ON THE EXISTENCE AND NONEXISTENCE OF STABLE SUBMANIFOLDS AND CURRENTS IN POSITIVELY CURVED MANIFOLDS AND THE TOPOLOGY OF SUBMANIFOLDS IN EUCLIDEAN SPACES

RALPH HOWARD AND SHIHSHU WALTER WEI

ABSTRACT. Let M^n be a compact Riemannian manifold isometrically immersed in the Euclidean space $\mathbb{R}^{n+m}$. Then a modification of a beautiful method first used by Lawson and Simons [22] is used to give a pointwise algebraic condition on the second fundamental form of M in $\mathbb{R}^{n+m}$ which implies that M has no complete stable submanifolds (or integral currents) in some given dimension $p \quad (1 \leq p \leq n-1)$. It is then shown that this condition is preserved under small deformation of the metric on M in the C^2 topology. Some results of these are (1) There is a C^2 neighborhood of the standard metric on the Euclidean sphere S^n (and other sufficiently convex hypersurfaces in $\mathbb{R}^{n+1}$), such that for any g in this neighborhood (S^n, g) has no stable submanifolds (or integral currents). (2) A characterization (and in some cases a classification) of stable submanifolds and integral currents of all the rank one symmetric spaces (extending the work of Lawson and Simons on the spheres and complex projective spaces [22]), and some information about what happens in this case under a small C^2 deformation of the metric. (3) There is a C^2 neighborhood $\mathcal{U}$ of the standard metric on the complete simply connected manifold $R^{n+m}(c)$ of constant sectional curvature $c \geq 0$ such that if M^n is a compact immersed submanifold of $(R^{n+m}(c), g)$ with mean curvature vector and the second fundamental form satisfying (5.17) for some $g \in \mathcal{U}$, $1 \leq p \leq \frac{n}{2}$, and $q = n-1$. Then (a) M has no stable p-integral or $(n-p)$-integral currents over any finitely abelian group G, (b) $H_p(M,G) = H_q(M,G) = 0$ for any G and if $p=1$ or $p=n-1$ then M is simply connected, and (c) If (5.17) holds for some $g \in \mathcal{U}$, and $p=1$, then M is *diffeomorphic* to S^n for all $n \geq 2$. Indeed, by a Theorem of Federer and Fleming [10] every non-zero integral homology class in a compact manifold N can be represented by a stable integral current; thus, the method can also be used to give pointwise conditions on the second fundamental form of a compact submanifold which forces some of the integral homology groups of M to vanish.

2010 *Mathematics Subject Classification.* 53C42, 53A10, 49Q05, 49Q15, 57N65, 14F40.

Key words and phrases. Stable Currents; Mass; Second Fundamental Forms; Mean Curvature; Stable Minimal Submanifolds; Homology Groups; Cohomology Groups with compact support.

Research of the second author supported in part by NSF (DMS-1240370), the OU Office of the Vice President for Research, and the OU College of Arts and Sciences.

1. Introduction

In this paper we will be concerned with how the geometry of a Riemannian manifold M^n effects the existence, or more importantly the nonexistence, of stable submanifolds and currents in M. Stability in our context means any deformation with compact support does not decrease the volume or mass (cf. (2.8)). In particular, we will give conditions on the second fundamental form of an immersed submanifold M of a Euclidean space which implies that M supports no stable submanifolds or currents in dimension p. These conditions are related to M being positively curved in a sense made precise by Proposition 3.1 below. This has strong implications about the topology of M because of the existence theorems of Federer and Fleming which state that every nonzero homology class (over a finitely generated Abelian group) in a compact manifold contains a stable current.

The earliest result of this type known to us is the result of Synge that a compact, orientable, even dimensional Riemannian manifold of positive sectional curvature has no stable closed geodesics. As any free homotopy class contains a stable geodesic, this implies the toplogical result that any such manifold must be simply connected. The first results about the nonexistence of stable submanifolds other than closed geodesics seems to be in the celebrated paper of Simons [31] on minimal varieties. He shows that the standard sphere S^n contains no stable submanifolds and that there are no oriented closed stable hypersurfaces in an oriented manifold with positive Ricci curvature. The first of these results was extended by Lawson and Simons [22] to show that S^n and its submanifolds whose second fundamental form are small enough have no stable currents. This gives very strong results about the topology of these submanifolds. On the basis of these results they made the

Conjecture A (Lawson and Simons [22])**.** *There are no closed stable submanifolds (or rectifiable currents) in any compact, simply connected, strictly $\frac{1}{4}$-pinched Riemannian manifold.*

We are able to verify this conjecture for several classes of positively curved manifolds.

Theorem 1. *There is a neighborhood in the C^2 topology of the usual metric on the Euclidean sphere S^n such that for any metric g in this neighborhood (S^n, g) has no stable rectifiable currents. (In fact, no stable varifolds).*

Theorem 2. *Let $M^n (n \geq 3)$ be a compact hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ which is pointwise δ-pinched for*

$$\delta = \frac{1}{4} + \frac{3}{n^2 + 4}$$

then M has no stable rectifiable currents (or varifolds) and $M$is diffeomorphic to a sphere.

On the basis of our results we feel a stronger conjecture is justified. (In what follows G is a finitely generated abelian group, and $H_p(M, G)$ is the p-th singular homology group of M with coefficients in G.)

Conjecture B. *Let M^n be a compact Riemannian manifold of positive sectional curvature. If $H_p(M, G) = 0$ (and if $p = 1$ also assume M is simply connected) then there are no stable rectifiable currents of dimension p over the group G in M.*

As evidence supporting this conjecture we have,

Theorem 3. *Let (M, g_0) be a compact simply connected rank one symmetric space with its usual metric g_0. Then there is a neighborhood of g_0 in the C^2 topology such that for any metric g in this neighborhood the Riemannian manifold (M, g) satisfies* Conjecture B.

Along the way to proving this we also characterize the stable currents in the simply connected rank one symmetric spaces.

Theorem 4. *Let $\mathbb{F} = \mathbb{H}$ or $\mathbb{C}\mathrm{ay}$ and $\mathcal{S} \in \mathcal{R}_p(\mathbb{FP}^n, G)$ be a stable current, where $\mathbb{FP}^n = \mathbb{HP}^n$ or $\mathbb{C}\mathrm{ay}\mathbb{P}^2$. Then*

(a) For $\|\mathcal{S}\|$ almost all $x \in \mathbb{FP}^n$, the approximate tangent space $T_x(\mathcal{S})$ is an $\mathbb{F}$-subspace of $T_x(\mathbb{FP}^n)$. There is also a set of smooth vector fields $V_1, \ldots, V_\ell$ on $\mathbb{FP}^n$ such that for every p with $1 \leq p \leq n \cdot \dim_{\mathbb{R}}(\mathbb{F})$ that is not divisible by $\dim_{\mathbb{R}}(\mathbb{F})$, the set $V_1, \ldots, V_\ell$ is universally mass decreasing in dimension p.

(b) If $\mathbb{FP}^n = \mathbb{HP}^n$, $p = 4k$, and the $(4k-1)$-dimensional Hausdorff measure of the singular set of $\mathcal{S}$ is zero, then there are a finite number $L_1, \ldots, L_\ell$ of $\mathbb{HP}^k$'s in $\mathbb{HP}^n$ and elements $a_1, \ldots, a_\ell \in G$ so that the current $\mathcal{S} = a_1 L_1 + \cdots + a_\ell L_\ell$. Thus,

(c) The only connected stable submanifolds of $\mathbb{HP}^n$ are the totally geodesic $\mathbb{HP}^k$'s, $1 \leq k \leq n$.

(d) If $\mathbb{FP}^n = \mathbb{HP}^n$, $G = \mathbb{Z}_2$, and $\mathcal{S}$ is a mass minimizing element of a nonzero $\mathbb{Z}_2$ homology class of dimension $4k$, then $\mathcal{S} = a_1 L_1 + \cdots + a_\ell L_\ell$, for some $a_i \in G$, finite number L_i of $\mathbb{HP}^k$'s in $\mathbb{HP}^n$, $1 \leq i \leq \ell$.

(e) If $\mathbb{FP}^n = \mathbb{C}\mathrm{ay}\mathbb{P}^2$, $p = 8$, and the 7-dimensional Hausdorff measure of the singular set of $\mathcal{S}$ is zero, then there are a finite number $L_1, \ldots, L_\ell$ of $\mathbb{C}\mathrm{ay}\mathbb{P}^1$'s in $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ and elements $a_1, \ldots, a_\ell \in G$ so that as the current $\mathcal{S} = a_1 L_1 + \cdots + a_\ell L_\ell$. Thus,

(f) The only connected stable submanifolds of $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ are the totally geodesic $\mathbb{C}\mathrm{ay}\mathbb{P}^1$'s.

For the Euclidean spheres and the complex projective spaces these have already done by Lawson and Simons [22]. The above classification results (a), (c) and (f) are also obtained by Ohnita [24]. The non-simply connected rank one symmetric spaces are the real projective spaces $\mathbb{RP}^n$. For these we show the following (cf.

section 7), where the case $\mathcal{S}$ is a p-dimensional stable minimal submanifold is also obtained by Ohnita [24].

Theorem 5. *Let $0 \neq \alpha \in H_p(\mathbb{RP}^n, \mathbb{Z}_2)$ and let $\mathcal{S}$ be a closed rectifiable current in α of least mass. Then, up to a rigid motion of $\mathbb{RP}^n$, $\mathcal{S}$ is just the standard imbedding of $\mathbb{RP}^p$ into $\mathbb{RP}^n$.*

Our basic method is to isometrically immerse the manifold we wish to study in a Euclidean space $\mathbb{R}^{n+m}$. Then for any parallel vector field v on $\mathbb{R}^{n+m}$ and compact p-dimensional submanifold (or rectifiable current) N^p of M^n we deform N along the flow $\varphi_t^{V^T}$ of the vector field V^T obtained by taking the orthogonal projections of V onto tangent spaces of M. In this case the formula for the second variation (which will be nonnegative when N is stable)

$$\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{vol}(\varphi_t^{V^T} N) \tag{1.1}$$

can be greatly simplified by use of the fundamental equations of Gauss and Weingarten in submanifold theory. The resulting formula still has one term (involving the covariant derivative of the Weingarten map) that is hard to understand, but if (1.1) is averaged over an orthonormal basis $\{v_1, \dots v_{n+m}\}$ of $\mathbb{R}^{n+m}$, this term drops out. The result is a pointwise algebraic condition on the second fundamental form of M (corresponding to the average of (1.1) over $v_1 \dots v_{n+m}$ being negative) which implies M has no stable currents in dimension p. This method is in contrast to the method of Lawson-Simons [22] in which the gradient vector fields of the first eigenvalues of the Laplacian are used to deform p-rectifiable currents in the unit sphere S^n. Our method which can be viewed as an extrinsic average variational method [36], does not require the symmetry of the ambient manifold M^n and the deformation vector fields agree with the conformal gradient vector fields when $M^n = S^n$. This method has the advantages that it also works when M is not compact and the resulting criterion for the nonexistence of stable currents is preserved under small deformations of the metric in the C^2 topology. Therefore we can also conclude that (M, g) has no stable currents for g sufficiently close to the original metric. The method also gives (what seem to us) striking results about the topology of submanifolds of Euclidean space. For example in sections 4 and 5 it is shown, among other things, that the following hold.

Theorem 6. *Let M^n be a compact immersed hypersurface in $\mathbb{R}^{n+1}$ with principal curvatures $k_1 \leq \cdots \leq k_n$. Assume for some $1 \leq p \leq n-1, q = n-p$ that*

(a) $0 < k_1 + \cdots + k_p$

(b) $k_{q+1} + \cdots + k_n < k_1 + \cdots + k_p + k_1 + \cdots + k_q$,

then $H_p(M, G) = H_q(M, G) = 0$ and if $p = 1$ or $n-1$ then $\pi_1(M) = 1$. If (a) and (b) hold for $1 \leq p \leq \frac{n}{2}$ or $\frac{n}{2} \leq p \leq n$ then M is homeomorphic to a sphere.

Theorem 7. *Let M^n be a compact immersed submanifold of the simply connected Riemannian space form $R^{n+m}(c)$ of dimension $n+m$ and constant sectional curvature $c \geq 0$. Let h be the second fundamental form and H the mean curvature vector of M in $R^{n+m}(c)$. If for some $p \leq \frac{n}{2}$ the inequality*

$$||h||^2 < \frac{n^2}{n-p}||H||^2 + 2pc \tag{1.2}$$

holds at all points of M then $H_k(M, G) = 0$ for $p \leq k \leq n-p$ and if it holds for $p = 1$ then M is a homeomorphic to a sphere.

In the process of proving this Theorem, we have proved the following:

Corollary. *With the notation of Theorem 7, if for $1 \leq p \leq \frac{n}{2}$ and $q = n-p$,*

$$\sum_{i=1}^{p} \sum_{\ell=p+1}^{n} \Big(2||h(e_i, e_\ell)||^2 - \langle h(e_i, e_i), h(e_\ell, e_\ell)\rangle\Big) < pqc \tag{5.19}$$

at all points where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_xM. then

(a) there are no stable currents in $\mathcal{R}_p(M, G)$ or $\mathcal{R}_q(M, G)$ for any finitely generated abelian group G. In particular M has no closed stable minimal submanifolds of dimension p or $n-p$.

(b) $H_p(M, G) = H_q(M, G) = 0$ and if $p = 1$ or $p = n-1$ then M is simply connected.

(c) If (5.19) hold for $p = 1$ or for $p = n-1$ then M is a topological sphere. Furthermore, when $n = 2$ or $n = 3$ M is diffeomorphic to a sphere.

These results (a) and (b) are also obtained by Xin [38]. In fact, Theorem 1 in [38] treats the Euclidean space case (c=0), where the right hand side of (5.19) becomes 0. As an application, this recaptures a Trace Formula of Lawson-Simons on the unit Euclidian sphere case (c=1), where the right hand side of (5.19) becomes $pq \cdot 1$. (cf. also [35, p.538] for the results (a), (b) and (c)).

Theorem 8. *There is a C^2 neighborhood $\mathcal{U}$ of the standard metric g_0 on $R^{n+m}(c)$, $c \geq 0$ such that if M^n is a compact immersed submanifold of $R^{n+m}(c)$ with the mean curvature vector H and the second fundamental form h satisfying*

$$q\|h\|^2 < n^2\|H\|^2 + 2pqc \tag{5.17}$$

with respect to some g in this neighborhood $\mathcal{U}$, $1 \leq p \leq \frac{n}{2}$, and $q = n-1$. Then for any finitely generated abelian group G,

(a) M has no stable submanifolds of dimension p or $n-p$ $\big($or stable rectifiable G-currents of degree p or $n-p$ $\big)$.

(b) $H_p(M, G) = H_q(M, G) = 0$ and if $p = 1$ or $p = n-1$ then M is simply connected.

(c) If (5.17) *holds for some* $g \in \mathcal{U}$ *and* $p = 1$*, i.e.*

$$\|h\|^2 < \frac{n^2}{n-1}\|H\|^2 + 2c\,, \tag{5.20}$$

then M *is* diffeomorphic *to* S^n *for all* $n \geq 2$.

This Theorem generalizes a pioneering Theorem of H.B. Lawson and J. Simons [22] when $g = g_0\,, R^{n+m}(c) = S^{n+m}(1)$, and $n \geq 5$, M is homeomorphic to S^n. Furthermore, the results (c) in the case $g = g_0$, $R^{n+m}(c) = S^{n+m}(1)$, and $n \geq 4$ are due to G. Huisken [19] and B. Andrews [2] for codimension $m = 1$, and to J.R. Gu and H. W. Xu [14] for arbitrary codimensions $m \geq 1$ based on the work of S. Brendle [4].

The inequality (5.20) is optimal. As presented in [35], we have the following immediate optimal result.

Proposition. *Let* M *be a closed surface in a Euclidean sphere with the second fundamental form* h *satisfying* $\|h\|^2 < 2$. *Then* M *is* diffeomorphic *to a sphere* S^2 *or* $\mathbb{RP}^2$ *depending on* M *is orientable or not.*

This result is sharp as the length of the second fundamental form of Clifford Torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ in $S^3(1)$ satisfies $\|h\|^2 = 2$. The case $n = 2$, $\|h\|^2 < 1$ is due to Lawson-Simons: Let M be a compact (orientable) manifold of dimension n immersed in S^N with second fundamental form h satisfying $\|h\|^2 < \min\{n - 1, \sqrt{n-1}\}$. Then M is a homotopy sphere [22, Corollary 2].

It is our hope that the methods used here, and especially the trace formulas of Section 3, will find other uses in studying the topology of submanifolds and also that the examples given here shed light on the relation between curvature properties of manifolds and the existence of stable submanifolds. These methods can also be used to study other variational problems in Riemannian geometry, for example the nonexistence of nonconstant stable harmonic maps between manifolds. We have done this in a subsequent paper [18].

Many Theorems in this paper were proved and presented in 1983 (cf. e.g. [35]). A preprint was circulated and has been quoted in the literature and at international conferences, listed as "preprint" (cf. e.g. [25], [29], [30]). The authors went on to pursue other projects. Meanwhile these methods have been used, extended or generalized to other situations such as harmonic maps ([18],[23]), Yang-Mills Fields ([20]), p-harmonic maps ([37]), F-harmonic maps ([3]), Finsler geometry ([28]), etc. The notions of strongly unstable, super-strongly unstable, p-super-strongly unstable, F-super-strongly unstable manifolds, etc are introduced and studied. This paper is an improved and enlarged update of the preprint.

2. Universally mass decreasing sets of vector fields.

Let M be a smooth complete Riemannian manifold with metric $g(\ ,\ ) = \langle\ ,\ \rangle$. If G is a finitely generated abelian group then we are interested in elements of the group $\mathfrak{R}_p(M,G)$ of p-rectifiable currents in M over the group G. To establish our notation we give an informal description of $\mathfrak{R}_p(M,G)$ and refer the reader to [11] or [8] for the exact definitions. Let σ_p be the standard p-dimensional simplex with its usual volume form Ω_{σ_p}. *A singular p-dimensional Lipschitz simplex c in M is a* Lipschitz continuous map $c : \sigma_p \to M$. The variation measure $||c||$ of c is the Borel measure on M defined on continuous real valued functions φ on M by

$$\int_M \varphi(y) d||c||(y) = \int_{\sigma_p} \varphi(c(x))|(Jc)(x)|\Omega_{\sigma_p}(x) \tag{2.1}$$

where Jc is the Jacobian of c. (Recall that by Rademacher's theorem [8, p.216] a Lipschitz map has a well-defined Jacobian almost everywhere.) The mass of $\mathcal{M}(c)$ of c is defined to be

$$\mathcal{M}(c) = \int_M 1\, d||c|| = \int_{\sigma_p} |(Jc)(x)|\Omega_{\sigma_p}(x)\,. \tag{2.2}$$

Thus if $c : \sigma_p \to M$ is a smooth imbedding then integration with respect to $||c||$ is just integration over $c[\sigma_p]$ with the volume form induced on it as a submanifold of M, and $\mathcal{M}(c)$ is the p-dimensional volume of $c[\sigma_p]$. *A singular p-dimensional Lipschitz chain over the group G is a finite sum.*

$$s = \sum_k g_k c_k \tag{2.3}$$

where each $g_k \in G$ and each c_k is singular p-dimensional Lipschitz simplex. Let $C_p(M;G)$ be the group of all p-dimensional Lipschitz chains over G modulo the equivalence relation $\sim$ such that $s_1 \sim s_2$ if and only if "s_1 and s_2 triangulate the same subset of M." In the case $G = \mathbb{Z}$, the integers, this can be made precise by

$$s_1 \sim s_2 \quad \text{if and only if} \quad \int_{s_1} \omega = \int_{s_2} \omega$$

for all smooth p-forms ω on M. (This defines $C_p(M;\mathbb{Z})$, the general case can then be defined by $C_p(M;G) = G \otimes_{\mathbb{Z}} C_p(M;\mathbb{Z})$.)

Assume that G has a translation invariant norm $|\cdot|$ (every finitely generated abelian group has at least one). When $G = \mathbb{Z}$. we will always assume that $|\cdot|$ is the usual absolute value and when $G = \mathbb{Z}_\ell$ (the integers modulo ℓ) then $|\cdot|$ will always be taken to by $|\alpha| = \min\{|k| : k \in \alpha\}$. Then define the variation measure

$||s||$ and the mass $\mathcal{M}(s)$ of s given by equation (2.3) by

$$||s|| = \sum_k |g_k| ||c_k|| \tag{2.4}$$

$$\mathcal{M}(s) = \int_M 1\, d||s|| = \sum_k |g_k| \mathcal{M}(c_k) \tag{2.5}$$

then $C_p(M;G)$ is a metric space with respect to the distance $\rho(s_1, s_2) = \mathcal{M}(s_1 - s_2)$. Whence $\mathcal{R}_p(M,G)$, the group of *rectifiable currents of degree p in M over the group G*, can be defined to be the completion of $C_p(M;G)$ with respect to the distance function ρ. For our purposes we only need to know that each $\mathcal{S} \in \mathcal{R}_p(M;G)$ has associated with it a variation measure $||\mathcal{S}||$ and a mass $\mathcal{M}(\mathcal{S})$ (and if $\mathcal{S} = s$ is of the form (2.3) these are given by (2.4) and (2.5)) such that for $||\mathcal{S}||$ almost all $x \in M$, $\mathcal{S}$ has a well-defined p-dimensional approximate tangent space $T_x(\mathcal{S})$ which is a subspace to T_xM (see [8, chap. 4].)

We now summarize the part of the variational theory of currents we need. Let V be a smooth vector field on M and let φ_t^V be the flow (or one parameter pseudogroup) of V. Then for small t we can deform $\mathcal{S} \in \mathcal{R}_p(M,G)$ along the flow of V to get a new current $\varphi_{t*}^V \mathcal{S}$. If $\mathcal{S} = s$ is given by (2.3) then

$$\varphi_{t*}^V \mathcal{S} = \sum_k g_k\, \varphi_t^V \circ c_k \tag{2.6}$$

Definition *An element* $\mathcal{S} \in \mathcal{R}_p(M,G)$ is minimal (or critical) *if and only if for every smooth vector field on* M*,*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{M}(\varphi_{t*}^V \mathcal{S}) = 0 \tag{2.7}$$

It is stable *if and only if for every smooth vector field* V *there is a* $\delta > 0$ *such that*

$$\mathcal{M}(\varphi_{t*}^V \mathcal{S}) \geq \mathcal{M}(\mathcal{S}) \tag{2.8}$$

For stable currents $\mathcal{S}$ there is the *stability inequality*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi_{t*}^V \mathcal{S}) \geq 0 \tag{2.9}$$

The fundamental result on the existence of stable current is due to Federer and Fleming [10] when $G = \mathbb{Z}$ and Fleming [11] for finite G. They define a subgroup $\mathcal{I}_p(M,G)$ of $\mathcal{R}_p(M,G)$ (the group of integral currents in M over G) and boundary operators $\partial : \mathcal{I}_p(M,G) \to \mathcal{I}_{p-1}(M,G)$ (which coincides with the usual boundary operator on the singular p-dimensional Lipschitz chains) such that $\partial \circ \partial = 0$.

They then show there is a natural isomorphism of the homology groups $H_*(\mathcal{I}_*(M,G))$ with the singular homology groups $H_*(M,G)$. The basic existence theorem for stable currents is then the following, which gives a deep relation between geometry, topology and the calculus of variations.

Compactness Theorem (Federer and Fleming [10] and Fleming [11])**.** . *Let M be a compact Riemannian manifold and G a finitely generated abelian group. Then every nonzero homology class $\alpha \in H_p(\mathcal{I}_*(M,G)) \cong H_p(M,G)$ contains a stable element of $\mathcal{R}_p(M,G)$. In fact there is an $\mathcal{S} \in \alpha$ of least mass in the sense that $0 < \mathcal{M}(\mathcal{S}) \leq \mathcal{M}(\mathcal{S}')$ for all $\mathcal{S}' \in \alpha$ (and this $\mathcal{S}$ is clearly stable).*

Corollary. *(Generalized principle of Synge). If there are no nonzero stable currents in $\mathcal{R}_p(M,G)$ then $H_p(M,G) = 0$. If $p = 1$ and $G = \mathbb{Z}$ then not only does $H_1(M;\mathbb{Z})$ vanish but M is also simply connected.*

Proof. All of the corollary except the statement about M being simply connected follows at once from the compactness theorem. If $\pi_1(M) \neq 1$ then, as is well known, every free homotopy class of loops in M contains a closed geodesic of minimum length. This geodesic represents a stable current in $\mathcal{R}_1(M,\mathbb{Z})$. □

To use the generalized principle of Synge to study the topology of a Riemannian manifold a method of relating the geometry of the manifold to the nonexistence of stable currents is needed. This is provided by the first and second variation formulas for the mass integrand. First some notation is needed. Let ∇ be the Riemannian connection on M defined by the Riemannian metric on M. For any smooth vector field V on M define a tensor field of type $(1,1)$ (i.e. a field of linear endomorphisms of tangent spaces) by

$$\mathcal{A}^V(X) = \nabla_X V \tag{2.10}$$

Extend $\mathcal{A}^V$ to the full tensor algebra as a derivation. Then $\mathcal{A}^V$ is given on decomposable element of $\Lambda^p(T_xM)$ by

$$\mathcal{A}^V(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} x_1 \wedge \cdots \wedge \mathcal{A}^V x_i \wedge \cdots \wedge x_p\,. \tag{2.11}$$

If $\mathcal{S} \in \mathcal{R}_p(M,G)$ and x is a point at which $\mathcal{S}$ has an approximate tangent space then set

$$\overrightarrow{\mathcal{S}_x} = e_1 \wedge \cdots \wedge e_p$$

where $\{e_1, \cdots, e_p\}$ is an orthonormal basis of $T_x(\mathcal{S})$. (This is only well-defined up to a sign, but in all the formulas in which $\overrightarrow{\mathcal{S}}$ appears are invariant under the substitution $e_1 \wedge \cdots \wedge e_p \mapsto -e_1 \wedge \cdots \wedge e_p$.) The first and second variation formulas (due to Lawson and Simons [22]) are

$$\frac{d}{dt}\Big|_{t=0} \mathcal{M}(\varphi^V_{t*}\mathcal{S}) = \int_M \langle \mathcal{A}^V(\overrightarrow{\mathcal{S}}), \overrightarrow{\mathcal{S}}\rangle \, d||\mathcal{S}|| \tag{2.12}$$

(2.13)

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi^V_{t*}\mathcal{S}) = \int_M (-\langle \mathcal{A}^V\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle^2 \quad + \langle \mathcal{A}^V\mathcal{A}^V\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle + ||\mathcal{A}^V\overrightarrow{\mathcal{S}}||^2 + \langle (\nabla_V \mathcal{A}^V)\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle) \, d||\mathcal{S}|| \,.$$

If the linear map $\mathcal{A}^V_x$ is self adjoint for all x then $||\mathcal{A}^V\overrightarrow{\mathcal{S}}||^2 = \langle \mathcal{A}^V\mathcal{A}^V\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle$ and the second variation formula can be rewritten as

$$\begin{aligned} &\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi^V_{t*}\mathcal{S}) \\ &= \int_M (-\langle \mathcal{A}^V\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle^2 \quad + 2\langle \mathcal{A}^V\mathcal{A}^V\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle + \langle (\nabla_V \mathcal{A}^V)\overrightarrow{\mathcal{S}}, \overrightarrow{\mathcal{S}}\rangle) \, d||\mathcal{S}|| \,. \end{aligned} \tag{2.14}$$

It is convenient to introduce some notation for the integrand in the second variation formula. Let $g(\ ,\ ) = \langle\ ,\ \rangle$ be the metric on M. Then for any nonzero decomposable p-vector $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M and any vector field V on M define $\mathcal{V}(V, \xi; g)$ to be the integrand in (2.13), that is

(2.15)

$$\mathcal{V}(V,\xi;g) = -\frac{g(\mathcal{A}^V\xi,\xi)^2}{g(\xi,\xi)^2} \quad + \frac{g(\mathcal{A}^V\mathcal{A}^V\xi,\xi)}{g(\xi,\xi)} + \frac{g(\mathcal{A}^V\xi,\mathcal{A}^V\xi)}{g(\xi,\xi)} + \frac{g((\nabla_V\mathcal{A}^V)\xi,\xi)}{g(\xi,\xi)} \,.$$

If $g(\ ,\ )$ is written as $\langle\ ,\ \rangle$, $||\xi|| = 1$ and the dependence on g is suppressed from the notation this becomes

$$\mathcal{V}(V,\xi) = -\langle \mathcal{A}^V\xi,\xi\rangle^2 + \langle \mathcal{A}^V\mathcal{A}^V\xi,\xi\rangle + ||\mathcal{A}^V\xi||^2 + \langle (\nabla_V\mathcal{A}^V)\xi,\xi\rangle \tag{2.16}$$

and if $\mathcal{A}^V$ is self-adjoint

$$\mathcal{V}(V,\xi) = \langle \mathcal{A}^V\xi,\xi\rangle^2 + 2\langle \mathcal{A}^V\mathcal{A}^V\xi,\xi\rangle + \langle (\nabla_V\mathcal{A}^V)\xi,\xi\rangle \,. \tag{2.17}$$

Definition. *A finite set $\{V_1 \ldots V_\ell\}$ of smooth vector fields on the Riemannian manifold (M, g) is* universally mass decreasing in dimension p *on (M, g) if and only if for every nonzero decomposable p-vector $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M*

$$\sum_{i=1}^{\ell} \mathcal{V}(V_i, \xi, g) < 0 \tag{2.18}$$

Theorem 9. *Let (M, g_0) be a complete Riemannian manifold and assume there is a set of vector fields $\{V_1, \cdots, V_\ell\}$ on M that are universally mass decreasing in dimension p on (M, g_0). Then (M, g_0) has no stable currents in $\mathcal{R}_p(M, G)$ for any* *finitely generated abelian group G. Thus if M is compact $H_p(M,G)=0$ and if M is compact and $p=1$ then M is simply connected. Moreover there is a neighborhood of g_0 in the C^2 topology (if M is not compact we must use the strong C^2 topology see* [16] *for the definition) such that for every g in this neighborhood the Riemannian manifold (M,g) has no stable currents in $\mathcal{R}_p(M,G)$.*

Proof of Theorem 9. Let $\mathcal{S}$ be any element of $\mathcal{R}_p(M,G)$ then from the definition of a universally mass decreasing set of vector fields and the second variation formula,

$$\sum_{i=1}^{\ell} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi_{t*}^{V_i}\,\mathcal{S}) = \int_M \sum_{i=1}^{\ell} \mathcal{V}(V_i, \overrightarrow{\mathcal{S}}, g_0)\, d||\mathcal{S}|| < 0.$$

Thus, $\frac{d^2}{dt^2}|_{t=0}\mathcal{M}(\varphi_{t*}^{V_i}\,\mathcal{S}) < 0$, for some i. This contradicts the stability inequality (2.9) and thus $\mathcal{S}$ is not stable. The results on the topology of M now follow from the generalized principle of Synge.

To prove the last part of the theorem it is enough to show there is a neighborhood $\mathcal{U}$ of g_0 in the C^2 topology such that for every $g \in \mathcal{U}$ the set $\{V_1, \cdots V_\ell\}$ is still universally mass decreasing on (M,g). Let $\mathfrak{M}(M)$ be the space of smooth Riemannian metrics on M with the strong C^2 topology and let $G_p(M)$ be the bundle of p-planes tangent to M. Consider the function on $G_p(M)\times\mathfrak{M}(M)$ given by

$$(\xi, g) \mapsto \sum_{i=1}^{\ell} \mathcal{V}(V_i, \xi, g). \tag{2.19}$$

If this is continuous then

$$\mathcal{U} = \{g \in \mathfrak{M}(M) : \sum_{i=1}^{\ell} \mathcal{V}(V_i,\xi,g) < 0 \text{ for all } \xi \in G_p(M)\}$$

is the required neighborhood of g_0. To show that the function given by (2.19) is continuous it is enough to show that for any smooth vector field V the function $(g,\xi)\mapsto \mathcal{V}(V,\xi,g)$ is continuous.

Let $x^1, \dots x^n$ be local coordinates on M and let $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ be the components of g in this coordinate system. Let the Christoffel symbols Γ_{ij}^k be given as usual by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}$. Then by a well known formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{n} g^{k\ell}\Big(\frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\Big)\,. \tag{2.20}$$

(where $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$). If the vector field V is locally given by $V=\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ and the components $(\mathcal{A}^V)_i^j$ and $(\nabla_V \mathcal{A}^V)_i^j$ are given by

$$(\mathcal{A}^V)\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} (\mathcal{A}^V)_i^j \frac{\partial}{\partial x^j}\,, \quad (\nabla_V\mathcal{A}^V)\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} (\nabla_V\mathcal{A}^V)_i^j \frac{\partial}{\partial x^j}$$

then a little calculation shows that

$$(\mathcal{A}^V)^j_i = \frac{\partial v^j}{\partial x^i} + \sum_{k=1}^n v^k \Gamma^j_{ik}\,. \tag{2.21}$$

$$(\nabla_V \mathcal{A}^V)^j_i = \sum_{k=1}^n v^k \frac{\partial a^j_i}{\partial x^k} + \sum_{k,\ell=1}^n (a^\ell_i v^k \Gamma^j_{k\ell} - a^j_\ell v^k \Gamma^\ell_{ki})\,, \tag{2.22}$$

where $a^j_i = (\mathcal{A}^V)^j_i$. Indeed,

$$\begin{aligned}
(\nabla_V \mathcal{A}^V)\frac{\partial}{\partial x^i} &= \nabla_V\big(\mathcal{A}^V(\frac{\partial}{\partial x^i})\big) - \mathcal{A}^V(\nabla_V \frac{\partial}{\partial x^i}) \\
&= \nabla_{\sum_{k=1}^n v_k \frac{\partial}{\partial x^k}}(\sum_{j=1}^n a^j_i \frac{\partial}{\partial x^j}) - \nabla_{\nabla_{\sum_{k=1}^n v_k \frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}} V) \\
&= \sum_{j,k=1}^n v^k \frac{\partial a^j_i}{\partial x^k}\frac{\partial}{\partial x^j} + \sum_{k,\ell=1}^n (a^j_i v^k \Gamma^\ell_{kj}\frac{\partial}{\partial x^\ell} - v^k \Gamma^\ell_{ki} \nabla_{\frac{\partial}{\partial x^\ell}} V)\,, \\
&= \sum_{j=1}^n \left(\sum_{k=1}^n v^k \frac{\partial a^j_i}{\partial x^k} + \sum_{k,\ell=1}^n (a^\ell_i v^k \Gamma^j_{k\ell} - a^j_\ell v^k \Gamma^\ell_{ki})\right)\frac{\partial}{\partial x^j}\,,
\end{aligned}$$

Putting (2.20) into (2.21) and (2.22) and the result of that into (2.15) gives $\mathcal{V}(V,\xi,g)$ as a rational function of the g_{ij} and their first two derivatives. Thus $\mathcal{V}(V,\xi,g)$ is clearly a continuous function of g in the strong C^2 topology. This completes the proof. □

Using the trace formulas of the next section we will show latter (sections 4 and 6) that if M is any compact simply connected rank one symmetric space then there is a set of vector fields $V_1,\dots,V_\ell$ on M that is universally mass decreasing in any dimension p such that $H_p(M;\mathbb{Z})=0$. Using this along with The Main Theorem yields

Theorem 10. *Let (M,g_0) be a compact simply connected rank one symmetric space with its usual metric. Then there is a neighborhood of g_0 in the C^2 topology such that for any metric g in this neighborhood (M,g) has no nonzero stable currents $\mathcal{R}_p(M,G)$ for any p with $H_p(M,\mathbb{Z})=0$.*

Remarks. (1) The last theorem proves Theorems 1 and 3 of the introduction.

(2) The first and second variation formulas hold in the forms given by (2.12) and (2.13) for arbitrary varifolds on M (see section 2 of [22]). Thus the Theorems of this section can be extended to conclude there are no stable varifolds on M in the appropriate dimensions.

3. Trace formulas for immersed submanifolds of Euclidean Space

In this section M^n will be an n-dimensional Riemannian manifold isometrically immersed in the Euclidean space $\mathbb{R}^{n+m}$. We now fix our notation for the imbedding invariants on M. The normal bundle of M in $\mathbb{R}^{n+m}$ will be denoted by $T^{\perp}M$, the Riemannian connection on M by ∇ and the standard Riemannian connection on $\mathbb{R}^{n+m}$ by $\widetilde{\nabla}$. Let h be the second fundamental form of M in $\mathbb{R}^{n+m}$. Then for each $x \in M$, h_x is a symmetric bilinear map from $T_xM \times T_xM$ to $T_x^{\perp}M$. If $X\,,Y$ are vector fields on M then $\nabla\,,\overline{\nabla}$ and h are related by the Gauss equation

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.1}$$

where $X\,,Y$ are smooth vector fields on M. For each $x \in M$ and $\eta \in T_x^{\perp}M$ let A^{η} be the Weingarten map for the direction η. It is a self-adjoint linear map $A^{\eta} : T_xM \to T_xM$ and is related to the second fundamental form by

$$\langle h(x,y), \eta\rangle = \langle A^{\eta}x, y\rangle \tag{3.2}$$

for $x, y \in T_xM$ and $\eta \in T_x^{\perp}M$. Denote by $\nabla^{\perp}$ the induced connection on the normal bundle, that is if η is a section of $T^{\perp}M$ and X is a vector field on M then $\nabla^{\perp}{}_X\,\eta$ is the orthogonal projection of $\overline{\nabla}_X Y$ onto $T^{\perp}M$. It is related to A and $\overline{\nabla}$ by the Weingarten equation

$$\overline{\nabla}_X \eta = \nabla^{\perp}{}_X\,\eta - A^{\eta}X\,. \tag{3.3}$$

The connections ∇ and $\nabla^{\perp}$ induce a connection $\overline{\nabla}$ (the connection of van der Wearden-Bortolotti) on all the tensor bundles constructed from TM and $T^{\perp}M$. In particular A can be viewed as a smooth section of $Hom(T^{\perp}M, Hom(TM,TM))$ and if X, Y are tangent fields on M, η is a section of $T^{\perp}M$ then by definition

$$\nabla_X(A^{\eta}Y) = (\overline{\nabla}_X A)(Y) + A^{\nabla^{\perp}\eta}(Y) + A^{\eta}(\nabla_X Y)$$

so that

$$\nabla_X(A^{\eta}) = (\overline{\nabla}_X A)^{\eta} + A^{\nabla^{\perp}{}_X\,\eta}\,. \tag{3.4}$$

We identify all tangent vectors to $\mathbb{R}^{n+m}$ with elements of $\mathbb{R}^{n+m}$ in the usual way. If v is an element of $\mathbb{R}^{n+m}$ then define a smooth field v^T of tangent vectors on M and a section of $T^{\perp}M$ by

$$v^T(x) = \text{orthogonal projection of} \quad v \quad \text{onto} \quad T_xM\,. \tag{3.5}$$

$$v^{\perp}(x) = \text{orthogonal projection of} \quad v \quad \text{onto} \quad T_x^{\perp}M\,. \tag{3.6}$$

Lemma 3.1. *Let $v \in \mathbb{R}^{n+m}$ and $X \in \Gamma(TM)$. Then*

$$\mathcal{A}^{V^T} = A^{v^{\perp}} \quad \text{and thus} \quad \mathcal{A}^{V^T} \quad \text{is self-adjoint}\,, \tag{3.7}$$

$$\nabla^{\perp}{}_X\,v^{\perp} = -h(X, v^{\perp})\,, \tag{3.8}$$

$$\nabla_{V^T}\mathcal{A}^{V^T} = (\overline{\nabla}_{V^T}A)^{V^{\perp}} - A^{h(v^T,v^T)}\,. \tag{3.9}$$

Also equations (3.7) *and* (3.9) *hold when* $\mathcal{A}^{V^T}$, $A^{v^\perp}$, *and* $A^{h(v^T,v^T)}$ *are extended to* $A^p(TM)$ *as derivations.*

Proof. The vector $v \in \mathbb{R}^{n+m}$ is identified with a parallel vector field on $\mathbb{R}^{n+m}$ and thus $\overline{\nabla}_X\, v = 0$. By the Gauss equation (3.11) $\nabla_x v^T = (\overline{\nabla}_x v^T)^T$ and by the Weingarten equation (3.3) $A^{v^\perp}X = -(\nabla_x\, v^\perp)^\perp$. Thus

$$\mathcal{A}^{V^T}(X) = \nabla_X\, v^T = \big(\overline{\nabla}_X(v - v^\perp)\big)^T = 0 - (\overline{\nabla}_X v^\perp)^T = A^{v^\perp}X.$$

This proves (3.7). To prove (3.8) use (by (3.1)) $(\overline{\nabla}_x v^T)^T = \nabla_x v^T$,

$$\nabla^\perp_X\, v^\perp = \overline{\nabla}_X\, v^\perp = \overline{\nabla}_X\,(v - v^T) = 0 - (\overline{\nabla}_X\, v^T)^\perp = -h(X, v^T)\,.$$

To prove (3.9) use (3.7), (3.8), and (3.4)

$$\begin{aligned}\nabla_{V^T}\,(\mathcal{A}^{V^T}) = \nabla_X(A^{V^T}) &= (\overline{\nabla}_{V^T} A)^{V^\perp} + A^{\nabla^\perp_{v^T}\, v^\perp} \\ &= (\overline{\nabla}_{V^T} A)^{V^\perp} - A^{h(v^T,v^T)}\,.\end{aligned}$$

The result about the extensions to $\wedge^p(TM)$ is straightforward. $\square$

We now introduce some notation. If $\mathcal{S} \in \mathcal{R}_p(M,G)$ then define $\mathcal{I}_\mathcal{S}$ to be the function on $\mathbb{R}^{n+m}$ given by

$$\mathcal{I}_\mathcal{S}(v) = \frac{d^2}{dt^2}\Big|_{t=0}\, \mathcal{M}(\varphi^{V^T}_{t*}\,\mathcal{S})\,. \tag{3.10}$$

It follows at once from the second variation formula (2.13) that $\mathcal{I}_\mathcal{S}$ is a quadratic form on $\mathbb{R}^{n+m}$. For each decomposable p-vector $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M define another quadratic form on $\mathbb{R}^n$ by

$$Q_\xi(v) = \mathcal{V}(v^T, \xi)\,, \tag{3.11}$$

where $\mathcal{V}(v,\xi)$ is given by equation (2.17). Then as $\mathcal{A}^{V^T}$ is self-adjoint equation (2.14) yields

$$\mathcal{I}_\mathcal{S}(v) = \int_M Q_{\mathcal{S}_x}(v)\, d||\mathcal{S}||(x)\,. \tag{3.12}$$

Theorem 11 (Trace Formulas)**.** *With the notation just introduced*

$$\operatorname{trace}(\mathcal{I}_\mathcal{S}) = \int_M \operatorname{trace}(Q_{\vec{\mathcal{S}}_x})\, d||\mathcal{S}||(x)\,. \tag{3.13}$$

and if $\mathcal{S}$ *is stable then* $\operatorname{trace}(\mathcal{I}_\mathcal{S}) \geq 0$. *If* $\{e_1, \ldots e_{n+m}\}$ *is an orthonormal basis of* $\mathbb{R}^{n+m}$ *such that* $\{e_1, \ldots, e_n\}$ *is a basis of* T_xM , $\{e_{n+1}, \ldots, e_{n+m}\}$ *is a basis of* $T^\perp_x M$ *and* $\xi = e_1 \wedge \cdots \wedge e_p$ *then trace*(Q_ξ) *either of the formulas*

$$\operatorname{trace}(Q_\xi) = \sum_{k=n+1}^{n+m} \big(-\langle A^{e^k}\xi, \xi\rangle^2 + 2\langle A^{e_k}A^{e_k}\xi,\xi\rangle - \langle \operatorname{tr}(A^{e_k})A^{e_k}\xi,\xi\rangle\big) \tag{3.14}$$

(here each A^{e_k} has been extended to $\wedge^p TM$ as a derivation and tr(A^{e_k}) is the trace of A^{e_k} as a linear map $A^{e_k} : TM \to TM$) or

$$\text{trace}(Q_\xi) = \sum_{i=1}^{p} \sum_{\ell=p+1}^{n} \big(2||h(e_i, e_\ell)||^2 - \langle h(e_i, e_i), h(e_\ell, e_\ell)\rangle\big). \tag{3.15}$$

Finally if $\xi^\perp = e_{p+1} \wedge \cdots \wedge e_n$ there is a duality

$$\text{trace}(Q_\xi) = \text{trace}(Q_{\xi^\perp}). \tag{3.16}$$

Corollary. *Let M^n be a complete Riemannian manifold isometrically immersed in $\mathbb{R}^{n+m}$.*

(a) If for some p the inequality trace$(Q_\xi) < 0$ holds for all unit decomposable p-vectors $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M and $\{v_1, \cdots, v_{n+m}\}$ is an orthonormal basis of $\mathbb{R}^{n+m}$ then $V_1 = v_1^T, \cdots, V_{n+m} = v_{n+m}^T$ is a set of vector fields which is universally mass decreasing in both dimension p and dimension $n-p$. Thus there are no stable currents in $\mathcal{R}_p(M, G)$ for any finitely generated abelian group G. In particular M has no closed stable minimal submanifolds of dimension p or $n-p$.

(b) If in addition to the hypothesis of (a) *M is also compact then $H_p(M, G) = H_q(M, G) = 0$ and if $p = 1$ or $p = n-1$ then M is simply connected.*

(c) If M is compact and the hypothesis of (a) hold for $1 \le p \le \frac{n}{2}$ or for $\frac{n}{2} \le p \le n-1$ then M is a topological sphere.

Remarks. (a) In some sense this result is as sharp as passable for we will show in section 6 that there is an isometric immersion of the real projective space $\mathbb{RP}^n$ into a Euclidean space in such a way that trace$(Q_\xi) = 0$ for every decomposable p-vector tangent to $\mathbb{RP}^n$ and all $1 \le p \le n-1$. But $H_p(M, \mathbb{Z}_2) \ne 0\,, 1 \le p \le n-1$. Thus $\mathcal{R}_p(M, \mathbb{Z}_2)$ contains stable currents (in section 7 we will show that the natural imbedding of $\mathbb{RP}^p$ into $\mathbb{RP}^n$ is stable viewed as an element of $\mathcal{R}_p(M, \mathbb{Z}_2)$.)

(b) As with the results in the last section both Theorem 11 and Corollary 3 can be extended to varifolds.

(c) We note that when M is not compact that Corollary 3 (a) does not rule out the existence of noncompact stable minimal submanifolds N of M, where in the noncompact case stable means that for every compact subset K of M and every smooth vector field V supported in K that vol$(K \cap \varphi^V_{t*} N) \ge$ vol$(K \cap N)$.

(d) In the case that M^n is a submanifold of the sphere $S^{n+m-1} \subset \mathbb{R}^{n+m}$ then the trace formula (3.15) is in the paper of Lawson and Simons [22].

Proof of Corollary 3 from Theorem 11. If $\{v_1, \cdots, v_{n+m}\}$ is an orthonormal basis of $\mathbb{R}^{n+m}$ then

$$\text{trace}(Q_\xi) = \sum_{i=1}^{n+m} Q_\xi(v_i) = \sum_{i=1}^{n+m} \mathcal{V}(v_i^T, \xi)$$

and thus if $\operatorname{trace}(Q_\xi) < 0$ for all unit p-vectors ξ tangent to M then $v_1^T, \cdots, v_{n+m}^T$ is clearly universally mass decreasing in dimension p $\big($ and also in dimension $n-p$ by the duality (3.16) $\big)$. Therefore parts (a) and (b) of the lemma follow from the Main Theorem 9. To prove (c) note that the hypothesis, along with (b), imply that $\pi_1(M) = 1$ and $H_p(M, \mathbb{Z}) = 0$ for $1 \le p \le n-1$. The Hurewicz Isomorphism Theorem [33, p.393-400] then implies M is a homotopy sphere. Therefore M is a topological sphere (classical for $n = 2$, Smale [32] for $n \ge 5$, and Friedmann [12] for $n = 4$). When $n = 3$ equation (3.18) below implies the Ricci tensor of M is positive and a result of Perelman [26, 27] or Hamilton [15] implies that any simply connected three dimensional manifold with positive Ricci tensor is diffeomorphic to the Euclidean sphere S^3. This completes the proof. $\square$

Proof of Theorem 11. . By the second variation formula (2.14) (recall $\mathcal{A}^{v^T}$ is self-adjoint by Lemma 3.1)

$$\mathcal{I}_{\mathcal{S}}(v) = \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi_{t*}^{v^T}\, \mathcal{S}) = \int_M \mathcal{V}(v^T, \vec{\mathcal{S}}_x)\, d||\mathcal{S}||x = \int_M Q_{\vec{\mathcal{S}}}(v)\, d||\mathcal{S}||\,.$$

Therefore equation (3.14) follows by summing the last equation over any orthonormal basis of $\mathbb{R}^{n+m}$ We now compute $\operatorname{trace}(Q_\xi)$. Let $\{e_1, \cdots, e_{n+m}\}$ be an orthonormal basis of $\mathbb{R}^{n+m}$ chosen as in the statement of Theorem 11. Then using equation (2.17) and lemma 3.1

$$\begin{aligned}
\operatorname{trace}(Q_\xi) &= \sum_{i=1}^{n+m} Q_\xi(e_i) = \sum_{i=1}^{n+m} \mathcal{V}(e_i^T, \xi) \\
&= \sum_{i=1}^{n+m} (-\langle \mathcal{A}^{e_i^T}\xi, \xi\rangle^2 + 2\langle \mathcal{A}^{e_i^T} A^{e_i^T}\xi, \xi\rangle + \langle (\nabla_{{e_i}^T}\mathcal{A}^{{e_i}^T})\xi, \xi\rangle) \\
&= \sum_{i=1}^{n+m} (-\langle A^{e_i^\perp}\xi, \xi\rangle^2 + 2\langle A^{e_i^\perp} A^{e_i^\perp}\xi, \xi\rangle + \langle (\overline{\nabla}_{{e_i}^T} A)^{{e_i}^\perp}\xi, \xi\rangle) - \sum_{i=1}^{n+m} \langle A^{h(e_i^T, e_i^T)}\xi, \xi\rangle \\
&= \sum_{k=n+1}^{n+m} (-\langle A^{e^k}\xi, \xi\rangle^2 + 2\langle A^{e_k} A^{e_k}\xi, \xi\rangle) - \sum_{i=1}^{n} \langle A^{h(e_i, e_i)}\xi, \xi\rangle\,.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{n} \langle A^{h(e_i, e_i)}\xi, \xi\rangle &= \sum_{i=1}^{n} \sum_{k=n+1}^{n+m} \langle A^{\langle h(e_i, e_i), e_k\rangle e_k}\xi, \xi\rangle \\
&= \sum_{k=n+1}^{n+m} \sum_{i=1}^{n} \langle h(e_i, e_i), e_k\rangle \langle A^{e_k}\xi, \xi\rangle \\
&= \sum_{k=n+1}^{n+m} \sum_{i=1}^{n} \langle A^{e_k} e_i, e_i\rangle \langle A^{e_k}\xi, \xi\rangle \\
&= \sum_{k=n+1}^{n+m} \operatorname{trace}(A^{e_k}) \langle A^{e_k}\xi, \xi\rangle
\end{aligned}$$

The last two equations imply (2.14). We now prove (3.15) from (3.14). First

$$
\begin{aligned}
-\sum_{k=n+1}^{n+m} \langle A^{e_k}\xi, \xi\rangle^2 &= -\sum_{k=n+1}^{n+m} \left(\sum_{i=1}^{p} \langle e_1 \wedge \cdots \wedge A^{e_k} e_i \wedge \cdots \wedge e_p, e_1 \wedge \cdots \wedge e_p \rangle \right)^2 \\
&= -\sum_{k=n+1}^{n+m} \left(\sum_{i=1}^{p} \langle A^{e_k} e_i\,, e_i \rangle \right)^2 \\
&= -\sum_{k=n+1}^{n+m} \sum_{i,j=1}^{p} \langle A^{e_k} e_i\,, e_i \rangle \langle A^{e_k} e_j\,, e_j \rangle \\
&= -\sum_{i,j=1}^{p} \sum_{k=n+1}^{n+m} \langle h(e_i\,, e_i), e_k \rangle \langle h(e_j\,, e_j), e_k \rangle \\
&= -\sum_{i,j=1}^{p} \langle h(e_i\,, e_i), h(e_j\,, e_j) \rangle
\end{aligned}
$$

Second

$$
\begin{aligned}
&2\sum_{k=1}^{n+m} \langle A^{e_k} A^{e_k}\xi, \xi\rangle = 2\sum_{k=1}^{n+m} \langle A^{e_k}\xi, A^{e_k}\xi\rangle \\
&= 2\sum_{k=n+1}^{n+m} \langle \sum_{i=1}^{p} e_1 \wedge \cdots \wedge A^{e_k} e_i \wedge \cdots \wedge e_p, \sum_{j=1}^{p} e_1 \wedge \cdots \wedge A^{e_k} e_j \wedge \cdots \wedge e_p \rangle \\
&= 2\sum_{k=n+1}^{n+m} \sum_{i,j=1}^{p} \langle A^{e_k} e_i, e_i \rangle \langle A^{e_k} e_j, e_j \rangle + 2\sum_{k=n+1}^{n+m} \sum_{\substack{1 \le i \le p \\ p+1 \le \ell \le n}} \langle A^{e_k} e_i, e_\ell \rangle^2 \\
&= 2\sum_{i,j=1}^{p} \sum_{k=n+1}^{n+m} \langle h(e_i, e_i), e_k \rangle \langle h(e_j, e_j), e_k \rangle + 2\sum_{\substack{1 \le i \le p \\ p+1 \le \ell \le n}} \sum_{k=n+1}^{n+m} \langle h(e_i, e_\ell), e_k \rangle^2 \\
&= 2\sum_{i,j=1}^{p} \langle h(e_i, e_i), h(e_j, e_j) \rangle + 2\sum_{\substack{1 \le i \le p \\ p+1 \le \ell \le n}} ||h(e_i, e_\ell)||^2
\end{aligned}
$$

Third

$$
\begin{aligned}
&-\sum_{k=n+1}^{n+m} \langle \operatorname{tr}(A^{e_k})A^{e_k}\xi, \xi\rangle \\
&= -\sum_{k=n+1}^{n+m}\sum_{t=1}^{n}\langle A^{e_k}e_t, e_t\rangle\langle\sum_{i=1}^{p} e_1\wedge\cdots\wedge A^{e_k}e_i\wedge\cdots\wedge e_p, e_1\wedge\cdots\wedge e_p\rangle \\
&= -\sum_{t=1}^{n}\sum_{i=1}^{p}\sum_{k=n+1}^{n+m}\langle h(e_t\,, e_t), e_k\rangle\langle h(e_i\,, e_i), e_k\rangle \\
&= -\sum_{t=1}^{n}\sum_{i=1}^{p}\langle h(e_t\,, e_t), h(e_i\,, e_i)\rangle \\
&= -\sum_{i,j=1}^{p}\langle h(e_i\,, e_i), h(e_j\,, e_j)\rangle - \sum_{\substack{1\le i\le p\\ p+1\le \ell\le n}}\langle h(e_i\, e_i), h(e_\ell\, e_\ell)\rangle
\end{aligned}
$$

Using these in (3.14) yields (3.15). Finally (3.16) follows at once from (3.15). This completes the proof. □

We close this section by giving a lower bound on trace(Q_ξ) in terms of the sectional curvatures of M. This will show that Theorem 11 and Corollary 3 can only apply to get results on the topology of a manifold if M is positively curved in the sense that the sum of sectional curvatures (without the minus sign) on the right of (3.17) is positive.

Proposition 3.1. *With the notation used in Theorem 11 let* $\{e_1, \cdots, e_n\}$ *be an orthonormal basis of* T_xM *and set* $\xi = e_1, \wedge\cdots\wedge e_p$. *Then*

$$
\begin{aligned}
\operatorname{trace}(Q_\xi) &= -\sum_{\substack{1\le i\le p\\ p+1\le \ell\le n}} K(e_i\,, e_\ell) + \sum_{\substack{1\le i\le p\\ p+1\le \ell\le n}} ||h(e_i\,, e_\ell)||^2 \\
&\ge -\sum_{\substack{1\le i\le p\\ p+1\le \ell\le n}} K(e_i\,, e_\ell)
\end{aligned}
\tag{3.17}
$$

where $K(e_i, e_\ell)$ *is the sectional curvature of the two-plane spaned by* e_i *and* e_ℓ. *Let* Ric(,) *be the Ricci tensor of* M. *Then if* $p = 1$ (3.17) *becomes*

$$
\operatorname{trace}(Q_\xi) \ge -\operatorname{Ric}(e_1, e_1) \tag{3.18}
$$

and if $p = n-1$ *it becomes*

$$
\operatorname{trace}(Q_\xi) \ge -\operatorname{Ric}(e_n, e_n) \tag{3.19}
$$

Proof. The Gauss equation is

$$K(e_i, e_\ell) = \langle h(e_i, e_i), h(e_\ell, e_\ell)\rangle - ||h(e_i, e_\ell)||^2 .$$

Using this in (3.15) yields (3.17). Equations (3.18) and (3.19) now follow from the definition of the Ricci tensor. □

4. Applications to the topology of hypersurfaces

In this section we will apply the results of the last section to the case M^n is an immersed submanifold of the Euclidean space $\mathbb{R}^{n+1}$. We assume that M has the induced metric and that it is complete. Fix a unit normal field η along M. *We do not assume that η is continuous as we do not wish to assume that M is orientable.* Let $k_1, \cdots, k_n$ be the principal curvatures of M corresponding to the choice of η. That is $k_1, \cdots k_n$ are the eigenvalues of the Weingarten map A_η. Order the principal curvatures so that

$$k_1 \leq \cdots \leq k_n .$$

Theorem 12. *Let $1 \leq p \leq n-1$ and set $q = n-p$. Assume that at every point of M the principal curvatures of M satisfy*

(a) $0 < k_1 + \cdots + k_p$
(b) $k_{q+1} + \cdots + k_n < k_1 + \cdots + k_p + k_1 + \cdots + k_q$.

Then for any decomposable unit p vector $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M

$$\begin{aligned} &\operatorname{trace}(Q_\xi) = \operatorname{trace}(Q_{\xi^\perp}) \\ &\leq -(k_1 + \cdots + k_p)\big(k_1 + \cdots + k_p + k_1 + \cdots + k_q - (k_{q+1} + \cdots + k_n)\big) \\ &< 0. \end{aligned} \tag{4.1}$$

Therefore there are no stable currents in $\mathcal{R}_p(M;G)$ or $\mathcal{R}_q(M;G)$ for any finitely generated abelian group G. If M is compact then $H_p(M,G) = H_q(M,G) = 0$ and if also $p = 1$ or $p = n-1$, M is simply connected. If (a) and (b) hold for $1 \leq p \leq \frac{n}{2}$ or $\frac{n}{2} \leq p \leq n-1$ then M is a topological sphere.

Proof. Once the inequality (4.1) is proven everything else follows from *Theorem* 11 and its corollary. Write A for A^η. Because A is self-adjoint there is an orthonormal basis $\{v_1, \cdots, v_n\}$ of T_xM such that $Av_i = k_i v_i$. For each sequence $I = \{i_1, \ldots, i_p\}$ with $1 \leq i_1 < \cdots < i_p \leq n$ let $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$ and $k_I = k_{i_1} + \cdots + k_{i_p}$. Then

$$Av_I = k_I v_I \tag{4.2}$$

and $\{v_I\}$ is an orthonormal basis of $\wedge^p(T_xM)$. Write ξ in terms of the basis $\{v_I\}$

$$\xi = \sum_I x_I v_I . \tag{4.3}$$

Then

$$\langle A\xi,\xi\rangle = \sum_I x_I^2 k_I.$$

and $\sum_I x_I^2 = 1$ as ξ is a unit vector. Thus $\langle A\xi,\xi\rangle$ is a convex combination of the k_I's so that

$$k_1 + \cdots + k_p \le \langle A\xi,\xi\rangle \le k_{q+1} + \cdots + k_n.$$

Using the formula (3.14) for $\operatorname{trace}(Q_\xi)$, that by (a) $k_I > 0$ for all I, equation (4.3) and the last inequality

$$\begin{aligned}
\operatorname{trace}(Q_\xi) &= -\langle A\xi,\xi\rangle^2 + 2\langle AA\xi,\xi\rangle - \langle \operatorname{tr}(A)A\xi,\xi\rangle \\
&= \langle\big(-\langle A\xi,\xi\rangle A + 2AA - \operatorname{tr}(A)A\big)\xi,\xi\rangle \\
&= \langle\big(-\langle A\xi,\xi\rangle A + 2AA - \operatorname{tr}(A)A\big)\sum_I x_I v_I, \sum_J x_J v_J\rangle \\
&= \sum_I \big(-\langle A\xi,\xi\rangle k_I + 2k_I^2 - \operatorname{tr}(A)k_I\big)x_I^2 \\
&= \sum_I \big(-\langle A\xi,\xi\rangle + 2k_I - \operatorname{tr}(A)\big)k_I x_I^2 \\
&\le \sum_I \big(-(k_1+\cdots+k_p) + 2(k_{q+1}+\cdots+k_n) - (k_1+\cdots+k_n)\big)k_I x_I^2 \\
&= -\big(k_1+\cdots+k_p + k_1+\cdots+k_q - (k_{q+1}+\cdots+k_n)\big)\sum_I k_I x_I^2 \\
&\le -\big(k_1+\cdots+k_p + k_1+\cdots+k_q - (k_{q+1}+\cdots+k_n)\big)(k_1+\cdots+k_p) \\
&< 0.
\end{aligned}$$

This completes the proof. □

In light of the Conjectures in the introduction it is of interest to relate the last result to the intrinsic geometry of M.

Definition 4.1. Let $0 < \delta < 1$. Then a Riemannian manifold M is *pointwise δ-pinched* if and only if at each point of M there is a positive real number $r(x)$ such that for every two-plane P tangent to M at x

$$\delta r(x) \le K(P) \le r(x)$$

where $K(P)$ is the sectional curvature of the two-plane P.

Theorem 13. *Let M^n be a complete Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$ as a hypersurface. Let $1 \le p \le \frac{n}{2}$ and set $q = n - p$. Then*

(a) If M is pointwise δ-pinched for some δ satisfying

$$n^2\delta^2 - (p^2-1)\delta - 1 > 0\,, \tag{4.4}$$

in particular, if

$$\delta = \frac{p^2}{n^2} + \frac{n^2-p^2}{n^2(p^2+1)} \tag{4.5}$$

then M has no stable currents in $\mathcal{R}_p(M,G)$ or $\mathcal{R}_q(M,G)$ for any finitely generated abelian group G.

(b) If $n = 2k+1$ is odd and M is pointwise δ-pinched for δ greater than the positive root of $nx^2 - (k^2-1)x - 1 = 0$, in particular if

$$\delta \geq \frac{1}{4} - \frac{4k^3 - 11k^2 - 12k - 3}{4(2k+1)^2(k^2+1)} \tag{4.6}$$

($\delta = \frac{1}{4}$ works when $k \geq 4$) or if $n = 2k \geq 4$ is even and M is pointwise δ-pinched for δ greater than the positive root of $n^2x^2 - (k^2-1)x^2 - 1 = 0$, in particular, if

$$\delta \geq \frac{1}{4} + \frac{3}{4(k^2+1)} = \frac{1}{4} + \frac{3}{n^2+4} \tag{4.7}$$

then there are no stable currents in $\mathcal{R}_p(M;G)$ for $p = 1, \cdots, n-1$.

(c) If M is pointwise δ-pinched for

$$\delta = \frac{1}{4} + \frac{3}{n^2+4}$$

$n \geq 2$, then M is diffeomorphic to a sphere.

Lemma 4.1. *Let $n \geq 3$ and $0 < k_1 \leq k_2 \leq \cdots \leq k_n$ be n positive numbers. Assume that for some $r > 0$, $0 < \delta < 1$ that $\delta r \leq k_i k_j \leq r$ for $1 \leq i < j \leq n$. If $1 \leq p \leq n-1$ and $q = n-p$ then*

$$n^2\delta^2 - (p^2-1)\delta^2 - 1 > 0 \tag{4.8}$$

implies

$$k_{q+1} + \cdots + k_n < k_1 + \cdots + k_p + k_1 + \cdots + k_q\,. \tag{4.9}$$

Proof. Recall the inequality

$$\frac{1}{\binom{k}{2}} \sum_{1 \leq i < j \leq k} x_i x_j \leq \frac{1}{k^2}(x_1 + \cdots + x_k)^2\,.$$

This implies

$$r\delta = \frac{1}{\binom{p}{2}} \sum_{1 \leq i < j \leq p} r\delta \leq \frac{1}{\binom{p}{2}} \sum_{1 \leq i < j \leq p} k_i k_j \leq \frac{1}{p^2}(k_1 + \cdots + k_p)^2\,.$$

Therefore $p\sqrt{r\delta} \leq k_1 + \cdots + k_p$. Likewise $q\sqrt{r\delta} \leq k_1 + \cdots + k_q$. Hence

$$n\sqrt{r}\sqrt{\delta} = p\sqrt{r\delta} + q\sqrt{r\delta} \leq k_1 + \cdots + k_p + k_1 + \cdots + k_q\,. \tag{4.10}$$

As $n \geq 3$

$$k_n^2 = \frac{k_n k_1 k_n k_2}{k_1 k_2} \leq \frac{r \cdot r}{r\delta} = \frac{r}{\delta}$$

and if $i < n\,, k_i^2 \leq k_i k_{i+1} \leq r$. Thus

$$\begin{aligned}(k_{q+1} + \cdots + k_n)^2 &= k_{q+1}^2 + \cdots + k_n^2 + 2 \sum_{1 \leq i < j \leq p} k_i k_j \\ &\leq (p-1)r + \frac{r}{\delta} + p(p-1)r \\ &= \big((p^2-1) + \frac{1}{\delta}\big) r\,.\end{aligned} \tag{4.11}$$

The two inequalities (4.10) and (4.11) show that (4.9) is implied by

$$\sqrt{r}\sqrt{p^2 - 1 + \frac{1}{\delta}} < n\sqrt{r}\sqrt{\delta}$$

and this inequality is easily seen to be equivalent to (4.8). □

Lemma 4.2. *If* $p \leq \frac{n}{2}$ *and*

$$\delta = \frac{p^2}{n^2} + \frac{n^2 - p^2}{n^2(p^2+1)} \tag{4.12}$$

then

$$n^2\delta^2 - (p^2-1)\delta - 1 > 0.$$

Proof. Let the convex function $f(x) = n^2x^2 - (p^2-1)x - 1$. Then the tangent line to $y = f(x)$ at the point where $x = \frac{p^2}{n^2}$ is

$$y - \left(\frac{p^2}{n^2} - 1\right) = (p^2+1)\left(x - \frac{p^2}{n^2}\right). \tag{4.13}$$

The graph of $y = f(x)$ lies above any of its tangent lines and $f(\frac{p^2}{n^2}) \neq 0$. Therefore if δ is the x-intercept of the line (4.13) then $f(\delta) > 0$. But the x-intercept of (4.13) is easily seen to be given by (4.12). This completes the proof. □

Proof of Theorem 13. If $e_1, \cdots, e_n$ are the eigenvectors of A^η at $x \in M$, say $A^\eta e_i = k_i e_i$, then the Gauss equation yields that $k_i k_j$ is the sectional curvature of the two-plane spanned by e_i and e_j. Therefore part (a) of Theorem 13 follows from the last two lemmas and Theorem 12. To prove part (b) first assume $n = 2k+1$ is odd and that $p \leq \frac{n}{2}$. Then $p \leq k$. Thus

$$n^2\delta^2 - (p^2-1)\delta - 1 \geq n^2\delta^2 - (k^2-1)\delta - 1$$

and so if δ is greater than the positive root of $n^2x^2 - (k^2-1)x - 1 = 0$ then the hypothesis (4.4) of part (a) hold for $1 \leq p \leq \frac{n}{2}$. Whence there are no stable currents in $\mathcal{R}_p(M, G)$ for $1 \leq p \leq n-1$. If we let $n = 2k+1, p = k$ in (4.5) then the result is given by (4.6). This completes the proof of part (b) in the case n is odd. The proof when n is even is similar. The hypothesis (4.4) of part (c) implies that M is a compact manifold with pointwise $\frac{1}{4}$-pinched sectional curvature. For $n \geq 4$ by [5, Theorem 1], M admits a metric of a constant curvature and therefore is diffeomorphic to a spherical space form. By (b), M is simply-connected and hence is diffeomorphic to a sphere. For $n = 3$, the result follows from [15, 26, 27], and for $n = 2$, the result is classical. $\square$

5. Topological vanishing theorems for submanifolds of Euclidean space

In this section the generalized Synge lemma and the trace formulas of Theorem 11 will be used to study the topology of compact immersed submanifolds of Euclidean space. At least part of the motivation for this is the classical theorem that if M^n is a compact immersed submanifold of $\mathbb{R}^{n+m}$ that is totally umbilic (that is each of the Weingarten maps A^η is proportional to the identity map) then M^n is the standard imbedding of a sphere of constant sectional curvature into $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+m}$. It is therefore reasonable that if the immersion is close to being totally unbilic then M is a topological sphere. Among other things we will make this precise.

For the rest of this section M^n will be a compact Riemannian manifold isometrically immersed in $\mathbb{R}^{n+m}$. For each $x \in M$ we split the normal bundle $T_x^\perp M$ into an orthogonal direct sum

$$T_x^\perp M = E_x^1 \oplus E_x^2 \tag{5.1}$$

with $\dim(E_x^1) = m_1$, $\dim(E_x^2) = m_2$ and $m_1 + m_2 = m$. It is not necessary for this decomposition to be smooth, however in all the applications we have in mind this will be the case. Now we split the second fundamental form h of M in $\mathbb{R}^{n+m}$ into two pieces

$$\begin{aligned} h_x^1(X,Y) &= E_x^1 \quad \text{component of} \quad h_x(X,Y) \\ h_x^2(X,Y) &= E_x^2 \quad \text{component of} \quad h_x(X,Y). \end{aligned}$$

It follows easily that

$$||h(X,Y)||^2 = ||h^1(X,Y)||^2 + ||h^2(X,Y)||^2$$

and

$$\langle h(X,X), h(Y,Y)\rangle = \langle h^1(X,X), h^1(Y,Y)\rangle + \langle h^2(X,X), h^2(Y,Y)\rangle.$$

Therefore if $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_xM, $\xi = e_1 \wedge \cdots \wedge e_p$, $q = n-p$ and $T_\xi^{p,q}$ is defined by

$$T_\xi^{p,q}(h^\alpha) = \sum_{i=1}^{p} \sum_{\ell=p+1}^{n} \Big(2||h^\alpha(e_i, e_\ell)||^2 - \langle h^\alpha(e_i,e_i), h^\alpha(e_\ell, e_\ell)\rangle \Big)$$

for $\alpha = 1, 2$. Then, with the notation of Theorem 11,

$$\operatorname{trace}(Q_\xi) = T_\xi^{p,q}(h^1) + T_\xi^{p,q}(h^2)\,. \tag{5.2}$$

If $\{e_{n+1}, \cdots, e_{n+m}\}$ is an orthonormal basis of E^1_x and A is the Weingarten map of M in $\mathbb{R}^{n+m}$ then the calculations in the proof of Theorem 11 which show the equivalence of equations (3.14) and (3.15) also show

$$T^{p,q}_\xi(h^1) = \sum_{k=n+1}^{n+m_1} \Big(- \langle A^{e^k}\xi, \xi\rangle^2 + 2\langle A^{e_k} A^{e_k}\xi, \xi\rangle - \langle \operatorname{trace}(A^{e_k}) A^{e_k}\xi, \xi\rangle \Big) . \tag{5.3}$$

For each $x \in M$ define a "quasi-norm" $||\cdot||_{p,q}$ on the space of symmetric bilinear maps from $T_xM \times T_xM$ to E^α_x , $\alpha = 1, 2$ by

$$||B||^2_{p,q} = \sup_\xi |T^{p,q}_\xi(B)| . \tag{5.4}$$

where the supremum is taken over all decomposable p-vectors $\xi = e_1 \wedge \cdots \wedge e_p$ with $e_1, \cdots, e_p$ orthonormal. In section 4 of [22] it is shown that

$$||B||^2_{p,q} \le \max\Big\{1, \frac{\sqrt{pq}}{2}\Big\} ||B||^2 \tag{5.5}$$

where $||\cdot||$ is the standard norm given by

$$||B||^2 = \sum_{i,j=1}^{n} ||B(e_i, e_j)||^2 .$$

We thus have the following inequalities

$$\begin{aligned} \operatorname{trace}(Q_\xi) &= T^{p,q}_\xi(h^1) + T^{p,q}_\xi(h^2) \\ &\le T^{p,q}_\xi(h^1) + ||h^2||^2_{p,q} \\ &\le T^{p,q}_\xi(h^1) + \max\Big\{1, \frac{\sqrt{pq}}{2}\Big\} ||h^2||^2 . \end{aligned}$$

This, along with the results of sections 2 and 3, imply

Proposition 5.1. *Let M^n be a compact Riemannian manifold isometrically immersed in $\mathbb{R}^{n+m}$. With the notation of the last paragraph, if for all unit decomposable p-vectors $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to M either of the inequalities*

$$||h^2||^2_{p,q} < -T^{p,q}_\xi(h^1) \tag{5.6}$$

or

$$\max\Big\{1, \frac{\sqrt{pq}}{2}\Big\} ||h^2||^2 < -T^{p,q}_\xi(h^1) \tag{5.7}$$

holds then for any finitely generated abelian group G, $H_p(M, G) = H_q(M, G) = 0$ and if $p = 1$ or $p = n-1$ M is also simply connected. If (5.6) or (5.7) holds for $1 \le p \le \frac{n}{2}$ or $\frac{n}{2} \le p \le n-1$ then M is a topological sphere.

To make use of the last proposition to get results about the topology of a compact immersed submanifold of $\mathbb{R}^{n+m}$ it is only necessary to find an orthogonal splitting $T^\perp_x M = E^1_x \oplus E^2_x$ such that h^1 is well behaved in the sense that $T^{p,q}_\xi(h^1) < 0$ and h^2 is small relative to h^1. Then (5.7) will hold. As an example we make precise our remarks on the topology of submanifolds of $\mathbb{R}^{n+m}$ which are close to being totally umbilic.

Recall that if η is a unit section (i.e. $||\eta|| \equiv 1$) of the normal bundle of M^n then the principal curvatures $k_1(\eta) \leq \cdots \leq k_n(\eta)$ corresponding to η are the eigenvalues of A^η. For a unit section η of $T^\perp M$ let

$$h^{\eta^\perp}(X,Y) = \text{orthogonal projection of } h(X,Y) \text{ onto } \eta^\perp \ .$$

We remark that M is totally umbilic if and only if there is a unit section η of $T^\perp M$ such that for all $x \in M$, $A^{\eta(x)} = r(x)\,\mathrm{Id}$ for some $r(x) > 0$ and $h^\perp_\eta \equiv 0$.

Theorem 14. *Let M^n be a compact immersed submanifold of $\mathbb{R}^{n+m}$, and let $1 \leq p \leq n-1$. Set $q = n-p$ and assume there is a unit section η of $T^\perp M$ such that at all points of M $\big($setting $k_i = k_i(\eta)\big)$*

(a) $k_1 + \cdots + k_p > 0$

(b) $k_{q+1} + \cdots + k_n < k_1 + \cdots + k_p + k_1 + \cdots + k_q$.

(c) one of the two inequalities

$$(5.8)\quad ||h^{\eta^\perp}||^2_{p,q} < (k_1 + \cdots + k_p)\big(k_1 + \cdots + k_p + k_1 + \cdots + k_q - (k_{q+1} + \cdots + k_n)\big)$$

or

$$(5.9)\quad \max\Big\{1, \frac{\sqrt{pq}}{2}\Big\}||h^{\eta^\perp}||^2 < (k_1+\cdots+k_p)\big(k_1+\cdots+k_p+k_1+\cdots+k_q-(k_{q+1}+\cdots+k_n)\big)$$

holds.

Then $H_p(M,G) = H_q(M,G) = 0$ and if $p=1$ or $p = n-1$, M is simply-connected. If $(a),(b),(c)$ hold for $1 \leq p \leq \frac{n}{2}$ (or $\frac{n}{2} \leq p \leq n-1$), then M is a topological sphere.

Proof. In the last proposition let $E^1_x = \mathrm{span}\{\eta(x)\}$ and $E^2_x =$ orthogonal complement of $\mathrm{span}\{\eta(x)\}$ in $T^\perp_x M$. Then using equation (5.3)

$$\begin{aligned} T^{p,q}_\xi(h^1) &= -\langle A^\eta\xi,\xi\rangle^2 + 2\langle A^\eta A^\eta \xi,\xi\rangle - \mathrm{trace}(A^\eta)\langle A^\eta\xi,\xi\rangle \\ &\leq -(k_1+\cdots+k_p)\big(k_1+\cdots+k_p+k_1+\cdots+k_q-(k_{q+1}+\cdots+k_n)\big) \end{aligned}$$

where the proof of the inequality is the same as the proof of the inequality (4.1). The result now follows from the last proposition. □

Remarks. (1) In the results of the last paragraph if we take M^n to be an immersed submanifold of $S^{n+m-1} = \{x \in \mathbb{R}^{n+m} : \|x\| = 1\}$ and take η to be the inward-pointing unit normal to S^{n+m-1} then A^η is the identity map on TM and $h^{\eta^\perp}$ is the second fundamental form B of M in S^{n-1}. The conditions of part (c) then become $\|B\|^2_{p,q} < pq$ or $\|B\|^2 < \min\{pq, 2\sqrt{pq}\}$. Therefore our result implies Theorem 4 of paragraph 4 in Lawson and Simons. [22]. (Note that due to the subsequent results of Perelman [26, 27] or Hamilton [15], and Freedman [12] that the corollary to their theorem for $n \geq 5$ can be strengthened to the conclusion that $\|B\|^2 <$

$\min\{n-1, 2\sqrt{n-1}\}$ implies M^n is homeomorphic to a sphere for all n. In fact, M is diffeomorphic to a sphere.)

(2) In light of the last remarks and the examples in section 4 of [22] it follows that Theorem 14 is sharp in the sense that if $pq \geq 4$ there is an imbedding of $M = S^p(r) \times S^q(s)$ ($r^2 = \frac{\sqrt{p}}{\sqrt{p}+\sqrt{q}}$, $s^2 = \frac{\sqrt{q}}{\sqrt{p}+\sqrt{q}}$) into $\mathbb{R}^{n+2}$ and a smooth section η of $T^\perp M$ so that (5.8) and (5.9) both hold with "$<$" replaced by "$\leq$" but neither $H_p(M,G)$ nor $H_q(M,G)$ vanishes.

(3) As another application of the above formulas we extend Theorem 12 to the case of hypersurfaces in simply connected manifolds of positive constant sectional curvature. For any real number c let $R^n(c)$ be the complete simply connected Riemannian manifold of constant sectional curvature c and dimension n. If $c > 0$ then $R^n(c) = S^n(r)$, the sphere of radius $r = \frac{1}{\sqrt{c}}$ in $\mathbb{R}^{n+1}$.

Applying the same technique as before, one can prove the following:

Theorem 15. *Let M^n be a compact hypersurface in $R^{n+1}(c)$ where $c > 0$. Let $k_1 \leq \cdots \leq k_n$ be the principal curvatures of M^n in $R^{n+1}(c)$ and $1 \leq p \leq n-1$. Set $q = n-p$. Assume that at each point of M^n that*

(a) $k_1 + \cdots + k_p \geq 0$

(b) $k_{q+1} + \cdots + k_n \leq k_1 + \cdots + k_p + k_1 + \cdots + k_q$

(c) $(k_1 + \cdots + k_p)\big(k_1 + \cdots + k_p + k_1 + \cdots + k_q - (k_{q+1} + \cdots + k_n)\big) < pq\,c.$

Then $H_p(M,G) = H_q(M,G) = 0$ and if $p = 1$ or $p = n-1$, M is simply connected. If (a) and (b) hold for $1 \leq p \leq \frac{n}{2}$ or $\frac{n}{2} \leq p \leq n-1$, then M is homeomorphic to a sphere.

Problem. *Find an extension of the last theorem to the case where $c < 0$.*

Let M^n be a submanifold of the Riemannian manifold $\overline{M}^{n+m}$ and let h be the second second fundamental form of M^n in $\overline{M}^{n+m}$. Then the mean curvature vector H of M in $\overline{M}$ is defined to be

$$H_x = \frac{1}{n}\mathrm{trace}(h_x) = \frac{1}{n}\sum_{i=1}^{n} h_x(e_i, e_i)$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_xM. It is well known that $n\|H_x\|^2 \leq \|h_x\|^2$ with equality if and only if M is totally umbilic in $\overline{M}$ at x. Thus if $\overline{M}^{n+m} = \mathbb{R}^{n+m}$ and equality holds at each point of M then M^n is isometric to a sphere. Conversely we will show that if $\overline{M} = \mathbb{R}^{n+m}$ that $(n-1)\|h\|^2 < n^2\|H\|^2$ implies M is a topological sphere.

Lemma 5.1. *Let v_{ij}, where $1 \leq i,j \leq k$ be k^2 vectors in an inner product space. Then,*

$$\left\|\sum_{i=1}^{k} v_{ii}\right\|^2 \leq k\sum_{i,j=1}^{k}\|v_{ij}\|^2 \tag{5.10}$$

with equality if and only if $v_{ij} = 0$ *for* $i \neq j$ *and* $v_{11} = \cdots = v_{kk}$.

Proof. This follows from the easily verified identity

$$k \sum_{i,j=1}^{k} \|v_{ij}\|^2 - \left\| \sum_{i=1}^{k} v_{ii} \right\|^2 = \frac{1}{2} \sum_{i,j=1}^{k} \|v_{ii} - v_{jj}\|^2 + k \sum_{i \neq j} \|v_{ij}\|^2 \geq 0$$

$\square$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_x M$ and $\xi = e_1 \wedge \cdots \wedge e_p$, $\xi^\perp = e_{p+1} \wedge \cdots \wedge e_n$ then define $h|_\xi$ to be the restriction of h to $\mathrm{span}\{e_1, \ldots, e_p\} \times \mathrm{span}\{e_1, \ldots, e_p\}$ and $h|_{\xi^\perp}$ similarly. Clearly

$$\|h|_\xi\|^2 = \sum_{i,j=1}^{p} \|h(e_i, e_j)\|^2, \ \|h|_{\xi^\perp}\|^2 = \sum_{\ell,s=p+1}^{n} \|h(e_\ell, e_s)\|^2 \tag{5.11}$$

and

$$\|h|_\xi\|^2 + \|h|_{\xi^\perp}\|^2 \leq \|h\|^2. \tag{5.12}$$

Proposition 5.2. *With notation as above let* $q = n - p$. *Then*

$$T_\xi^{p,q}(h) \leq \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(\frac{p}{2} - 1\right) \|h|_\xi\|^2 + \left(\frac{q}{2} - 1\right) \|h|_{\xi^\perp}\|^2. \tag{5.13}$$

If $1 \leq p \leq \frac{n}{2}$

$$T_\xi^{p,q}(h) \leq \frac{1}{2}(q\|h\|^2 - n^2\|H\|^2). \tag{5.14}$$

Proof. Write $h_{ij} = h(e_i, e_j)$. Then $\|h\|^2 = \sum_{s,t=1}^{n} \|h_{st}\|^2$ implies that

$$2 \sum_{\substack{1 \leq i \leq p \\ p+1 \leq \ell \leq n}} \|h_{i\ell}\|^2 = \|h\|^2 - \sum_{i,j=1}^{p} \|h_{ij}\|^2 - \sum_{\ell,s=p+1}^{n} \|h_{\ell s}\|^2 \tag{5.15}$$

and squaring

$$nH = \sum_{t=1}^{n} h_{tt} = \sum_{i=1}^{p} h_{ii} + \sum_{\ell=p+1}^{n} h_{\ell\ell}$$

implies

$$\begin{aligned} \left\langle \sum_{i=1}^{p} h_{ii}, \sum_{\ell=p+1}^{n} h_{\ell\ell} \right\rangle &= \sum_{\substack{1 \leq i \leq p \\ p+1 \leq \ell \leq n}} \langle h_{ii}, h_{\ell\ell} \rangle \\ &= \frac{n^2}{2}\|H\|^2 - \frac{1}{2}\left\| \sum_{i=1}^{p} h_{ii} \right\|^2 - \frac{1}{2}\left\| \sum_{\ell=p+1}^{n} h_{\ell\ell} \right\|^2. \end{aligned} \tag{5.16}$$

In the following we use (5.10) (several times), (5.15), and (5.16).

$$
\begin{aligned}
T^{p,q}_\xi(h) &= \sum_{\substack{1\le i\le p\\ p+1\le \ell\le n}} \left(2\|h_{i\ell}\|^2 - \langle h_{ii}, h_{\ell\ell}\rangle\right)\\
&= \|h\|^2 - \frac{n^2}{2}\|H\|^2 - \sum_{i,j=1}^{p}\|h_{ij}\|^2 - \sum_{\ell,s=p+1}^{n}\|h_{\ell s}\|^2 + \frac{1}{2}\left\|\sum_{i=1}^{p} h_{ii}\right\|^2 + \frac{1}{2}\left\|\sum_{\ell=p+1}^{n} h_{\ell\ell}\right\|^2\\
&\le \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(-\frac{1}{p}+\frac{1}{2}\right)\left\|\sum_{i=1}^{p} h_{ii}\right\|^2 + \left(-\frac{1}{q}+\frac{1}{2}\right)\left\|\sum_{\ell=p+1}^{n} h_{\ell\ell}\right\|^2\\
&\le \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(-\frac{1}{p}+\frac{1}{2}\right)p\sum_{i=1}^{p}\|h_{ii}\|^2 + \left(-\frac{1}{q}+\frac{1}{2}\right)\sum_{\ell,s=p+1}^{n}\|h_{\ell s}\|^2\\
&= \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(\frac{p}{2}-1\right)\|h|_\xi\|^2 + \left(\frac{q}{2}-1\right)\|h|_{\xi^\perp}\|^2
\end{aligned}
$$

The use of (5.10) in the second inequality is allright even in the case that $(-\frac{1}{p}+\frac{1}{2})\,\big(\text{or}\,(-\frac{1}{q}+\frac{1}{2}\,)\big)$ is negative for in that case $p = 1\,(\text{or } q = 1\,)$ and equality holds in (5.10). To prove (5.14) note that $1 \le p \le \frac{n}{2}$ implies $\frac{p}{2} \le \frac{q}{2}$. Thus using (5.13)

$$
\begin{aligned}
T^{p,q}_\xi(h) &\le \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(\frac{p}{2}-1\right)\|h|_\xi\|^2 + \left(\frac{q}{2}-1\right)\|h|_{\xi^\perp}\|^2\\
&\le \|h\|^2 - \frac{n^2}{2}\|H\|^2 + \left(\frac{q}{2}-1\right)(\|h|_\xi\|^2 + \|h|_{\xi^\perp}\|^2)\\
&\le \frac{1}{2}(q\|h\|^2 - n^2\|H\|^2).
\end{aligned}
$$

□

Theorem 16. *Let $R^{n+m}(c)$ be the complete simply connected Riemannian manifold of constant sectional curvature $c \ge 0$ and let M^n be a compact immersed submanifold of $R^{n+m}(c)$. Let $1 \le p \le \frac{n}{2}$ and $q = n - p$. Assume that the mean curvature vector H and the second fundamental form h of M satisfy*

$$q\|h\|^2 < n^2\|H\|^2 + 2pqc \tag{5.17}$$

at all points. Then $H_k(M, G) = 0$ for $p \le k \le n-p$. If (5.17) holds when $p = 1$ then M is homeomorphic to a sphere.

Problem. *Does Theorem 16 also hold when $c < 0$?*

Proof of Theorem 16. In the case $c = 0$ then $R^{n+m}(c) = \mathbb{R}^{n+m}$ and the trace formulas of section 3 apply. Thus for any decomposable unit p-vector ξ we have by (5.14) that

$$\operatorname{trace}(Q_\xi) = T^{p,q}_\xi(h) \le \frac{1}{2}(q\|h\|^2 - n^2\|H\|^2)$$

so the result follows from Corollary 3. In the case $c > 0$ let $r = \frac{1}{\sqrt{c}}$ and let $S^{n+m}(r)$ be the sphere of radius r. Then $M^n \subseteq R^{n+m}(c) = S^{n+m}(r) \subseteq \mathbb{R}^{n+m+1}$. Let $\overline{h}$ be the second fundamental form of M in $\mathbb{R}^{n+m+1}$ and h_1 the restriction of the second fundamental form of $R^{n+m}(c) = S^{n+m}(r)$ to TM. Then if η is the inward-pointing unit normal along $S^{n+m}(r)$ that $h_1(X, Y) = c\,\langle X, Y\rangle$. It follows that $T^{p,q}_\xi(h_1) = -pqc$ for all decomposable p-vectors ξ tangent to M. Also, $\overline{h} = h + h_1$ and the ranges of h and h_1 are everywhere orthogonal. Thus, by equation (5.2), for any decomposable p-vector ξ tangent to M

$$\begin{aligned}
\operatorname{trace}(Q_\xi) &= T^{p,q}_\xi(h) + T^{p,q}_\xi(h_1) \\
&= \sum_{i=1}^{p}\sum_{\ell=p+1}^{n}\left(2||h(e_i, e_\ell)||^2 - \langle h(e_i, e_i), h(e_\ell, e_\ell)\rangle_{\mathbb{R}^{n+m}(c)}\right) - pqc \\
&\le \frac{1}{2}(q\|h\|^2 - n^2\|H\|) - pqc \quad \Big(\text{equality holds in (5.10)}\Big) \\
&< 0
\end{aligned} \tag{5.18}$$

and now the result again follows from Corollary 3. □

Remark 5.1. The constants involved in Theorem 16 are the best possible as we now show. Assume $c > 0$, let $r = \frac{1}{\sqrt{c}}$ and $\alpha, \beta > 0$ with $\alpha^2 + \beta^2 = r^2$. Let $1 \le p \le \frac{n}{2}$ and set $q = n - p$. If $M(\alpha) = S^p(\alpha) \times S^q(\beta) \subseteq S^{n+1}(r) = R^{n+1}(c)$ the calculations show that on $M(\alpha)$ that

$$\|h\|^2 = \frac{p}{\alpha^2} + \frac{q}{\beta^2} - nc$$

and

$$n^2\|H\|^2 = \frac{p^2}{\alpha^2} + \frac{q^2}{\beta^2} - n^2c.$$

If we take limits as $\alpha \to r$ so that $\frac{1}{\alpha^2} \to c$ we find

$$\begin{aligned}
q\|h\|^2 - n^2\|H\|^2 &= \frac{pq}{\alpha^2} + \frac{q^2}{\beta^2} - qnc - \frac{p^2}{\alpha^2} - \frac{q^2}{\beta^2} + n^2c \\
&\to (pq - qn - p^2 + n^2)c = 2pqc.
\end{aligned}$$

But $H_p(M(\alpha), G) \neq 0$. Thus it is impossible to make the constants $2pqc$ or n^2 any larger or q any smaller in (5.17) and still have the conclusion of Theorem 16 hold. Similar examples also work when $c = 0$.

In the process of proving Theorem 16, we have proved the following:

Corollary. *With the notations M^n, $R^{n+m}(c)$, and h of Theorem 16, if for $1 \le p \le \frac{n}{2}$ and $q = n - p$,*

$$\sum_{i=1}^{p}\sum_{\ell=p+1}^{n}\left(2||h(e_i, e_\ell)||^2 - \langle h(e_i, e_i), h(e_\ell, e_\ell)\rangle\right) < pqc \tag{5.19}$$

at all points where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_xM, then (a) there are no stable currents in $\mathcal{R}_p(M, G)$ or $\mathcal{R}_q(M, G)$ for any finitely generated abelian *group G. In particular M has no closed stable minimal submanifolds of dimension p or $n-p$. (b) $H_p(M,G) = H_q(M,G) = 0$ and if $p=1$ or $p=n-1$ then M is simply connected. (c) If (5.19) holds for $p=1$ or for $p=n-1$ then M is a topological sphere. Furthermore, when $n=2$ or $n=3$ M is diffeomorphic to a sphere.*

Proof. In the case $c=0$, the result follows from Corollary 3. In the case $c>0$ let $r = \frac{1}{\sqrt{c}}$. Then $M^n \subseteq R^{n+m}(c) = S^{n+m}(r) \subseteq \mathbb{R}^{n+m+1}$. It follows from (5.18) and the assumption (5.19) that

$$\operatorname{trace}(Q_\xi) = \sum_{i=1}^{p} \sum_{\ell=p+1}^{n} \Big(2||h(e_i,e_\ell)||^2 - \langle h(e_i,e_i), h(e_\ell,e_\ell)\rangle_{\mathbb{R}^{n+m}(c)} \Big) - pqc < 0.$$

Now the result again follows from Corollary 3. □

Theorem 17. *There is a C^2 neighborhood $\mathcal{U}$ of the standard metric g_0 on $R^{n+m}(c)$, $c \geq 0$ such that if M^n is a compact immersed submanifold of $(R^{n+m}(c), g)$ with the mean curvature vector H and the second fundamental form h satisfying (5.17) for some $g \in$ mathcalU, $1 \leq p \leq \frac{n}{2}$, and $q = n-1$. Then*

(a) M has no stable p-currents or $(n-p)$-currents over any finitely generated abelian group G,

(b) $H_p(M,G) = H_q(M,G) = 0$ and if $p=1$ or $p=n-1$ then M is simply connected, and

(c) If (5.17) holds for some $g \in \mathcal{U}$ and $p=1$, i.e.

$$\|h\|^2 < \frac{n^2}{n-1}\|H\|^2 + 2c\,, \tag{5.20}$$

then M is diffeomorphic *to S^n for all $n \geq 2$.*

Proof. Case 1: $g = g_0$. The assertions (a) and (b) follow from Theorem 16. The assertions (c), when $n \geq 4$ follows from Huisken [19] and B. Andrews [2] for codimension $m=1$, and J. R. Gu and H.W. Xu [14] for arbitrary codimensions $m \geq 1$. When $n=3$, the Gauss equation, or (3.18) and (3.19) imply that M has positive Ricci curvature and hence by a Theorem of Hamilton, M is diffeomorphic to S^3. When $n=2$, M is diffeomorphic to S^2. This follows from the Gauss- Bonnet Theorem.

Case 2: g is in some neighborhood of g_0. Denote by $\bar{K}(\pi)$ the sectional curvature of $(R^{n+m}(c), g)$ for 2-plane $\pi\big(\subset T_x(R^{n+m}(c),g)\big)$. Set

$$\bar{K}_{\max}(x) := \max_{\pi \subset T_x(R^{n+m}(c),g)} \bar{K}(\pi) \quad \text{and} \quad \bar{K}_{\min}(x) := \min_{\pi \subset T_x(R^{n+m}(c),g)} \bar{K}(\pi)$$

We note when $p=1$, (5.18) takes the form

$$\big(\operatorname{trace}(Q_\xi) \leq \big)\frac{1}{2}((n-1)\|h\|^2 - n^2\|H\|) - (n-1)c < 0 \quad \text{on} \quad (R^{n+m}(c), g_0),$$

and coincides with

$$\left(\|h\|^2 - \frac{8}{3}\big(\bar{K}_{\min} - \frac{1}{4}\bar{K}_{\max}\big) - \frac{n^2\|H\|^2}{n-1} = \right)\|h\|^2 - 2c - \frac{n^2\|H\|^2}{n-1} < 0 \tag{5.21}$$

on $(R^{n+m}(c), g_0)$. In view of the proof of Theorem 9 and (5.18), $\operatorname{trace}(Q_\xi)$, $\|h\|^2 - \frac{n^2}{n-1}\|H\|^2 - 2c$ and $\|h\|^2 - \frac{8}{3}\big(\bar{K}_{\min} - \frac{1}{4}\bar{K}_{\max}\big) - \frac{n^2}{n-1}\|H\|^2$ are continuous functions of g in the C^2 strong topology. Thus, by the continuity and (5.21) we can choose a neighborhood $\mathcal{U}$ of g_0, such that for every $g \in \mathcal{U}$,

$$\operatorname{trace}(Q_\xi) < 0 \quad \text{on} \quad (R^{n+m}(c), g)$$

and

$$\|h\|^2 - \frac{8}{3}\big(\bar{K}_{\min} - \frac{1}{4}\bar{K}_{\max}\big) - \frac{n^2\|H\|^2}{n-1} < 0 \tag{5.22}$$

on $(R^{n+m}(c), g)$.

Arguing in the same way as in the proof of Theorem 16, the assertions (a), (b) and the topological sphere theorem follow. To prove the differentiable sphere theorem (c), we consider the following cases: (1) if $n \geq 4$, then by a Theorem of H.W. Xu and J.R. Gu [39, Theorem 4.1], that is built on the work of Simon Brendle [4, Theorem 2] on the convergence of Ricci flow, inequality (5.22) implies that the submanifold M, of $(R^{n+m}(c), g)$ is diffeomorphic to a space form. As in the proof of Theorem 16 the compact manifold M is simply- connected and hence M is diffeomorphic to S^n. (2) if $n = 3$, then argue as before, M has positive Ricci curvature and M is diffeomorphic to S^3, by a Theorem of Hamilton [15]. (3) if $n = 2$, then M has positive Gaussian curvature and Gauss-Bonnet Theorem implies that M is diffeomorphic to S^2. This completes the proof. □

As presented in 1983 (cf. [35]), we have the following immediate optimal result.

Proposition 5.3. *Let M be a closed surface in a Euclidean sphere with the second fundamental form h satisfying $\|h\|^2 < 2$. Then M is diffeomorphic to a sphere S^2 or $\mathbb{RP}^2$ depending on M is orientable or not.*

Proof. If $\|h\|^2 < 2$ then (5.20) holds. Now the assertion is an immediate consequence of Theorem 17. □

6. Stable currents in the rank one symmetric spaces

In this section we classify the stable currents in the compact simply connected rank one symmetric spaces. We recall that these are the spheres S^n, the complex projective spaces $\mathbb{CP}^n$, the quaternionic projective spaces $\mathbb{HP}^n$, and the Cayley plane $\mathbb{C}\mathrm{ay}\mathbb{P}^2$. The spheres were done in section 4. We start by giving a description of the usual imbeddings of the projective spaces in Euclidean space as sets of matrices.

Let $\mathbb{F}$ be one of the following: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the Cayley numbers $\mathbb{C}\mathrm{ay}$. Then $\mathbb{F}$ is a division algebra over the real numbers $\mathbb{R}$ (nonassociative if $\mathbb{F} = \mathbb{C}\mathrm{ay}$) that has an involutive antiautomorphism $a \mapsto \overline{a}$ (called conjugation) such that the set of elements fixed by conjugation is the field of real numbers imbedded in $\mathbb{F}$ as the scalar multiples of the identity element 1. The real part of a and the norm of a are defined to be

$$\mathrm{Re}(a) = \frac{1}{2}(a + \overline{a}), \ |a|^2 = a\overline{a} = \overline{a}a. \tag{6.1}$$

An inner product is given on $\mathbb{F}$ by $\langle a, b\rangle = \mathrm{Re}(a\overline{b})$.

If $A = [a_{ij}]$ is an $m \times n$ matrix with elements in $\mathbb{F}$ then A^* is the conjugate transpose of A, that is $A^* = [b_{ij}]$ with $b_{ij} = \overline{a}_{ji}$. Let $H(n+1, \mathbb{F})$ be the real vector space of $(n+1)$ by $(n+1)$ Hermitian matrics over $\mathbb{F}$, that is $A \in H(n+1, \mathbb{F})$ if and only if $A^* = A$. *If* $\mathbb{F} = \mathbb{C}\mathrm{ay}$ *we will always assume that* $n = 2$. Define a positive definite inner product on $H(n+1, \mathbb{F})$ by

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Re}\ \mathrm{trace}(AB^*) = \frac{1}{2}\mathrm{Re}\ \mathrm{trace}(A^*B). \tag{6.2}$$

Let $\mathbb{F}\mathbb{P}^n$ be the set of rank one idempotents in $H(n+1, \mathbb{F})$, that is $A^2 = A$ and $\mathrm{trace}(A) = 1$. Given $\mathbb{F}\mathbb{P}^n$ the metric it inherits as a submanifold of $H(n+1, \mathbb{F})$. To see this is isometric to the usual model of $\mathbb{F}\mathbb{P}^n$, at least when $\mathbb{F} \neq \mathbb{C}\mathrm{ay}$, let $U(n+1, \mathbb{F})$ be the group of $(n+1)$ by $(n+1)$ matrices g over $\mathbb{F}$ that satisfy $gg^* = g^*g = 1$. This group acts on $H(n+1, \mathbb{F})$ by the rule $g(A) = gAg^*$. This action preserves the inner product on $H(n+1, \mathbb{F})$ and maps $\mathbb{F}\mathbb{P}^n$ onto itself. Moreover $U(n+1, \mathbb{F})$ is transitive on $\mathbb{F}\mathbb{P}^n$ (for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ this follows from standard normal forms theorems, for the case $\mathbb{F} = \mathbb{H}$ see the appendix to the paper [34]). Moreover if we let

$$P_0 = \begin{bmatrix} \mathrm{I}_{1\times i} & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{bmatrix} \tag{6.3}$$

then the subgroup of $U(n+1, \mathbb{F})$ fixing P_0 is $U(I, \mathbb{F}) \times U(n, \mathbb{F})$. This shows that $\mathbb{F}\mathbb{P}^n$ is isometric to the homogeneous space $U(n+1, \mathbb{F})/(U(I, \mathbb{F}) \times U(n, \mathbb{F}))$ with an invariant metric. But this is the usual model of $\mathbb{F}\mathbb{P}^n$ as a symmetric space. When $\mathbb{F} = \mathbb{C}\mathrm{ay}$ and $n = 2$ things are more complicated. The vector space $H(3, \mathbb{C}\mathrm{ay})$ becomes a Jordon algebra under the product $A \circ B = \frac{1}{2}(AB + BA)$. The automorphism group of this algebra is the compact exceptional Lie group F_4, which for reasons of uniformity we denote by $U(3, \mathbb{C}\mathrm{ay})$. This group preserves the inner product of $H(3, \mathbb{C}\mathrm{ay})$ and is transitive on $\mathbb{C}\mathrm{ay}\mathbb{P}^2$. The subgroup of $F_4 = U(3, \mathbb{C}\mathrm{ay})$ fixing P_0 (given by (6.3)) is the group spin(9). Thus $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ is $F_4/\mathrm{spin}(9)$ with an invariant metric. See [34] and the references given there for details.

Lemma 6.1. *(a) The tangent space to* $\mathbb{F}\mathbb{P}^n$ *at* P_0 $\big($*given by* (6.3)$\big)$ *is*

$$T_{P_0}(\mathbb{F}\mathbb{P}^n) = \left\{ \begin{bmatrix} 0_{1\times 1} & X^* \\ X & 0_{n\times n} \end{bmatrix} : X \in \mathbb{F}^n \right\} \tag{6.4}$$

(here $\mathbb{F}^n$ *is the space of column vectors of length* n *over* $\mathbb{F}$*).*

(b) If the vector $X \in \mathbb{F}^n$ is identified with

$$x = \begin{bmatrix} 0_{1\times 1} & X^* \\ X & 0_{n\times n} \end{bmatrix} \in T_{P_0}(\mathbb{FP}^n) \tag{6.5}$$

then the second fundamental form of $\mathbb{FP}^n$ in $H(n+1,\mathbb{F})$ is given at P_0 by

$$h(X,Y) = \begin{bmatrix} -X^*Y - Y^*X & 0_{1\times n} \\ 0_{n\times 1} & XY^* + YX^* \end{bmatrix} \tag{6.6}$$

Proof. If $c(t)$ is a curve in $\mathbb{FP}^n$ with $c(0) = P_0$ then the lemma is proven by taking derivatives of the relation $c(t) = c(t)c(t) = c(t)c(t)^*$. See [7] for details. $\square$

Definition 6.1. Identify $\mathbb{F}^n$ with $T_{P_0}(\mathbb{FP}^n)$ by (6.4) and for $1 \le k \le n-1$ let $\mathbb{F}^k$ be the subspace of $\mathbb{F}^n = T_{P_0}(\mathbb{FP}^n)$ given by

$$\mathbb{F}^k = \left\{ \begin{bmatrix} X_{k\times 1} \\ 0_{(n-k)\times 1} \end{bmatrix} : X \in \mathbb{F}^n \right\}.$$

Then a subspace W tangent to $\mathbb{FP}^n$ at some point A will be called *an $\mathbb{F}$-subspace of $T_A(\mathbb{FP}^n)$* if and only if there is an element $g \in U(n+1,\mathbb{F})$ such that $g_{*A}W = \mathbb{F}$ for some k with $1 \le k \le n-1$.

Therefore every subspace of a tangent space to $\mathbb{RP}^n$ is an $\mathbb{R}$-subspace, the $\mathbb{C}$-subspaces of tangent spaces to $\mathbb{CP}^n$ are the complex subspaces in the usual sense (i.e. invariant under the almost complex structure) and all have even dimension over $\mathbb{R}$, the $\mathbb{H}$-subspaces of tangent spaces to $\mathbb{HP}^n$ are the quaternionic subspaces in the usual sense (see [13]) and all have dimension divisible by four over $\mathbb{R}$, and all $\mathbb{C}$ay-subspaces of $T(\mathbb{C}\mathrm{ay}\mathbb{P}^2)$ have dimension eight over $\mathbb{R}$. In the case $\mathbb{F} \ne \mathbb{R}$ it is easy to give an intrinsic definition of an $\mathbb{F}$-subspace. Normalize the metric on $\mathbb{FP}^n$ so that the maximal sectional curvatures are 4 and the minimal sectional curvatures are 1. Then a subspace W tangent to $\mathbb{FP}^n$ is an $\mathbb{F}$-subspace if and only if for each independent pair of vectors X, Y tangent to $\mathbb{FP}^n$ at A with $K(X,Y) = 4$, $X \in W$ implies that $Y \in W$. $\big(K(X,Y) =$ sectional curvature of span $\{X,Y\}.\big)$ A fact that we will use several times is that a subspace W is an $\mathbb{F}$-subspace if and only if its orthogonal complement is an $\mathbb{F}$-subspace.

Lemma 6.2. *(a) Let X, Y be orthonormal vectors tangent to $\mathbb{FP}^n$ at some point A. Then*

$$2\|h(X,Y)\|^2 - \langle h(X,X), h(Y,Y)\rangle \le 0 \tag{6.7}$$

with equality if and only if Y is orthogonal to the $\mathbb{F}$-subspace of $T_A(\mathbb{FP}^n)$ generated by X.

(b) Let $d = \dim_{\mathbb{R}}(\mathbb{F})$ and let $\{e_1, \ldots, e_{nd}\}$ be an orthonormal basis of $T_A(\mathbb{FP}^n)$ and for some $1 \le p \le nd$, set $\xi = e_1 \wedge \cdots \wedge e_p$. Then, with the notation of Theorem 11,

$$\operatorname{trace}(Q_\xi) \le 0 \tag{6.8}$$

with equality if and only if $e_1, \dots, e_p$ *span an* $\mathbb{F}$*-subspace of* $T_A(\mathbb{FP}^n)$*.*

Proof. Because $U(n+1,\mathbb{F})$ acts on both $\mathbb{FP}^n$ and $H(n+1,\mathbb{F})$ by isometries we can replace X, Y by g_*X, g_*Y for any $g \in U(n+1,\mathbb{F})$ without changing the value of the left side of (6.7). The group $U(n+1,\mathbb{F})$ is transitive on the set of unit vectors tangent to $\mathbb{FP}^n$ so that we may (by replacing X, Y by g_*X, g_*Y for the proper choice of g) assume that $A = P_0$ and

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ Y_1 \end{bmatrix}$$

where $1, y_1 \in \mathbb{F}$, $0, Y_1 \in \mathbb{F}^{n-1}$. In the following calculations we will use that for matrices A, B over $\mathbb{F}$ that $\mathrm{Re\ trace}(AB) = \mathrm{Re\ trace}(BA)$. To simplify notation set

$$h_1(X,Y) = -X^*Y - Y^*X, \quad h_2(X,Y) = XY^* + YX^*.$$

Then,

$$\begin{aligned} &2\|h(X,Y)\|^2 - \langle h(X,X), h(Y,Y)\rangle \\ &= \sum_{i=1} \big(2\|h_i(X,Y)\|^2 - \langle h_i(X,X), h_i(Y,Y)\rangle\big). \end{aligned} \tag{6.9}$$

Using that $X^*Y = y_1$, $Y^*X = \overline{y}_1$, $X^*X = 1$ and $Y^*Y = \|Y\|^2 = 1$;

$$\begin{aligned} &2\|h_1(X,Y)\|^2 - \langle h_1(X,X), h_1(Y,Y)\rangle \\ &= \mathrm{Re\ trace}\big((X^*Y + Y^*X)(X^*Y + Y^*X)^*\big) - \frac{1}{2}\mathrm{Re\ trace}\big(2(X^*X)(Y^*Y)\big) \\ &= \mathrm{Re}\big((y_1 + \overline{y}_1)(y_1 + \overline{y}_1)\big) - 2\mathrm{Re}\big((1)(1)\big) \\ &= \mathrm{Re}(y_1^2) + \mathrm{Re}(\overline{y}_1^2) + 2|y_1|^2 - 2\,. \end{aligned} \tag{6.10}$$

But because X is orthogonal to Y we have that $0 = \mathrm{Re}(X^*Y) = \mathrm{Re}(y_1)$. But it is an elementary fact that if $a \in \mathbb{F}$ with $\mathrm{Re}(a) = 0$ then $a^2 = -|a|^2$. $\big(2\mathrm{Re}(a) = a + \overline{a} = 0$ implies $a^2 = a(-\overline{a}) = -|a|^2\big)$. Whence $y_1^2 = \overline{y}_1^2 = -|y_1|^2$. Using this in (6.10) gives

$$2\|h_1(X,Y)\|^2 - \langle h_1(X,X), h_1(Y,Y)\rangle = -2. \tag{6.11}$$

Next,

$$\begin{aligned} &2\|h_2(X,Y)\|^2 - \langle h_2(X,X), h_2(Y,Y)\rangle \\ &= \mathrm{Re\ trace}\big((XY^* + YX^*)(XY^* + YX^*)^*\big) - \frac{1}{2}\mathrm{Re\ trace}\big((2XX^*)(2YY^*)\big) \\ &= \mathrm{Re\ trace}\big((XY^*)(XY^*) + (XY^*)(YX^*) + (YX^*)(XY^*) + (YX^*)(YX^*)\big) \\ &\qquad - 2\mathrm{Re\ trace}\big((XX^*)(YY^*)\big). \end{aligned} \tag{6.12}$$

Let $x_1 = 1$, $x_2 = \cdots = x_n = 0$ be the components of X and $y_1, \dots, y_n$ the components of Y. Then the components of XY^* are $(XY^*)_{ij} = x_i\overline{y}_j (= 0$ when

$i \neq 1$ and $= \overline{y}_j$ when $i = 1$) and likewise for XX^*, YY^*, etc. Using that $y_1^2 = \overline{y}_1^2 = -|y_1|^2$,

$$\begin{aligned}
\operatorname{trace}\big((XY^*)(XY^*)\big) &= \sum_{i,j=1}^{n} (x_1\overline{y}_j)(x_j\overline{y}_i) = \overline{y}_1\overline{y}_1 = -|y_1|^2 \\
\operatorname{trace}\big((YX^*)(YX^*)\big) &= \sum_{i,j=1}^{n} (y_i\overline{x}_j)(y_j\overline{x}_i) = (y_1)(y_1) = -|y_1|^2 \\
\operatorname{trace}\big((XY^*)(YX^*)\big) &= \sum_{i,j=1}^{n} (x_1\overline{y}_j)(y_j\overline{x}_i) = \sum_{j=1}^{n} y_j\overline{y}_j = |Y|^2 = 1 \\
\operatorname{trace}\big((YX^*)(XY^*)\big) &= \sum_{i,j=1}^{n} (y_i\overline{x}_j)(x_j\overline{y}_i) = \sum_{i=1}^{n} y_i\overline{y}_i = |Y|^2 = 1 \\
\operatorname{trace}\big((XX^*)(YY^*)\big) &= \sum_{i,j=1}^{n} (x_i\overline{x}_j)(y_j\overline{y}_i) = y_1\overline{y}_1 = |y_1|^2.
\end{aligned}$$

Note that the associative law has not been used so these calculations work when $\mathbb{F} = \mathbb{C}\mathrm{ay}$. Using these in (6.12) gives

$$2\|h_2(X,Y)\|^2 - \langle h_2(X,X), h_2(Y,Y)\rangle = 2 - 4|y_1|^2.$$

Putting this and equation (6.11) into (6.9) yields

$$2\|h(X,Y)\|^2 - \langle h(X,X), h(Y,Y)\rangle = -4|y_1|^2 \leq 0 \tag{6.13}$$

and equality holds if and only if $y_1 = 0$ which is equivalent to Y being orthogonal to the $\mathbb{F}$-subspace generated by X. This proves part (a) of the lemma.

By equation (3.15) and part (a)

$$\operatorname{trace}(Q_\xi) = \sum_{i=1}^{p} \sum_{\ell=p+1}^{nd} \big(2\|h(e_i,e_\ell)\|^2 - \langle h(e_i,e_i), h(e_\ell,e_\ell)\rangle\big) \leq 0 \tag{6.14}$$

with equality if and only if e_ℓ is orthogonal to the $\mathbb{F}$-subspace generated by e_i whenever $1 \leq i \leq p$ and $p+1 \leq \ell \leq nd$. Thus if $\operatorname{span}\{e_1,\ldots,e_p\}$ is an $\mathbb{F}$-subspace $\operatorname{trace}(Q_\xi) = 0$. Conversely, suppose that $\operatorname{span}\{e_1,\ldots,e_p\}$ is not an $\mathbb{F}$ subspace. Then there is a unit u in $\operatorname{span}\{e_1,\ldots,e_p\}$ so that the $\mathbb{F}$-subspace generated by u is not contained in $\operatorname{span}\{e_1,\ldots,e_p\}$. This implies there is a unit vector v in $\operatorname{span}\{e_1,\ldots,e_p\}^\perp = \operatorname{span}\{e_{p+1},\ldots,e_n\}$ that is not orthogonal to the $\mathbb{F}$-subspace generated by u. By relabelling we may assume that $e_1 = u$ and $e_{p+1} = v$. Then, e_{p+1} is not orthogonal to the $\mathbb{F}$-subspace generated by e_1 and so equality cannot hold in (6.14). This completes the proof of the lemma. □

We can now give our classification theorem, which is that stable currents are "$\mathbb{F}$-currents."

Theorem 18. *Let $\mathcal{S} \in \mathcal{R}_p(\mathbb{FP}^n, G)$ be a stable current. Then for $\|\mathcal{S}\|$ almost all $x \in \mathbb{FP}^n$ the approximate tangent space $T_x(\mathcal{S})$ is an $\mathbb{F}$-subspace of $T_x(\mathbb{FP}^n)$.*

There is also a set of smooth vector fields $V_1, \dots, V_\ell$ on $\mathbb{FP}^n$ such that for every p with $1 \le p \le n \cdot \dim_{\mathbb{R}}(\mathbb{F})$ that is not divisible by $\dim_{\mathbb{R}}(\mathbb{F})$ the set $\{V_1, \dots, V_\ell\}$ is universally mass decreasing in dimension p.

Proof. Let $l = \frac{d(n+1)(n+2)}{2} = \dim_{\mathbb{R}} M(n+1,\mathbb{F})$ and let $\{e_1, \dots, e_\ell\}$ be an orthonormal basis of $M(n+1,\mathbb{F})$. If $\mathcal{S}$ is stable then by Theorem 11 and the last lemma

$$0 \le \sum_{i=1}^{\ell} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{M}(\varphi_{t_*}^{e_i^T} \mathcal{S}) = \int_{\mathbb{FP}^n} \operatorname{trace}(Q_{\overrightarrow{\mathcal{S}_x}}) d\|\mathcal{S}\|(x) \le 0$$

and so $\operatorname{trace}(Q_{\overrightarrow{\mathcal{S}_x}}) = 0$ for $\|\mathcal{S}\|$ almost all x in $\mathbb{FP}^n$. By the last lemma this implies that $T_x(\mathcal{S})$ is an $\mathbb{F}$-subspace for $\|\mathcal{S}\|$ almost all $x \in \mathbb{FP}^n$. If p is not divisible by d then $\operatorname{trace}(Q_\xi) < 0$ for all unit decomposable p vectors $\xi = e_1 \wedge \cdots \wedge e_p$ tangent to $\mathbb{FP}^n$ as $\operatorname{span}\{e_1, \dots, e_p\}$ is never an $\mathbb{F}$-subspace. Thus, $\{e_1^T, \dots, e_\ell^T\}$ is universally mass decreasing in dimension p by Corollary 3. This completes the proof. $\square$

Remarks. (1) As is well known $H_p(\mathbb{FP}^n;\mathbb{Z}) \ne 0$ if and only if $0 \le p \le nd$ and p is divisible by d. Therefore Theorem 18 completes the proof of Theorem 10.

(2) As every subspace of $T(\mathbb{RP}^n)$ is an $\mathbb{R}$-subspace this implies $\operatorname{trace}(Q_\xi) = 0$ for all ξ as claimed in Remark 3 (1).

(3) In the case $\mathbb{F} = \mathbb{C}$ Theorem 18 is due to Lawson and Simsons [22] where they show how to use this theorem, along with a Theorem of Harvey and Shiffman on the structure of complex currents, to prove the only stable closed integral currents in $\mathbb{CP}^n$ are the algebraic cycles.

We now classify the stable currents in the quaternionic projective space $\mathbb{HP}^n$. By an $\mathbb{HP}^k$ in $\mathbb{HP}^n$ we mean any of the standard imbeddings of $\mathbb{HP}^k$ in $\mathbb{HP}^n$ as a totally geodesic submanifold. By the support of $\mathcal{S} \in \mathcal{R}_p(M,G)$ the support of the measure $\|\mathcal{S}\|$ is meant. If U is an open subset of M then a current $\mathcal{S} \in \mathcal{R}_p(M,G)$ is said to be *smooth* in U if and only if there are imbedded smooth oriented submanifolds $N_1, \dots, N_\ell$ of M (which are pairwise disjoint and if $\partial N_i \ne \emptyset$ then $\partial N_i \cap U = \emptyset$) and elements $a_1, \dots, a_\ell \in G$ such that the current $\mathcal{S} - (a_1 N_1 + \cdots + a_\ell N_\ell)$ has its support in $M \backslash U$. A point $x \in \operatorname{spt}(\mathcal{S})$ (the support of $\mathcal{S}$) is a *smooth point* of $\mathcal{S}$ if and only if x has a neighborhood U in M such that the $\mathcal{S}$ is smooth in U. The set of smooth points of $\mathcal{S}$ will be denoted by $\operatorname{smooth}(\mathcal{S})$. It is known [1] that $\operatorname{smooth}(\mathcal{S})$ is open and dense in $\operatorname{spt}(\mathcal{S})$. The set $\operatorname{sing}(\mathcal{S}) = \operatorname{spt}(\mathcal{S}) \backslash \operatorname{smooth}(\mathcal{S})$ is called the *singular* set of $\mathcal{S}$. We will denote the p-dimensional Hausdorff measure by $\mathcal{H}^p$ (see [8] for the definition).

Theorem 19. *Let $\mathcal{S} \in \mathcal{R}_{4k}(\mathbb{HP}^n, G)$ be a stable current and assume that*

$$\mathcal{H}^{4k-1}\big(\operatorname{sing}(\mathcal{S})\big) = 0.$$

Then there are a finite number $L_1, \dots, L_\ell$ of $\mathbb{HP}^k$'s in $\mathbb{HP}^n$ and elements $a_1, \dots, a_\ell \in G$ so that as a current $\mathcal{S} = a_1 L_1 + \cdots + a_\ell L_\ell$. Thus the only connected stable submanifolds of $\mathbb{HP}^n$ are the $\mathbb{HP}^k$'s.

To prove this Theorem, we need some lemmas.

Lemma 6.3 (A. Gray [13])**.** *Any smooth connected submainifold N of $\mathbb{HP}^k$ such that T_xN is always an $\mathbb{H}$-subspace of $T_x(\mathbb{HP}^n)$ is an open piece of some $\mathbb{HP}^k$.*

Lemma 6.4. *If M is an n-dimensional Riemannian manifold and $A \subseteq M$ disconnects M then $\mathcal{H}^{n-1}(A) > 0$.*

Proof. This follows from the isoperimetric inequalities in [8]. □

Lemma 6.5. *If $\mathcal{S}$ is a stable current of degree p then $\mathcal{H}^p\big(\mathrm{spt}(\mathcal{S})\big) < \infty$.*

Proof. See Proposition 3.13 of [21]. □

Proof of Theorem 19 . Let x_0 be a smooth point of $\mathrm{spt}(\mathcal{S})$ and let U be a maximal open (in $\mathrm{spt}(\mathcal{S})$) connected subset of $\mathrm{smooth}(\mathcal{S})$. Then by Theorem 18 and Lemma 6.3, U is an open subset of some $\mathbb{HP}^k$, say L. Let $\mathrm{bdy}(L)$ be the boundary of U in L. Then by the maximality of U it follows that $\mathrm{bdy}(U) \subseteq \mathrm{sing}(\mathcal{S})$. If U is not dense in L then $\mathrm{bdy}(U)$ disconnects L. But if this were the case then Lemma 6.4 would imply that

$$0 < \mathcal{H}^{4k-1}\big(\mathrm{bdy}(U)\big) \leq \mathcal{H}^{4k-1}\big(\mathrm{sing}(\mathcal{S})\big),$$

which contradicts our hypothesis. Thus U is dense in L and $\mathcal{H}^{4k-1}(L\backslash U) = 0$.

Therefore there is an element $a \in G$ such that $\mathrm{spt}(\mathcal{S} - aL) \subseteq L\backslash U$. Also $L \subseteq \mathrm{spt}(\mathcal{S})$. This shows that if $L_1, \ldots, L_m$ are $\mathbb{HP}^k$'s in $\mathbb{HP}^n$ so that $\mathrm{smooth}(\mathcal{S}) \cap L_i$ contains an open set of L_i then $\mathrm{spt}(\mathcal{S}) \supseteq L_1 \cup \cdots \cup L_m$ and thus

$$\infty > \mathcal{H}^{4k}(\mathcal{S}) \geq m\, \mathcal{H}^{4k}(\mathbb{HP}^k) = m\, \mathrm{vol}(\mathbb{HP}^k).$$

If follows that there are only a finite number $L_1, \ldots, L_\ell$ of $\mathbb{HP}^k$'s in $\mathbb{HP}^n$ that intersect $\mathrm{smooth}(\mathcal{S})$ in an open subset of $\mathrm{smooth}(\mathcal{S})$ and that there are $a_1, \ldots, a_\ell \in G$ such that

$$\mathcal{H}^{4k}\bigg(\mathrm{spt}\big(\mathcal{S} - (a_1L_1 + \cdots + a_\ell L_\ell)\big)\bigg) = 0.$$

This implies $\mathcal{S} = a_1L_1 + \cdots + a_\ell L_\ell$. The proof is complete. □

Remarks. (1) The present proof has the advantage that it works for all coefficients groups G and can also easily be extended to the case of varifolds.

(2) When $G = \mathbb{Z}_2$ and $\mathcal{S}$ has least mass in its homology class it is known [9] that $\mathcal{H}^{4k-1}(\mathrm{sing}(\mathcal{S})) = 0$. Therefore the last theorem implies that the mass minimizing elements of nonzero $\mathbb{Z}_2$ homology classes in $\mathbb{HP}^n$ are the $\mathbb{HP}^k$'s.

(3) There are several proofs of Lemma 6.3 in the literature but all of these we have seen are in the same spirit as Gray's original proof and make use of the existence of fields of almost complex structures on $\mathbb{HP}^n$ that have certain properties. It can be shown that no such almost complex structures exist on $\mathbb{C}\mathrm{ay}\mathbb{P}^2$.

Theorem 20. *Let $\mathcal{S} \in \mathcal{R}_2(\mathbb{C}\mathrm{ay}\mathbb{P}^2, G)$ be a stable current and assume that*

$$\mathcal{H}^7\big(\mathrm{sing}(\mathcal{S})\big) = 0.$$

Then there are a finite number $L_1, \dots, L_\ell$ of $\mathbb{C}\mathrm{ay}\mathbb{P}^1$'s in $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ and elements $a_1, \dots, a_\ell \in G$ so that as a current $\mathcal{S} = a_1 L_1 + \cdots + a_\ell L_\ell$. Thus the only connected stable submanifolds of $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ are the $\mathbb{C}\mathrm{ay}\mathbb{P}^1$'s.

Proof. Use the method in the proof of Theorem 19 and the analogy of Gray's Theorem (Lemma 6.3) that every smooth submanifold N^8 of $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ such that $T_x N$ is a $\mathbb{C}$ay-subspace of $T_x(\mathbb{C}\mathrm{ay}\mathbb{P}^2)$ for all $x \in N$ is totally geodesic (cf. [24], see also [25] for more general results), the assertion follows. $\square$

7. Mass minimizing currents modulo two in real projective spaces

In this section we will find all the currents in $\mathcal{R}_p(\mathbb{RP}^n, \mathbb{Z}_2)$ that minimize the mass in their homology class. Recall that for $0 \le p \le n$, $H_p(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2$.

Theorem 21. *Let α be the nonzero homology class in $H_p(\mathbb{RP}^n, \mathbb{Z}_2)$ $(1 \le p \le n-1)$ and let $\mathcal{S} \in \mathcal{R}_p(\mathbb{RP}^n, \mathbb{Z}_2)$ be a current in α of least mass. Then $\mathcal{S}$ is one of the standard imbeddings of $\mathbb{RP}^p$ into $\mathbb{RP}^n$ as a totally geodesic submanifold.*

Proof. Our main tool for this proof is the generalized Crofton Formula which relates volumes of a submanifold N of $\mathbb{RP}^n$ (and other homogeneous spaces) to the average number of points of intersection N has with a "moving plane." To be precise let $\mathrm{PG}(n, \ell)$ be the Grassman manifold of all $\mathbb{RP}^\ell$'s in $\mathbb{RP}^n$ with the volume form $\mathrm{d}L$ which is invariant under the natural action of the orthogonal group $O(n+1)$ $\big($which acts on $\mathbb{RP}^n$ and thus also on $\mathrm{PG}(n,\ell)\big)$. If N^p is any p-dimensional submanifold of $\mathbb{RP}^n$ of finite volume the Crofton formula

$$\int_{\mathrm{PG}(n,n-p)} \#(N \cap L)\, \mathrm{d}L = \gamma \,\mathrm{vol}(N) \tag{7.1}$$

holds. Here $\gamma = \gamma(n,p)$ only depends on n, p and the choice of the density $\mathrm{d}L$. ($\mathrm{d}L$ is unique up to a constant multiple.) This can be found in [6] or [17]. It is convenient to break the proof up into several steps. From here on $\mathcal{S}$ is as in the statement of the theorem.

Step 1. $\#(L \cap \mathrm{spt}(\mathcal{S})) \ge 1$ for all $L \in \mathrm{PG}(n, n-p)$.

Proof. Let s_k be a sequence of smooth chains in $\mathbb{RP}^n$ over $\mathbb{Z}_2$ with $\partial s_k = 0$ and such that $s_k \to \mathcal{S}$ in the flat topology. This sequence of chains exists by virtue of Theorem 15 in [11]. (The theorem there assumes that $\mathbb{RP}^n$ is imbedded in a Euclidean space and that the chains $\mathcal{P}_k$ converging to $\mathcal{S}$ are polyhedral chains in the Euclidean space. But will then be a tubular neighborhood U of $\mathbb{RP}^n$ and a smooth retraction $\pi : U \to \mathbb{RP}^n$. Then we can set $s_k = \pi_\# \mathcal{P}_k$.) Because $s_k \to \mathcal{S}$ and $\partial s_k = 0$ for all large enough k, and so we can assume for all k the homology class $[s_k]$ of s_k is $\alpha = [\mathcal{S}]$. It is a well known fact about the topology of $\mathbb{RP}^n$

that if $0 \neq \alpha \in H_p(\mathbb{RP}^n, \mathbb{Z}_2)$ and $0 \neq \beta \in H_{n-p}(\mathbb{RP}^n, \mathbb{Z}_2)$ then $\alpha \cap \beta \neq 0$ in the $\mathbb{Z}_2$ intersection ring of $\mathbb{RP}^n$. If $L \in \mathrm{PG}(n, n-p)$ then $0 \neq [L] \in H_{n-p}(M, \mathbb{Z}_2)$. Therefore for each k there is an $x_k \in L \cap \mathrm{spt}(s_k)$ and because L is compact we can assume $x_k \to x$ for some $x \in L$. Then $\mathcal{S}_k \to \mathcal{S}$ implies $x \in L \cap \mathrm{spt}(\mathcal{S}) \neq \emptyset$. □

Step 2. There is a smooth imbedded submanifold $\mathcal{S}$ of $\mathbb{RP}^n$ such that $\mathcal{S} \subseteq \mathrm{spt}(\mathcal{S})$ and the Hausdorff $p-2+\varepsilon$ dimensional measure $\mathcal{H}^{p-2+\varepsilon}$ of $\mathrm{spt}(\mathcal{S})\backslash S$ is zero for all $\varepsilon > 0$.

Proof. This is a Regularity Theorem of Federer [9]. □

Step 3. $\mathcal{M}(\mathcal{S}) = \mathrm{vol}(S) \geq \mathrm{vol}(\mathbb{RP}^p)$ with equality if and only if $\#(S \cap L) = 1$ for almost all $L \in \mathrm{PG}(n, n-p)$.

Proof. First note that $\mathcal{H}^{p-1}(\mathrm{spt}(\mathcal{S})\backslash S) = 0$ implies that $\mathrm{spt}(\mathcal{S}) \cap L = S \cap L$ for almost all $L \in \mathrm{PG}(n, n-p)$. It is elementary that if $\mathbb{RP}^p$ is imbedded in $\mathbb{RP}^n$ in the usual way that $\#(\mathbb{RP}^p \cap L) = 1$ for almost all $L \in \mathrm{PG}(n,p)$. Using this in Crofton's Formula (7.1)

$$\begin{aligned}
\gamma\, \mathcal{M}(\mathcal{S}) &= \gamma\, \mathrm{vol}(S) \int_{\mathrm{PG}(n,n-p)} \#(S \cap L)\, \mathrm{d}L \\
&\geq \int_{\mathrm{PG}(n,n-p)} 1\, \mathrm{d}L \\
&= \int_{\mathrm{PG}(n,n-p)} \#(\mathbb{RP}^p \cap L)\, \mathrm{d}L \\
&= \gamma\, \mathrm{vol}(\mathbb{RP}^p)
\end{aligned}$$

and equality holds if and only if $\#(S \cap L) = 1$ for almost all $L \in \mathrm{PG}(n, n-p)$. □

Step 4. Let N^p be an imbedded submanifold of $\mathbb{RP}^n$ of dimension p, $L_0 \in \mathrm{PG}(n, n-p)$ and $x_1, \ldots, x_\ell$ any points in $N \cap L_0$ where N and L_0 intersect transversely. For each i let U_i be a neighborhood of x_i in N. Then there is a neighborhood W of L_0 in $\mathrm{PG}(n, n-p)$ such that every element L of W intersects each U_i transversely in at least one point y_i. Thus if some element of $\mathrm{PG}(n, n-p)$ intersects N transversely in at least ℓ points, then there is an open subset of $\mathrm{PG}(n, n-p)$ whose elements all intersect N in at least ℓ points.

Proof. This is just a restatement in our context of a standard transversality result. See for example [16]. □

Step 5. The submanifold S of step 2 is contained in some $\mathbb{RP}^p$ of $\mathbb{RP}^n$.

Proof. Because $[\mathcal{S}] = [\mathbb{RP}^k]$ and $\mathcal{S}$ has least mass in its homology class equality must hold in step 3. Thus $\#(S \cap L) = 1$ for almost all $L \in P(n, n-p)$. Suppose, toward a contradiction, that S is not a subset of any $\mathbb{RP}^p$ in $\mathbb{RP}^n$. Then choose $x_1 \in S$ and let N_1 be the $\mathbb{RP}^p$ in $\mathbb{RP}^n$ that goes through x_1 and has the same tangent space at x_1 that S has. Because S is not contained in any $\mathbb{RP}^p$ there is an

$x_2 \in S \backslash N_1$. Let $\overline{x_1 x_2}$ be the geodesic through x_1 and x_2. Then, $\overline{x_1 x_2}$ is transverse to S at x_1 (for if not $\overline{x_1 x_2} \subseteq N_2$ and so $x_2 \in N_1$). It follows there is an element L_0 of $\mathrm{PG}(n, n-p)$ containing x_1 and x_2 which is transverse to S at x_1. If L_0 is also transverse to S at x_2 then step 4 implies there is an open subset of $\mathrm{PG}(n, n-p)$ which intersects S in at least two points. But open sets have positive measure and so this would contradict that equality holds in step 3. If L_0 is not transverse to S at x_2 then by step 4 we can choose a neighborhood W of L_0 in $G(n, n-p)$ such that every element of W intersects S transversely at some point near x_1. But W will contain at least one element L_1 that contains x_2 and is transverse to S at x_2. This L_1 will intersect S transversely in at least two points. As before this contradicts that equality holds in step 3. □

Step 6. Let N be the $\mathbb{RP}^p$ in $\mathbb{RP}^n$ with $S \subseteq N$ (as in step 5). Then $\mathcal{S} = N$ as currents.

Proof. By step 3, $\mathrm{vol}(S) \geq \mathrm{vol}(N) = \mathrm{vol}(\mathbb{RP}^p) = \mathrm{vol}(S)$ and thus S is dense in N. But S is also dense in $\mathrm{spt}(\mathcal{S})$. Therefore $\mathrm{spt}(\mathcal{S}) = N$. But currents over $\mathbb{Z}_2$ are determined by their supports. This completes the proof in step 6. □

This completes the Proof of Theorem 21. □

Acknowledgment

The authors wish to thank the referee for his or her encouraging comments and helpful remarks.

References

[1] W. K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[2] B. Andrews. Positively curved surfaces in the three-sphere. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 221–230. Higher Ed. Press, Beijing, 2002.

[3] M. Ara. Instability and nonexistence theorems for F-harmonic maps. *Illinois J. Math.*, 45(2):657–679, 2001.

[4] S. Brendle. A general convergence result for the Ricci flow in higher dimensions. *Duke Math. J.*, 145(3):585–601, 2008.

[5] S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[6] J. E. Brothers. Integral geometry in homogeneous spaces. *Trans. Amer. Math. Soc.*, 124:480–517, 1966.

[7] B.-Y. Chen. On the first eigenvalue of Laplacian of compact minimal submanifolds of rank one symmetric spaces. *Chinese J. Math.*, 11(4):1–15, 1983.

[8] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[9] H. Federer. The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Amer. Math. Soc.*, 76:767–771, 1970.

[10] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[11] W. H. Fleming. Flat chains over a finite coefficient group. *Trans. Amer. Math. Soc.*, 121:160–186, 1966.

[12] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[13] A. Gray. A note on manifolds whose holonomy group is a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. *Michigan Math. J.*, 16:125–128, 1969.

[14] J.-R. Gu and H.-W. Xu. The sphere theorems for manifolds with positive scalar curvature. *J. Differential Geom.*, 92(3):507–545, 2012.

[15] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[16] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[17] R. Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106(509):vi+69, 1993.

[18] R. Howard and S. W. Wei. Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space. *Trans. Amer. Math. Soc.*, 294(1):319–331, 1986.

[19] G. Huisken. Deforming hypersurfaces of the sphere by their mean curvature. *Math. Z.*, 195(2):205–219, 1987.

[20] S. Kobayashi, Y. Ohnita, and M. Takeuchi. On instability of Yang-Mills connections. *Math. Z.*, 193(2):165–189, 1986.

[21] H. B. Lawson, Jr. *Minimal varieties in real and complex geometry*. Les Presses de l'Université de Montréal, Montreal, Que., 1974. Séminaire de Mathématiques Supérieures, No. 57 (Été 1973).

[22] H. B. Lawson, Jr. and J. Simons. On stable currents and their application to global problems in real and complex geometry. *Ann. of Math. (2)*, 98:427–450, 1973.

[23] Y. Ohnita. Stability of harmonic maps and standard minimal immersions. *Tohoku Math. J. (2)*, 38(2):259–267, 1986.

[24] Y. Ohnita. Stable minimal submanifolds in compact rank one symmetric spaces. *Tohoku Math. J. (2)*, 38(2):199–217, 1986.

[25] Y. Ohnita. On stability of minimal submanifolds in compact symmetric spaces. *Compositio Math.*, 64(2):157–189, 1987.

[26] G. Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.

[27] G. Perelman. Ricci flow with surgery on three-manifolds, 2003.

[28] Y. Shen and S. W. Wei. The stability of harmonic maps on Finsler manifolds. *Houston J. Math.*, 34(4):1049–1056, 2008.

[29] Y.-B. Shen and Q. He. On stable currents and positively curved hypersurfaces. *Proc. Amer. Math. Soc.*, 129(1):237–246 (electronic), 2001.

[30] Y.-B. Shen and H.-Q. Xu. On the nonexistence of stable minimal submanifolds in positively pinched Riemannian manifolds. In *Geometry and topology of submanifolds, X (Beijing/Berlin, 1999)*, pages 274–283. World Sci. Publ., River Edge, NJ, 2000.

[31] J. Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.

[32] S. Smale. Generalized Poincaré's conjecture in dimensions greater than four. *Ann. of Math. (2)*, 74:391–406, 1961.

[33] E. H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[34] S.-s. Tai. Minimum imbeddings of compact symmetric spaces of rank one. *J. Differential Geometry*, 2:55–66, 1968.

[35] S. W. Wei. Generalized idea of Synge and its applications to topology and calculus of variations in positively curved manifolds. In *Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983)*, volume 45 of *Proc. Sympos. Pure Math.*, pages 533–539. Amer. Math. Soc., Providence, RI, 1986.

[36] S. W. Wei. An extrinsic average variational method. In *Recent developments in geometry (Los Angeles, CA, 1987)*, volume 101 of *Contemp. Math.*, pages 55–78. Amer. Math. Soc., Providence, RI, 1989.

[37] S. W. Wei and Y. Li. Generalized sharp Hardy type and Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds. *Tamkang J. Math.*, 40(4):401–413, 2009.

[38] Y. L. Xin. An application of integral currents to the vanishing theorems. *Sci. Sinica Ser. A*, 27(3):233–241, 1984.

[39] H.-W. Xu and J.-R. Gu. An optimal differentiable sphere theorem for complete manifolds. *Math. Res. Lett.*, 17(6):1111–1124, 2010.

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, U.S.A.

E-mail address: howard@math.sc.edu

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019-0315, U.S.A.

E-mail address: wwei@ou.edu