3 and M_z. Solving equation (4.9) for $\dot{\psi}$ gives:

$$\dot{\psi} = \frac{M_3 - \dot{\phi} I_3 \cos\theta}{I_3} \tag{4.11}$$

Substituting this into equation (4.8) gives:

$$\begin{aligned} M_z &= \dot{\phi}(I_1 \sin^2\theta + I_3 \cos^2\theta) + (M_3 - \dot{\phi} I_3 \cos\theta)\cos\theta \\ &= \dot{\phi}(I_1 \sin^2\theta + I_3 \cos^2\theta - I_3 \cos^2\theta) + M_3 \cos\theta \end{aligned}$$

Solving for $\dot{\phi}$ gives:

$$\dot{\phi} = \frac{M_z - M_3 \cos\theta}{I_1 \sin^2\theta} \tag{4.12}$$

To find $\dot{\psi}$ in terms of M_z and M_3, we substitute equation (4.12) into equation (4.11). Then,

$$\begin{aligned} \dot{\psi} &= \frac{M_3}{I_3} - \frac{M_z \cos\theta}{I_1 \sin^2\theta} + \frac{M_3}{I_1}\frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{M_3 I_1 \sin^2\theta - M_z I_3 \cos\theta + M_3 I_3 \cos^2\theta}{I_3 I_1 \sin^2\theta}. \end{aligned} \tag{4.13}$$

After substituting equations (4.12) and (4.13) into equation (4.10) for the kinetic energy, we get

$$\begin{aligned} E &= \frac{I_1}{2}\left[\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{I_1^2 \sin^2\theta}\right] \\ &+ \frac{I_3}{2}\left[\frac{M_3 I_1 \sin^2\theta - M_z I_3 \cos\theta + M_3 I_3 \cos^2\theta}{I_3 I_1 \sin^2\theta} + \frac{M_z \cos\theta - M_3 \cos^2\theta}{I_1 \sin^2\theta}\right]^2 \\ &\quad + mgl\cos\theta \\ &= \frac{I_1}{2}\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{2I_1 \sin^2\theta} \\ &+ \frac{I_3}{2}\left[\frac{M_3 I_1 \sin^2\theta - M_z I_3 \cos\theta + M_3 I_3 \cos^2\theta + M_z I_3 \cos\theta - M_3 I_3 \cos^2\theta}{I_3 I_1 \sin^2\theta}\right]^2 \\ &\quad + mgl\cos\theta \\ &= \frac{I_1}{2}\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{2I_1 \sin^2\theta} + \frac{I_3}{2}\left[\frac{M_3}{I_3}\right]^2 + mgl\cos\theta \\ &= \frac{I_1}{2}\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{2I_1 \sin^2\theta} + \frac{M_3^2}{2I_3} + mgl\cos\theta. \end{aligned}$$

Therefore, if we set

$$E_{\text{bar}} = \frac{I_1}{2}\dot{\theta}^2 + U_{\text{eff}}(\theta)$$

where

$$U_{\text{eff}} = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta,$$

we then have that

$$E = E_{\text{bar}} + \frac{M_3^2}{2I_2}.$$

Since $M_3^2/(2I_2)$ is constant, E and E_{bar} differ by a constant and since E is constant by Theorem 4.3.2, E_{bar} is constant. □

To study this one dimensional system, it is convenient to make the substitution $\cos\theta = u$ and to write

$$\begin{aligned}
\frac{M_z}{I_1} &= a, \\
\frac{M_3}{I_1} &= b, \\
\frac{2E_{\text{bar}}}{I_1} &= \alpha, \\
\frac{2mgl}{I_1} &= \beta > 0.
\end{aligned}$$

So,

$$\begin{aligned}
\dot{u} &= (-\sin\theta)\dot{\theta} \\
\dot{u}^2 &= (\sin^2\theta)(\dot{\theta})^2 \\
&= (1-u^2)(\dot{\theta})^2.
\end{aligned}$$

Making these substitutions into E_{bar}, we get

$$E_{\text{bar}} = \frac{I_1}{2}\frac{\dot{u}^2}{1-u^2} + \frac{(M_z - M_3 u)^2}{2I_1(1-u^2)} + mglu.$$

Since

$$\begin{aligned}
\dot{\theta}^2 &= \frac{\dot{u}^2}{1-u^2}, \\
\cos^2\theta = u^2 &= 1 - \sin^2\theta,
\end{aligned}$$

we can solve for $\dot{u}^2$ to find

$$\begin{aligned}
\dot{u}^2 &= \left[E_{\text{bar}} - mglu - \frac{(M_z - M_3u)^2}{2I_1(1-u^2)}\right]\frac{2(1-u^2)}{I_1} \\
&= \frac{2E_{\text{bar}}(1-u^2)}{I_1} - \frac{2mglu(1-u^2)}{I_1} - \frac{(M_z - M_3u)^2}{I_1^2} \\
&= \frac{2}{I_1}(1-u^2)(E_{\text{bar}} - mglu) - \frac{1}{I_1^2}(M_z - M_3u)^2 \\
&= (1-u^2)(\frac{2E_{\text{bar}}}{I_1} - \frac{2mglu}{I_1}) - (\frac{M_z}{I_1} - \frac{M_3}{I_1}u)^2 \\
&= (1-u^2)(\alpha - \beta u) - (a - bu)^2.
\end{aligned}$$

So the Law of Conservation of Energy E_{bar} can be rewritten as

$$\dot{u}^2 = f(u) \tag{4.14}$$

where

$$f(u) = (1-u^2)(\alpha - \beta u) - (a - bu)^2.$$

Obviously, $f(u)$ is a polynomial of degree three, $f(+\infty) = +\infty$, and $f(-\infty) = -\infty$. Also, $f(\pm 1) = -(a \mp b)^2$ and since $u = \cos\theta$, we have that u falls in the interval $-1 \leq u \leq 1$. Further, due to equation (4.14), we have $f(u) \geq 0$ for some $-1 \leq u \leq 1$. From this analysis, we can conclude that $f(u)$ has three roots, two of which, u_1 and u_2, lie in the interval $-1 \leq u \leq 1$, while one follows the inequality $u > 1$. The graph in Figure 4.3 illustrates this.

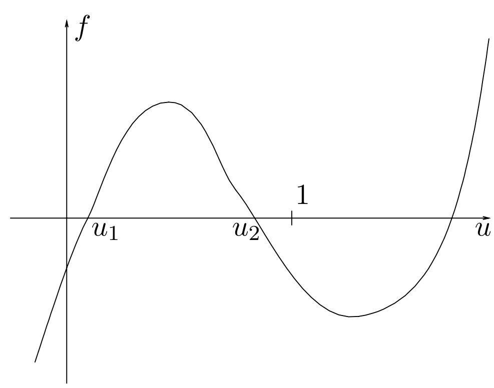

FIGURE 4.3. The cubic polynomial equation $f(u) = 0$ has three real roots, one of which is > 1. The other two are in the interval $[-1, 1]$.

Thus, the inclination θ of the axis of the top changes periodically between two limit values θ_1 and θ_2. This periodic change in the inclination θ of the axis is referred to as ***nutation***.

From the previous proof, we know that the law of variation of the azimuth ϕ is

$$\dot{\phi} = \frac{M_z - M_3\cos\theta}{I_1 \sin^2\theta}.$$

Using the substitutions of the previous proof ($\frac{M_z}{I_1} = a$, $\frac{M_3}{I_1} = b$, and $\cos\theta = u$), we have

$$
\begin{aligned}
\dot{\phi} &= \frac{a - bu}{1 - u^2} \\
&= \frac{a - b\cos\theta}{1 - \cos^2\theta} \\
&= \frac{a - b\cos\theta}{\sin^2\theta}, \qquad (4.15)
\end{aligned}
$$

which gives the variation of the azimuth of the axis.

The azimuthal motion is called ***precession***, which we now discuss in more detail. Looking from above the top, the point of intersection of the axis of the top with the unit sphere moves in a ring between two parallels θ_1 and θ_2 (from nutation), where $\cos\theta_1 = u_1$ and $\cos\theta_2 = u_2$.

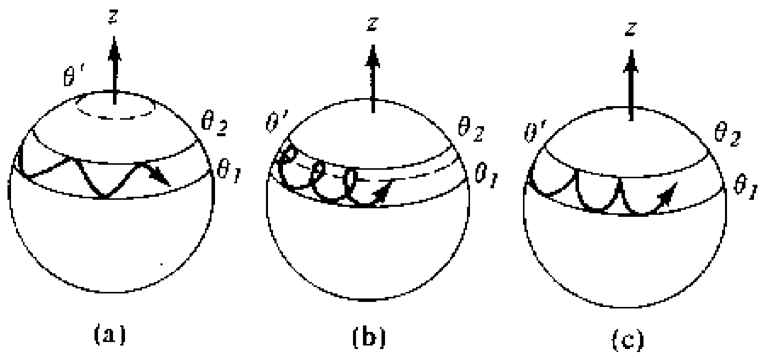

FIGURE 4.4. The three possibilities for the azimuthal motion (i.e., precession) of the axis of the top. When $\theta' > \theta_2$, the motion is monotone. When $\theta_1 < \theta' < \theta_2$, the monotonicity of the azimuthal changes each time the axis passes $\theta = \theta'$. When $\theta' = \theta_2$, the axis of the top "bounces" off the line $\theta = \theta'$ so that it traces out a curve with cusps.

The roots of equation (4.15) occur when $a = bu$ (or, in terms of θ, when $a = b\cos\theta$). Calling this root θ', we analyze several cases (see Figure 4.4):

1. The root θ' is not in the interval $[\theta_1, \theta_2]$. Then the expression $(a - b\cos\theta)/(\sin^2\theta)$ for $\dot{\phi}$ in (4.15) does not change sign and therefore ϕ is a monotone function of t. So, the axis of the top moves as pictured as in (a) of Figure 4.4.
2. The root θ' is in the open interval (θ_1, θ_2). Then $\dot{\phi}$ changes sign each time θ crosses the line $\theta = \theta'$. A change of sign of $\dot{\phi}$ corresponds to a change of monotonicity of ϕ each time the axis of the top moves past $\theta = \theta'$. This is pictured in (b) of Figure 4.4.
3. The last case is when $\theta' = \theta_2$. Then the axis of the top moves so that it has instantaneous velocity zero each time θ hits its maximum value of $\theta = \theta_2 = \theta'$. Thus, the curve traced on the unit sphere by the axis has cusps each time $\theta = \theta_2$ as in (c) of Figure 4.4

We have now given a detailed description of nutation (periodic motion of θ) and the azimuthal motion or precession. As the motion of the top consist of rotation around its own axis, nutation, and precession, this completes our analysis concerning the motion of a top.

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