TANGENT CONES AND REGULARITY OF
REAL HYPERSURFACES

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ABSTRACT. We characterize $\mathcal{C}^1$ embedded hypersurfaces of $\mathbf{R}^n$ as the only locally closed sets with continuously varying flat tangent cones whose measure-theoretic-multiplicity is at most $m < 3/2$. It follows then that any (topological) hypersurface which has flat tangent cones and is supported everywhere by balls of uniform radius is $\mathcal{C}^1$. In the real analytic case the same conclusion holds under the weakened hypothesis that each tangent cone be a hypersurface. In particular, any convex real analytic hypersurface $X \subset \mathbf{R}^n$ is $\mathcal{C}^1$. Furthermore, if X is real algebraic, strictly convex, and unbounded, then its projective closure is a $\mathcal{C}^1$ hypersurface as well, which shows that X is the graph of a function defined over an entire hyperplane.

1. INTRODUCTION

The *tangent cone* $T_p X$ of a set $X \subset \mathbf{R}^n$ at a point $p \in X$ consists of the limits of all (secant) rays which originate from p and pass through a sequence of points $p_i \in X \smallsetminus \{p\}$ which converges to p. These objects, which are generalizations of tangent spaces to smooth submanifolds, were first used by Whitney [38, 39] to study the singularities of real analytic varieties, and also play a fundamental role in geometric measure theory [14, 24]. Here we employ tangent cones to study the differential regularity of (topological) *hypersurfaces*, i.e., subsets of $\mathbf{R}^n$ which are locally homeomorphic to $\mathbf{R}^{n-1}$. Our first result is the following characterization for $\mathcal{C}^1$ hypersurfaces, i.e, hypersurfaces which may be represented locally as the graph of continuously differentiable functions. Here a *locally closed* set is the intersection of a closed set with an open set; further, $T_p X$ is *flat* when it is a hyperplane, and the (lower measure-theoretic) *multiplicity* of $T_p X$ is then just the $(n-1)$-dimensional lower density of X at p (see Section 2.2).

Theorem 1.1. *Let $X \subset \mathbf{R}^n$ be a locally closed set. Suppose that $T_p X$ is flat for each $p \in X$, and depends continuously on p. Then X is a union of $\mathcal{C}^1$ hypersurfaces. Further, if the multiplicity of each $T_p X$ is at most $m < 3/2$, then X is a hypersurface.*

Next we develop some applications of this result by looking for natural settings where the continuity of $p \mapsto T_p X$ follows automatically as soon as $T_p X$ is flat.

This is the case for instance when X is a *convex hypersurface*, i.e., the boundary of a convex set with interior points. More generally, the flatness of T_pX imply their continuity whenever X has *positive support* (Lemma 4.1), which means that through each point of X there passes a ball B of radius r, for some uniform constant $r > 0$, such that the interior of B is disjoint from X. If there pass two such balls with disjoint interiors through each point of X, then we say X has *double positive support*. The above theorem together with an observation of Hörmander (Lemma 4.2) yield:

Theorem 1.2. *Let $X \subset \mathbf{R}^n$ be a locally closed set with flat tangent cones and positive support. Suppose that either X is a hypersurface, or the multiplicity of each T_pX is at most $m < 3/2$. Then X is a $\mathcal{C}^1$ hypersurface; furthermore, if X has double positive support, then it is $\mathcal{C}^{1,1}$.*

Here $\mathcal{C}^{1,1}$ means that X may be represented locally as the graph of a function with *Lipschitz* continuous derivative [20, p. 97]. Examples of sets with positive support include the boundary of sets of *positive reach* introduced by Federer [13, 36]: A closed set $X \subset \mathbf{R}^n$ has positive reach if there exists a neighborhood of it in $\mathbf{R}^n$ of radius $r > 0$ such that every point in that neighborhood is closest to a *unique* point in X. For instance all convex sets have positive reach. Thus the above theorem generalizes the well-known fact that convex hypersurfaces with unique support hyperplanes; or, equivalently, convex functions with unique subdifferentials, are $\mathcal{C}^1$ (Lemma 4.5); later we will offer another generalization as well (Theorem 4.4). Also note that any hypersurface with positive reach automatically has double positive support, which we may think of as having a ball "roll freely" on both sides of X. This is important for $\mathcal{C}^{1,1}$ regularity as there are hypersurfaces with flat tangent cones and positive support which are not $\mathcal{C}^{1,1}$ (Example 7.2). Next we show that there are settings where the assumption in the above results that T_pX be flat may be weakened. A set $X \subset \mathbf{R}^n$ is *real analytic* if locally it coincides with the zero set of an analytic function. Using a version of the real nullstellensatz, we will show (Corollary 5.6) that if any tangent cone T_pX of a real analytic hypersurface is itself a hypersurface, then it is symmetric with respect to p. Consequently Theorem 1.2 yields:

Theorem 1.3. *Let $X \subset \mathbf{R}^n$ be a real analytic hypersurface with positive support. If T_pX is a hypersurface for all p in X, then X is $\mathcal{C}^1$. In particular, convex real analytic hypersurfaces are $\mathcal{C}^1$.*

Note that by a hypersurface in this paper we always mean a set which is locally homeomorphic to $\mathbf{R}^{n-1}$. The last result is optimal in several respects. First, the assumption that each T_pX be a hypersurface is necessary due to the existence of cusp type singularities which arise for instance in the planar curve $y^2 = x^3$. Further, there are real analytic hypersurfaces with flat tangent cones which are supported at each point by a ball of (nonuniform) positive radius, but are not $\mathcal{C}^1$ (Example 7.3). In particular, the assumption on the existence of positive support is necessary; although the case $n = 2$ is an exception (Section 5.4). Furthermore, there are real analytic convex hypersurfaces which are not $\mathcal{C}^{1,1}$, or even $\mathcal{C}^{1,\alpha}$, and thus the $\mathcal{C}^1$ conclusion in the above theorem is sharp (Example 7.2). Finally there are convex

semi-analytic hypersurfaces which are not $\mathcal{C}^1$ (Example 7.5). Next we show that the conclusions in the above theorem may be strengthened when X is *real algebraic*, i.e., it is the zero set of a polynomial function. We say that $X \subset \mathbf{R}^n$ is an *entire graph* if there exists a line $\ell \subset \mathbf{R}^n$ such that X intersects every line parallel to ℓ at exactly one point. In other words, after a rigid motion, X coincides with the graph of a function $f \colon \mathbf{R}^{n-1} \to \mathbf{R}$. The *projective closure* of X is its closure as a subset of $\mathbf{RP}^n$ via the *standard embedding* of $\mathbf{R}^n$ into $\mathbf{RP}^n$ (see Section 6.2). Also recall that a convex hypersurface is *strictly convex* provided that it does not contain any line segments.

Theorem 1.4. *Let $X \subset \mathbf{R}^n$ be a real algebraic convex hypersurface homeomorphic to $\mathbf{R}^{n-1}$. Then X is an entire graph. Furthermore, if X is strictly convex, then its projective closure is a $\mathcal{C}^1$ hypersurface.*

Thus, a real algebraic strictly convex hypersurface is not only $\mathcal{C}^1$ in $\mathbf{R}^n$ but also at "infinity". Interestingly, this phenomenon does not in general hold when X is the zero set of an analytic function (Example 7.6). So Theorem 1.4 demonstrates that there are genuine geometric differences between the categories of real analytic and real algebraic convex hypersurfaces. Further, the above theorem does not hold when X is *semi*-algebraic (Example 7.5), and the convexity assumption here is necessary, as far as the entire-graph-property is concerned (Example 7.7). Finally, the regularity of the projective closure does not hold without the strictness of convexity (Example 7.8). See [26, 31, 35] for some examples of convex hypersurfaces arising in algebraic geometry. *Negatively curved* real algebraic hypersurfaces have also been studied in [9]. Finally, some recent results on geometry and regularity of locally convex hypersurfaces may be found in [1, 2, 3, 16].

2. Preliminaries: Basics of Tangent Cones and Their Measure Theoretic Multiplicity

2.1.   Throughout this paper $\mathbf{R}^n$ denotes the *n*-dimensional Euclidean space with origin o, standard inner product $\langle \, , \, \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. The unit ball $\mathbf{B}^n$ and sphere $\mathbf{S}^{n-1}$ consist respectively of points x in $\mathbf{R}^n$ with $\|x\| \leq 1$ and $\|x\| = 1$. For any sets X, $Y \subset \mathbf{R}^n$, $X + Y$ denotes their Minkowski sum or the collection of all $x + y$ where $x \in X$ and $y \in Y$. Further for any constant $\lambda \in \mathbf{R}$, λX is the set of all λx where $x \in X$, and we set $-X := (-1)X$. For any point $p \in \mathbf{R}$, and $r > 0$, $B^n(p, r) := p + r\mathbf{B}^n$ denotes the (closed) *ball* of radius r centered at p. By a *ray* ℓ with origin p in $\mathbf{R}^n$ we mean the set of points given by $p + tu$, where $u \in \mathbf{S}^{n-1}$ and $t \geq 0$. We call u the *direction* of ℓ, or $\mathrm{dir}(\ell)$. The set of all rays in $\mathbf{R}^n$ originating from p is denoted by $\mathrm{Rays}(p)$. This space may be topologized by defining the distance between a pair of rays ℓ, $\ell' \in \mathrm{Rays}(p)$ as

$$\mathrm{dist}(\ell, \ell') := \| \mathrm{dir}(\ell) - \mathrm{dir}(\ell') \|.$$

Now for any set $X \subset \mathbf{R}^n$, and $p \in X$, we may define the *tangent cone* $T_p X$ as the union of all rays $\ell \in \mathrm{Rays}(p)$ which meet the following criterion: there exists a sequence of points $p_i \in X \smallsetminus \{p\}$ converging to p such that the corresponding

sequence of rays $\ell_i \in \mathrm{Rays}(p)$ which pass through p_i converges to ℓ. In particular note that $T_p X = \emptyset$ if and only if p is an isolated point of X.

We should point out that our definition of tangent cones coincides with Whitney's [38, p. 211], and is consistent with the usage of the term in geometric measure theory [14, p. 233]; however, in algebraic geometry, the tangent cones are usually defined as the limits of secant *lines* as opposed to *rays* [33, 25]. We refer to the latter notion as the *symmetric* tangent cone. More explicitly, let $(T_p X)^*$ denote the reflection of $T_p X$ with respect to p, i.e., the set of all rays $\ell \in \mathrm{Rays}(p)$ with $\mathrm{dir}(\ell) = -u$ where u is the direction of a ray in $T_p X$; then the *symmetric tangent cone* of X at p is defined as:

$$\widetilde{T}_p X := T_p X \cup (T_p X)^*.$$

Still a third usage of the term tangent cone in the literature, which should not be confused with the two notions we have already mentioned, corresponds to the zero set of the leading terms of the Taylor series of analytic functions f when $X = f^{-1}(0)$, and may be called the *algebraic tangent cone*. The relation between these three notions of tangent cones, which will be studied in Section 5, is the key to proving Theorem 1.3.

Next we will record a pair of basic properties of tangent cones. For any $\ell \in \mathrm{Rays}(p)$ and $\delta > 0$, let $C(\ell, \delta)$ be a neighborhood of ℓ of radius δ in $\mathrm{Rays}(p)$, i.e., the cone given by

$$C(\ell, \delta) := \left\{ \ell' \in \mathrm{Rays}(p) \mid \| \mathrm{dir}(\ell) - \mathrm{dir}(\ell') \| \leq \delta \right\}$$

The following observation is a quick consequence of the definitions which will be used frequently throughout this work:

Lemma 2.1. *Let $X \subset \mathbf{R}^n$, $p \in X$, and $\ell \in \mathrm{Rays}(p)$. Then $\ell \subset T_p X$, if, and only if, for every δ, $r > 0$,*

$$\left(X \smallsetminus \{p\} \right) \cap C(\ell, \delta) \cap B^n(p, r) \neq \emptyset.$$

In particular, if every open neighborhood of p in X intersects $\ell \smallsetminus \{p\}$, then $\ell \subset T_p X$. Furthermore, $T_p X$ is always a closed set. $\square$

Another useful way to think of $T_p X$ is as a certain limit of homotheties of X, which we describe next. For any $\lambda > 0$, let

$$X_{p,\lambda} := \lambda(X - p) + p$$

be the *homothetic expansion* of X by the factor λ centered at p. Then we have

Lemma 2.2. *Let $X \subset \mathbf{R}^n$, $p \in X$, and $r > 0$. Then for any $\epsilon > 0$ there exists $\lambda > 0$ such that for every $\lambda' \geq \lambda$,*

(1) $$X_{p,\lambda'} \cap B^n(p, r) \subset (T_p X + \epsilon \mathbf{B}^n) \cap B^n(p, r).$$

Proof. We may assume, for convenience, that $p = o$ and $r = 1$. Set $B(T_o X, \epsilon) := T_o X + \epsilon \mathbf{B}^n$, and $X_\lambda := X_{o,\lambda}$. Let $\epsilon < 1$ be given and suppose, towards a contradiction, that there is a sequence $\lambda_i \to \infty$ such that for each λ_i, there is a point $\lambda_i x_i \in X_{\lambda_i} \cap \mathbf{B}^n$ which does not belong to $B(T_o X, \epsilon)$. Since $\mathbf{B}^n$ is compact, $\lambda_i x_i$ has a limit point $\overline{x}$ in $\mathbf{B}^n$, which will be disjoint from the interior of $B(T_o X, \epsilon)$. Thus

the ray $\overline{\ell} \in \mathrm{Rays}(o)$ which passes through $\overline{x}$ does not belong to $T_o X$. On the other hand $\overline{\ell}$ is a limit of rays ℓ_i which originate from o and pass through x_i. Further note that since $\|\lambda_i x_i\| \le 1$ and $\lambda_i \to \infty$, we have $x_i \to o$. Thus $\overline{\ell}$ must belong to $T_o X$ and we have a contradiction. $\square$

It is now natural to wonder whether λ in Lemma 2.2 may be found so that the following inclusion, also holds for all $\lambda' \ge \lambda$:

$$(2) \qquad (X_{p,\lambda'} + \epsilon \mathbf{B}^n) \cap B^n(p,r) \subset T_p X \cap B^n(p,r).$$

For if (1) and (2) were both true, then it would mean that $X_{p,\lambda'}$ converges to $T_p X$ with respect to the *Hausdorff distance* [29] inside any ball centered at p. In fact when X is real analytic, this is precisely the case [38, p. 220]. This phenomenon, however, does not in general hold: consider for instance the set $X \subset \mathbf{R}$ which consists of the origin o and points $\pm 1/2^n$, $n = 1, 2, 3, \ldots$ (then $T_o X = \mathbf{R}$, while $X_{o,2^n} \cap \mathbf{B}^1 = X$). Still, there does exist a natural notion of convergence with respect to which $T_p X$ is the limit of the homothetic expansions $X_{p,\lambda}$: A set $X \subset \mathbf{R}^n$ is said to be the *outer limit* [29] of a sequence of sets $X_i \subset \mathbf{R}^n$, and we write $\limsup_{i \to \infty} X_i = X$, provided that for every $x \in X$ there exists a subsequence X_{i_j} which eventually intersects every open neighborhood of x.

Corollary 2.3. *For any set $X \subset \mathbf{R}^n$ and point $p \in X$, the tangent cone $T_p X$ is the outer limit of the homothetic expansions of X centered at p:*

$$T_p X = \limsup_{\lambda \to \infty} X_{p,\lambda}.$$

Proof. Lemma 2.2 quickly yields that $\limsup_{\lambda \to \infty} X_{p,\lambda} \subset T_p X$, and the reverse inclusion follows from Lemma 2.1, due to the invariance of $T_p X$ under homotheties. $\square$

2.2. Now we describe how one may assign a *multiplicity* to each tangent cone of a set $X \subset \mathbf{R}^n$. Let μ be a measure on $\mathbf{R}^n$. Then the (lower) multiplicity of $T_p X$ with respect to μ will be defined as

$$m_\mu(T_p X) := \liminf_{\lambda \to \infty} \frac{\mu\Big(X_{p,\lambda} \cap B^n(p,r)\Big)}{\mu\Big(T_p X \cap B^n(p,r)\Big)},$$

for some $r > 0$. Since $T_p X$ is invariant under homotheties of $\mathbf{R}^n$ centered at p, it follows that $m_\mu(T_p X)$ does not depend on r. If the Hausdorff dimension of $T_p X$ is an integer d, then we define the multiplicity of $T_p X$ with respect to the *Hausdorff measure* $\mathcal{H}^d$ as

$$m(T_p X) := m_{\mathcal{H}^d}(T_p X).$$

One may also note that $m(T_p X)$ is simply the ratio of the d-dimensional *lower densities* $\mathcal{D}^d$ of X and $T_p X$ at p. More explicitly, let us recall that for any set $X \subset \mathbf{R}^n$ at a point p

$$\mathcal{D}^d(X, p) := \liminf_{r \to 0} \frac{\mathcal{H}^d\big(X \cap B^n(p,r)\big)}{\mathcal{H}^d\big(r\mathbf{B}^d\big)}.$$

Now it is not difficult to check that

$$m(T_pX) = \frac{\mathcal{D}^d(X, p)}{\mathcal{D}^d(T_pX, p)}.$$

Note in particular that when T_pX is an affine d-dimensional space, then $\mathcal{D}^d(T_pX, p) = 1$ and therefore $m(T_pX) = \mathcal{D}^d(X, p)$. Figure 1 illustrates some examples of multi-
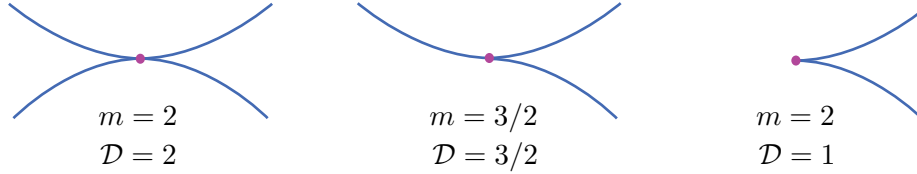

$$m = 2 \qquad\qquad m = 3/2 \qquad\qquad m = 2$$
$$\mathcal{D} = 2 \qquad\qquad \mathcal{D} = 3/2 \qquad\qquad \mathcal{D} = 1$$

Figure 1:

plicity and density of sets $X \subset \mathbf{R}^2$ at the indicated points.

3. Sets with Continuously Varying Flat Tangent Cones: Proof of Theorem 1.1

The main idea for proving Theorem 1.1 is to show that X can be represented locally as a multivalued graph with regular sheets. To this end, we first need to record a pair of basic facts concerning graphs with flat tangent cones. The first observation is an elementary characterization of the graphs of $\mathcal{C}^1$ functions in terms of the continuity and flatness of their tangent cones:

Lemma 3.1. *Let $U \subset \mathbf{R}^{n-1}$ be an open set, $f : U \to \mathbf{R}$ be a function, and set $\overline{a} := (a, f(a))$ for all $a \in U$. Then f is $\mathcal{C}^1$ on U, if $T_{\overline{a}}\operatorname{graph}(f)$ is a hyperplane which depends continuously on a and is never orthogonal to $\mathbf{R}^{n-1}$.*

Proof. First suppose that $n = 2$, and for any $a \in U$ let $m(a)$ be the slope of $T_{\overline{a}}\operatorname{graph}(f)$. Then for any sequence of points $x_i \in U$ converging to a, the slope of the (secant) lines through $\overline{a}$ and $\overline{x}_i$ must converge to $m(a)$. So $f'(a) = m(a)$, which shows that f is differentiable at a. Further, since $T_{\overline{a}}\operatorname{graph}(f)$ depends continuously on a, m is continuous. So f is $\mathcal{C}^1$. Now for the general case where $U \subset \mathbf{R}^{n-1}$, let e_i, $i = 1, \ldots n-1$, be the standard basis for $\mathbf{R}^{n-1}$. Then applying the same argument to the functions $f_i(t) := f(a + (t-a)e_i)$, shows that $\partial f/\partial x_i(a) = f_i'(a) = m_i(a)$ where $m_i(a)$ is the slope of the line in $T_{\overline{a}}\operatorname{graph}(f)$ which projects onto $a + te_i$. Thus all partial derivatives $\partial f/\partial x_i$ exist and are continuous on U; therefore, it follows that f is $\mathcal{C}^1$. $\square$

Our next observation yields a simple criterion for checking that the tangent cone of a graph is flat:

Lemma 3.2. *Let $U \subset \mathbf{R}^{n-1}$ be an open set, and $f : U \to \mathbf{R}$ be a continuous function. Suppose that, for some $a \in U$, $T_{\overline{a}}\operatorname{graph}(f)$ lies in a hyperplane H which is not orthogonal to $\mathbf{R}^{n-1}$. Then $T_{\overline{a}}\operatorname{graph}(f) = H$.*

Proof. Suppose, towards a contradiction, that there exists a point $x \in H$, which does not belong to $T_{\overline{a}} \operatorname{graph}(f)$, and let ℓ be the ray which originates from $\overline{a}$ and passes through x (ℓ is well-defined since x must necessarily be different from $\overline{a}$). Then, since $T_{\overline{a}} \operatorname{graph}(f)$ is invariant under homothetic expansions of $\mathbf{R}^n$ centered at $\overline{a}$, the open ray $\ell \smallsetminus \{\overline{a}\}$ must be disjoint from $T_{\overline{a}} \operatorname{graph}(f)$. So, by Lemma 2.1, there exists δ, $r > 0$ such that $C(\ell, \delta) \cap B^n(\overline{a}, r)$ is disjoint from $\operatorname{graph}(f) \smallsetminus \{\overline{a}\}$. Now let $\pi \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ be projection into the first $n-1$ coordinates. Then $\pi(\ell)$ is a ray in $\mathbf{R}^{n-1}$, since H is not orthogonal to $\mathbf{R}^{n-1}$. Let $x_i \in \pi(\ell)$ be a sequence of points converging to $\pi(\overline{a})$, then $\overline{x}_i := (x_i, f(x_i))$ converges to $\overline{a}$ by continuity of f. Consequently any ray ℓ' which is a limit of rays ℓ_i which pass through $\overline{x}_i$ and $\overline{a}$ must belong to $T_{\overline{a}} \operatorname{graph}(f)$. Then $\ell' \neq \ell$ and $\ell' \subset H$. So $\pi(\ell') \neq \pi(\ell)$, since H is not orthogonal to $\mathbf{R}^{n-1}$. But, since $\pi(\overline{x}_i) = x_i$, we must have $\pi(\ell_i) = \pi(\ell)$, which yields $\pi(\ell') = \pi(\ell)$, and we have a contradiction. $\qquad\square$

Next recall that a set $X \subset \mathbf{R}^n$ is *locally closed* if it is the intersection of an open set with a closed set. An equivalent formulation is that for every point p of X there exists $r > 0$ such that $V := B^n(p, r) \cap X$ is closed in X. Note that then the boundary ∂V of V as a subset of X lies on $\partial B^n(p, r)$ and therefore is disjoint from p.

Lemma 3.3. *Let $X \subset \mathbf{R}^n$ be a locally closed set. Suppose that for all $p \in X$, $T_p X$ is flat and depends continuously on p. Let H be a hyperplane which is not orthogonal to $T_p X$, for some $p \in X$, and $\pi \colon \mathbf{R}^n \to H$ be the orthogonal projection. Then there exists an open neighborhood U of p in X such that $\pi|_U$ is an open mapping.*

Proof. By continuity of $p \mapsto T_p X$, we may choose an open neighborhood U of p so small that no tangent cone of U is orthogonal to H. We claim that π is then an open mapping on U. To see this, let $q \in U$, and set

$$V := B^n(q, r) \cap X.$$

Choosing $r > 0$ sufficiently small, we may assume that $V \subset U$. Further, since X is locally closed, we may assume that V is compact, and it also has the following two properties: first, the boundary ∂V of V in X lies on $\partial B^n(q, r)$, which implies that

$$q \notin \partial V;$$

second, by Lemma 2.1, V intersects the line $L_q := \pi^{-1}(\pi(q))$ which passes through q and is orthogonal to H only at q, for otherwise L_q would have to belong to $T_q X$, which would imply that $T_q X$ is orthogonal to H. So we conclude that

$$\pi(q) \notin \pi(\partial V).$$

But recall that V is compact, and so ∂V is compact as well. Thus there exists, for some $s > 0$, a closed ball $B(\pi(q), s) \subset H$ such that

$$(3) \qquad\qquad B\big(\pi(q), s\big) \cap \pi(\partial V) = \emptyset.$$

Now suppose towards a contradiction that $\pi(q)$ is not an interior point of $\pi(V)$. Then $B(\pi(q), \delta) \not\subset \pi(V)$, for any $\delta > 0$. In particular there is a point

$$x \in B\big(\pi(q), s/2\big) \smallsetminus \pi(V).$$

Since $x \notin \pi(V)$, and $\pi(V)$ is compact, there exists a ball $B(x,t) \subset H$ which intersects $\pi(V)$ while the interior of $B(x,t)$ is disjoint from $\pi(V)$. Note that, since $q \in V$, we have $t \le \|x - \pi(q)\| \le s/2$, and so for any $y \in B(x,t)$,

$$\|y - \pi(q)\| \le \|y - x\| + \|x - \pi(q)\| \le t + s/2 \le s,$$

which yields

$$B(x,t) \subset B(\pi(q), s).$$

Thus $B(x,t) \cap \pi(\partial V) = \emptyset$ by (3). So the cylinder $C := \pi^{-1}(B(x,t))$ supports V at some point y in the interior of V. Then the tangent cone $T_y X = T_y V$ must be tangent to C, and hence be orthogonal to H, which is a contradiction, since $V \subset U$, and U was assumed to have no tangent cones orthogonal to H. $\square$

Now we are ready to prove the main result of this section:

Proof of Theorem 1.1. We will divide the argument into two parts. First, we show that X is a union of $\mathcal{C}^1$ hypersurfaces, and then we show that it is a hypersurface with the assumption on multiplicity.

Part I. Assume that $o \in X$ and $T_o X$ coincides with the hyperplane $x_n = 0$, which we refer to as $\mathbf{R}^{n-1}$. Let $\pi \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection into the first $n-1$ coordinates. Then, since the x_n-axis does not belong to $T_o X$, it follows from Lemma 2.1 that there exists an open neighborhood U of o in X which intersects the x_n-axis only at o. Further, by Lemma 3.3, we may choose U so small that $\pi(U)$ is open. So $\Omega := B^{n-1}(o, r) \subset \pi(U)$ for some $r > 0$. Now let $U' := \pi^{-1}(\Omega) \cap U$. Since X is locally closed, we may assume that U' is compact. Further, since $\pi(U') = \Omega$, we may define a function $f \colon \Omega \to \mathbf{R}$ by letting $f(x)$ be the supremum of the height (i.e., the nth-coordinate) of $\pi^{-1}(x) \cap U'$ for any $x \in \Omega$. Then $f(o) = 0$, so $o \in \operatorname{graph}(f)$. Further we claim that f is continuous. Then Lemma 3.2 yields that the tangent cones of $\operatorname{graph}(f)$ are hyperplanes. Of course the tangent hyperplanes of $\operatorname{graph}(f)$ must also vary continuously since $\operatorname{graph}(f) \subset X$. So f must be $\mathcal{C}^1$ by Lemma 3.1. Consequently, the graph of f is a $\mathcal{C}^1$ embedded disk in X containing o. This shows that each point p of X lies in a $\mathcal{C}^1$ hypersurface, since after a rigid motion we can always assume that $p = o$ and $T_p X = \mathbf{R}^{n-1}$.

So to complete the first half of the proof, i.e., to show that X is composed of $\mathcal{C}^1$ hypersurfaces, it remains only to verify that the function f defined above is continuous. To see this, first note that for every $x \in \pi(U')$, the set $\pi^{-1}(x) \cap U'$ is discreet, since none of the the tangent hyperplanes of U contain the line $\pi^{-1}(x)$, and thus we may apply Lemma 2.1. In particular, since U' is bounded, it follows that $\pi^{-1}(x) \cap U'$ is finite. So we may let $\overline{x}$ be the "highest" point of $\pi^{-1}(x) \cap U'$, i.e., the point in $\pi^{-1}(x)$ with the largest n^{th} coordinate. Now to establish the continuity of f, we just have to check that if $x_i \in \Omega$ form a sequence of points converging to $x \in \Omega$, then the corresponding points $\overline{x}_i$ converge to $\overline{x}$. To see this, first recall that $\overline{x}_i$ must have a limit point $\overline{x}' \in U'$, since U' is compact. So $\overline{x}'$ cannot lie above $\overline{x}$, by definition of $\overline{x}$. Suppose, towards a contradiction, that $\overline{x}'$ lies strictly below $\overline{x}$. By Lemma 3.3 there are then open neighborhoods V and V' of $\overline{x}$ and $\overline{x}'$ in U respectively such that $\pi(V)$ and $\pi(V')$ are open in $\mathbf{R}^{n-1}$. Further, we may suppose

that V' lies strictly below V. Note that the sequence $\bar{x}_i$ must eventually lie in V', and x_i must eventually lie in $\pi(V) \cap \pi(V')$. Thus, eventually $\pi^{-1}(x_i) \cap V$ will be higher than $\pi^{-1}(x_i) \cap V' = \bar{x}_i$, which is a contradiction.

Part II. Now we show that if the multiplicity of each tangent cone T_pX is at most $m < 3/2$, then X is a hypersurface, which will complete the proof. To this end let f and Ω be as in Part I, and define $g \colon \Omega \to \mathbf{R}$ by letting $g(x)$ be the infimum of the height of $\pi^{-1}(x) \cap U'$ for any $x \in \Omega$. Then it follows that g is also continuous by essentially repeating the argument in Part I given for f. Now it suffices to show that $f \equiv g$ on an open neighborhood of o. To see this first recall that we chose U' so that it intersects the x_n-axis, or $\pi^{-1}(o)$ only at o. Thus $f(o) = g(o)$. Now suppose, towards a contradiction, that there exists a sequence of points $x_i \in \Omega \setminus \{o\}$ converging to o such that $f(x_i) \neq g(x_i)$, and let $W_i := B^{n-1}(x_i, r_i)$ be the balls of largest radii centered at x_i such that $f \neq g$ everywhere on the interior of W_i, i.e., set r_i equal to the distance of x_i from the set of points where $f = g$. Then each W_i will have a boundary point o_i such that $f(o_i) = g(o_i)$. Note that since o is not in the interior of W_i, $r_i \leq \|o - x_i\|$, so by the triangle inequality:

$$\|o - o_i\| \leq \|o - x_i\| + \|x_i - o_i\| = \|o - x_i\| + r_i \leq 2\|o - x_i\|.$$

Thus, since x_i converges to o, it follows that o_i converges to o. Now if we set $\bar{o}_i := (o_i, f(o_i))$, then it follows that $T_{\bar{o}_i}X$ converges to $T_oX = \mathbf{R}^{n-1}$, by the continuity of f and $p \mapsto T_pX$. Further note that, if θ_i is the angle between $T_{o_i}X$ and $\mathbf{R}^{n-1}$, then, for any $A \subset \mathbf{R}^{n-1}$,

$$\mathcal{H}^{n-1}\Big(T_{\bar{o}_i}X \cap \pi^{-1}(A)\Big) = \frac{1}{\cos(\theta_i)}\mathcal{H}^{n-1}(A).$$

In particular, since $\theta_i \to 0$, we may choose for any $\epsilon > 0$, an index k large enough so that

$$\mathcal{H}^{n-1}\Big(T_{\bar{o}_k}X \cap \pi^{-1}\big(B^{n-1}(o_k, 1)\big)\Big) < (1 + \epsilon)\,\mathcal{H}^{n-1}\Big(B^{n-1}(o_k, 1)\Big).$$

In addition, since $T_{\bar{o}_k}X$ converges to $\mathbf{R}^{n-1}$, for any $\delta > 0$ we may assume that k is so large that

$$T_{\bar{o}_k}X \cap \pi^{-1}\Big(B^{n-1}(o_k, 1)\Big) \subset B^n(o_k, 1 + \delta).$$

See Figure 2.

Now fix k and let us assume, after a translation, that $o := \bar{o}_k$. Then $B^{n-1}(o_k, 1) = \mathbf{B}^{n-1}$ and $B^n(o_k, 1 + \delta) = (1 + \delta)\mathbf{B}^n$. So we may rewrite the last two displayed expressions as:

$$(4) \qquad \mathcal{H}^{n-1}\Big(T_oX \cap \pi^{-1}\big(\mathbf{B}^{n-1}\big)\Big) < (1 + \epsilon)\,\mathcal{H}^{n-1}(\mathbf{B}^{n-1}),$$

$$(5) \qquad T_oX \cap \pi^{-1}(\mathbf{B}^{n-1}) \subset (1 + \delta)\mathbf{B}^n.$$

Further, if we set $W := W_k$, then $o \in \partial W$, and $f \neq g$ everywhere in the interior of W; therefore, the projection $\pi \colon X \to \mathbf{R}^{n-1}$ is at least 2 to 1 over the interior of W.

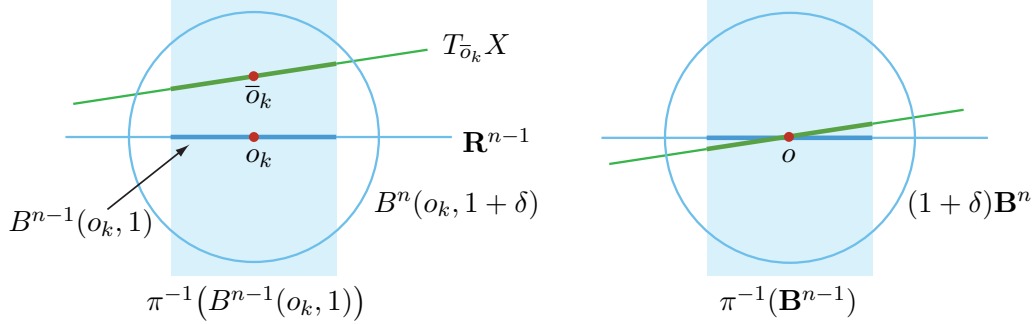Figure 2:

So the projections $\pi\colon X_{o,\lambda} \to W_{o,\lambda}$ are at least 2 to 1 as well for all $\lambda \geq 1$. But note that $W_{o,\lambda}$ eventually fills half of $\mathbf{B}^{n-1}$ (since $o \in \partial W$). Consequently,

$$(6) \qquad \liminf_{\lambda \to \infty} \mathcal{H}^{n-1}\Big(X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1})\Big) \geq \frac{3}{2}\mathcal{H}^{n-1}(\mathbf{B}^{n-1}),$$

due to the basic fact that orthogonal projections, since they are uniformly Lipschitz, do not increase the Hausdorff measure of a set, e.g., see [12, Lemma 6.1]. Also, here we have used the fact that, by Lemma 3.3, $\pi(X)$ contains an open neighborhood of o and thus $\pi(X_{o,\lambda}) = (\pi(X))_{o,\lambda}$ eventually covers $\mathbf{B}^{n-1}$ as λ grows large. Combining (6) with (4), now yields that

$$
\begin{aligned}
(7) \quad \liminf_{\lambda \to \infty} \mathcal{H}^{n-1}\Big(X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1})\Big) \;&\geq\; \frac{3}{2(1+\epsilon)}\mathcal{H}^{n-1}\Big(T_oX \cap \pi^{-1}(\mathbf{B}^{n-1})\Big) \\
&\geq\; \frac{3}{2(1+\epsilon)}\mathcal{H}^{n-1}\Big(T_oX \cap \mathbf{B}^n\Big),
\end{aligned}
$$

since $\mathbf{B}^n \subset \pi^{-1}(\mathbf{B}^{n-1})$. Next note that by (5)

$$T_oX \cap \pi^{-1}(\mathbf{B}^{n-1}) + \delta\mathbf{B}^n \subset (1+2\delta)\mathbf{B}^n.$$

Further, by Lemma 2.2, for sufficiently large λ,

$$X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1}) \subset (T_oX + \delta\mathbf{B}^n) \cap \pi^{-1}(\mathbf{B}^{n-1}) \subset T_oX \cap \pi^{-1}(\mathbf{B}^{n-1}) + \delta\mathbf{B}^n.$$

Thus it follows that, for sufficiently large λ, $X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1})$ lies in $(1+2\delta)\mathbf{B}^n$, and so we may write

$$X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1}) \subset X_{o,\lambda} \cap (1+2\delta)\mathbf{B}^n.$$

This in turn yields that

$$\liminf_{\lambda \to \infty} \mathcal{H}^{n-1}\Big(X_{o,\lambda} \cap (1+2\delta)\mathbf{B}^n\Big) \geq \liminf_{\lambda \to \infty} \mathcal{H}^{n-1}\Big(X_{o,\lambda} \cap \pi^{-1}(\mathbf{B}^{n-1})\Big).$$

The last inequality together with (7) now shows

$$\liminf_{\lambda \to \infty} \mathcal{H}^{n-1}\Big(X_{o,\lambda} \cap \mathbf{B}^n\Big) \geq \frac{3}{2(1+\epsilon)(1+2\delta)^{n-1}}\mathcal{H}^{n-1}\Big(T_oX \cap \mathbf{B}^n\Big).$$

So, recalling that ϵ and δ may be chosen as small as desired, we conclude that the multiplicity of $T_{\bar{o}_k}X$ becomes arbitrarily close to $3/2$ as k grows large. In particular it eventually exceeds any given constant $m < 3/2$, which is a contradiction, as we had desired. $\qquad\square$

4. Regularity of Sets with Positive Support and Reach: Proof of Theorem 1.2 and Another Result

4.1. The proof of the first half of Theorem 1.2, i.e., the $\mathcal{C}^1$ regularity of X, quickly follows from Theorem 1.1 via the next observation:

Lemma 4.1. *Let $X \subset \mathbf{R}^n$ be a set with flat tangent cones and positive support. Then the tangent cones of X vary continuously.*

Proof. First note that the support ball B_p of X of radius r at p must also support T_pX by Lemma 2.1. Thus, since T_pX is a hyperplane, B_p must be the unique support ball of radius r for X at p, up to a reflection through T_pX. Next note that if p_i are a sequence of points of X which converge to a point p of X, then B_{p_i} must converge to a support ball of X at p of radius r. Thus the limit of B_{p_i} must coincide with B_p or its reflection. Since $T_{p_i}X$ are tangent to B_{p_i}, it then follows that $T_{p_i}X$ must converge to the tangent hyperplane of B_p at p, which is just T_pX. Thus T_pX varies continuously. $\qquad\square$

4.2. To prove the second half of Theorem 1.2 which is concerned with the $\mathcal{C}^{1,1}$ regularity of X, we need the following lemma. A ball *slides freely* in a convex set $K \subset \mathbf{R}^n$, if for some uniform constant $r > 0$, there passes through each point of ∂K a ball of radius r which is contained in K. The proof of the following observation may be found in [20, p. 97]:

Lemma 4.2 (Hörmander)**.** *Let $K \subset \mathbf{R}^n$ be a convex set. If a ball slides freely inside K, then ∂K is a $\mathcal{C}^{1,1}$ hypersurface.* $\qquad\square$

Now the $\mathcal{C}^{1,1}$ regularity of X in Theorem 1.2 follows by means of a Möbius transformation which locally sends hypersurfaces with double positive support to convex hypersurfaces in which a ball slides freely:

Lemma 4.3. *Let $X \subset \mathbf{R}^n$ be a hypersurface with flat tangent cones and double positive support by balls of uniform radius r. Let B be any of these balls, p be the point of contact between B and X, and $i(X)$ be the inversion of X with respect to ∂B. Then an open neighborhood of p in $i(X)$ lies on the boundary of a convex set in which a ball slides freely.*

Proof. By Lemma 4.1, tangent hyperplanes of X vary continuously. So by Theorem 1.1, X is $\mathcal{C}^1$. Consequently, after replacing r with a smaller value, and X with a smaller neighborhood of p, we may assume that the interiors of the balls which lie on one side of X never intersect the interiors of the balls which lie on the other side (e.g., this follows from the tubular neighborhood theorem [34]). Furthermore, since X is $\mathcal{C}^1$, there exists a continuous unit normal vector field $n\colon X \to \mathbf{S}^{n-1}$. After a translation, rescaling, and possibly replacing n by $-n$, we may assume that $r = 1$,

$B = \mathbf{B}^n$, and $c(p) := p + n(p) = o$. Now for each $x \in X$ let B_x be the support ball of X at x with center at $c(x)$ and B'_x be the other support ball of X through x. Since c is continuous, and o is in the interior of B_p, there is an open neighborhood U of p in X so that o is in the interior of B_x for all $x \in U$; see the diagram on the left hand side of Figure 3.
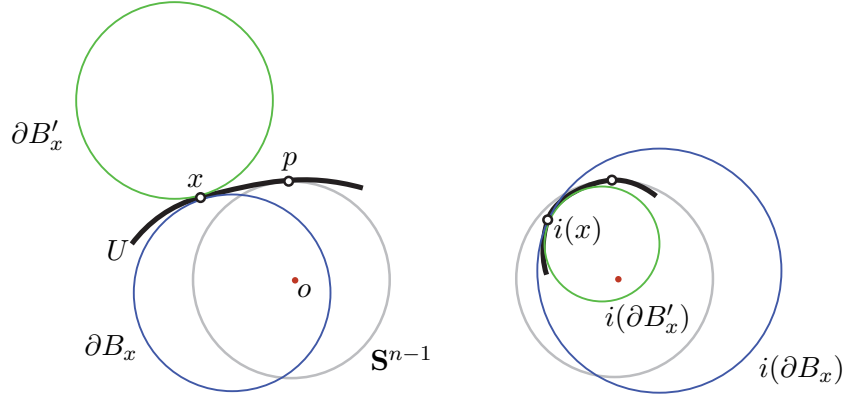

Figure 3:

Now consider $i(U)$ where $i \colon \mathbf{R}^n \smallsetminus \{o\} \to \mathbf{R}^n$ is inversion through $\partial B = \mathbf{S}^{n-1}$, i.e., $i(x) := x/\|x\|$. Note that if S is any topological sphere which contains o inside it, then i maps the outside of S to the inside of $i(S)$ (the "inside" of S is the closure of the bounded component of $\mathbf{R}^n \smallsetminus S$, and the "outside" of S is the closure of the other component). Thus, since U lies outside ∂B_x, and o is inside ∂B_x, it follows that $i(U)$ lies inside $i(\partial B_x)$; see the diagram on the right hand side of Figure 3. So through each point $i(x)$ of $i(U)$ there passes a ball bounded by $i(\partial B_x)$ which contains $i(U)$. Let K be the intersection of all these balls. Then K is a convex set and $i(U) \subset \partial K$.

Next recall that, by assumption, B'_x lies outside ∂B_y for all $x, y \in X$. Thus $i(B'_x)$ lies inside $i(\partial B_y)$, which in turn yields that $i(B'_x) \subset K$. Then since the radii of $i(B'_x)$ depend continuously on x, we may assume, after replacing U by a possibly smaller open neighborhood of p, that the radii of $i(B'_x)$ are all greater than a uniform constant $r' > 0$. Let K' be the union of all balls of radius r' in K. Then it is easy to check that K' is a convex set with $i(U) \subset \partial K'$, and a ball (of radius r') slides freely inside K'. So, by Lemma 4.2, partial K', and in particular $i(U)$ must be $\mathcal{C}^{1,1}$, which in turn yields that U is $\mathcal{C}^{1,1}$, since i is a local diffeomorphism, and finishes the proof. $\square$

4.3.   We close this section with one more regularity result for sets of positive reach which does not a priori assume that the boundary of the set is a hypersurface:

Theorem 4.4. *Let $K \subset \mathbf{R}^n$ be a set of positive reach. Suppose that the tangent cone of K at each boundary point $p \in \partial K$ is a half-space. Then ∂K is $\mathcal{C}^1$.*

This result may be regarded as another generalization of the fact that the boundary of a convex set with unique support hyperplanes is a $\mathcal{C}^1$ hypersurface. Indeed we use the inversion trick used in Lemma 4.3 to reduce the proof to the convex case:

Lemma 4.5. *Let $K \subset \mathbf{R}^n$ be a convex set with interior points, $p \in \partial K$, and suppose that there exists an open neighborhood U of p in ∂K such that through each point q of U there passes a unique support hyperplane of K. Then U is $\mathcal{C}^1$.*

The above fact follows from Theorem 1.2, since for any convex set $K \subset \mathbf{R}^n$ with interior points, ∂K is a hypersurface (e.g. see [7, p. 3]), and has positive support; further, by Lemma 5.7 below, $T_p(\partial K) = \partial(T_p K)$ which shows that $T_p(\partial K)$ is flat. We also include here a more analytic proof using basic properties of convex functions:

Proof. After a translation we may assume that $p = o$ and the support hyperplane of K at p coincides with the hyperplane of the first $(n-1)$ coordinates. Then, assuming it is sufficiently small, U coincides with the graph of a convex function $f \colon \Omega \to \mathbf{R}$ for some convex neighborhood Ω of o in $\mathbf{R}^{n-1}$. To show that f is $\mathcal{C}^1$, it suffices to check that it is differentiable at each point of Ω [32, Thm 1.5.2]. Further, f is differentiable at a point if and only if it has only one subgradient at that point [32, Thm 1.5.12]. But the subgradient is unique at a point of Ω if and only if the epigraph of f has a unique normal at the corresponding point [32, Thm 1.5.11], which is the case due to uniqueness of the support hyperplanes. $\square$

Now we are ready to prove the last result of this section:

Proof of Theorem 4.4. First recall that through each point $p \in \partial K$ there passes a support ball B_p of radius r for some uniform constant $r > 0$. Since $T_p K$ is a half-space, it follows that B_p is the unique support ball of K of radius r passing through p, since the interior of B_p must be disjoint from $T_p K$ by Lemma 2.1. Further note that if $p_i \in \partial K$ are any sequence of points converging to p, then any limit point of the sequence of balls B_{p_i} will be a support ball of K at p of radius r, which must be B_p due to the uniqueness of B_p; therefore, B_{p_i} converges to the support ball at p. So $p \mapsto B_p$ is continuous.

Now fix a point $p \in \partial K$. After a translation and rescaling we may assume that $B_p = \mathbf{B}^n$. In particular, the center of B_p is o. By continuity of the support balls there exists an open ball V centered at p in $\mathbf{R}^n$ such if we set

$$U := V \cap \partial K,$$

then B_x contains o for every $x \in U$. Next consider the inversion i with respect to $\partial B_p = \mathbf{S}^{n-1}$. We claim that if

$$K' := K \cap \overline{V},$$

where $\overline{V}$ denotes the closure of V, then $i(K')$ is convex. Indeed we show that through each point of $\partial(i(K')) = i(\partial K')$ there passes a sphere which contains $i(K')$. To see this note that if $x \in i(\partial K')$, then either $x \in i(\partial V)$ or $x \in i(V)$. If $x \in i(\partial V)$, then note that $i(\partial V)$ bounds the ball $i(\overline{V})$ which contains $i(K')$, since $\overline{V}$ contains K'. If, on the other hand, $x \in i(V)$, note that

$$i(\partial K') \cap i(V) = i(\partial K' \cap V) = i(\partial K \cap V) = i(U),$$

and recall that, as we already showed in the proof of Lemma 4.3, through each point $i(x)$ of $i(U)$ there passes a sphere $i(\partial B_x)$ which contains $i(K)$ and therefore $i(K')$ inside it. So we conclude that $i(K')$ is convex. Next note that for all $x \in \partial K'$

$$(8) \qquad\qquad T_{i(x)}i(K') = di_x(T_x K'),$$

where di_x is the differential map or the jacobian of i at x, which has full rank for all $x \in \mathbf{R}^n \smallsetminus \{o\}$ which includes U. Furthermore, if $x \in U$, then $T_x K' = T_x K$ which is a half space, and so $di_x(T_x K')$ is a half-space as well, since i_x has full rank. Thus (8) shows that $T_{i(x)}i(K')$ is a half-space for all $x \in U$. So $i(K')$ has a unique support hyperplane at all points of $i(U)$. But note that $i(U)$ is open in $i(\partial K')$; because, $i: \partial K' \to \partial K$ is a homeomorphism (and therefore must preserve open sets by the theorem on the invariance of domain). Now it follows from Lemma 4.5 that $i(U)$ is $\mathcal{C}^1$, which in turn yields that so is U, since i is a local diffeomorphism. $\qquad\square$

5. Tangent Cones of Real Analytic Hypersurfaces: Proof of Theorem 1.3

Theorem 1.3 follows quickly from Theorem 5.5 proved in Section 5.2 below which relates the three notions of tangent cones we discussed earlier in Section 2. To prove Theorem 5.5, we first need to study in Section 5.1 the sign of real analytic functions in the neighborhood of their zero sets.

5.1. We say that a function $f: \mathbf{R}^n \to \mathbf{R}$ changes sign at a point $p \in \mathbf{R}^n$ provided that in every open neighborhood of p in $\mathbf{R}^n$ there are points where $f > 0$ and $f < 0$. If f changes sign at every point of a set $X \subset \mathbf{R}^n$, then we say that f changes sign on X. The following observation shows that in certain cases we may assume that the function defining an analytic set changes sign on that set:

Proposition 5.1. *Let $X \subset \mathbf{R}^n$ be a hypersurface which coincides with the zero set $Z(f) := f^{-1}(0)$ of an analytic function $f: \mathbf{R}^n \to \mathbf{R}$. Then for every point $p \in X$ there exists an open neighborhood U of p in $\mathbf{R}^n$, and an analytic function $g: U \to \mathbf{R}$ such that $Z(g) = X \cap U$, and g changes sign on $X \cap U$.*

To prove this result, let us suppose that $p = o$, and let $\mathcal{C}_o^\omega$ denote the ring of germs of analytic functions at o. It is well-known that $\mathcal{C}_o^\omega$ is Noetherian and is a unique factorization domain, e.g., see [41, p. 148]. So f is the product of finitely may irreducible factors f_i in $\mathcal{C}_o^\omega$ (abusing notation, we denote here the functions and their germs by the same symbols). Further, note that $Z(f)$ is the union of $Z(f_i)$, and $\dim(Z(f)) = n - 1$ by assumption. Thus, by Łojasiewicz's structure theorem for real analytic varieties [22, Thm. 6.3.3 (c)], we may assume that there exists a factor g of f such that

$$(9) \qquad\qquad \dim\big(Z(g)\big) = n - 1.$$

Indeed, each $Z(f_i)$ is an analytic set which admits a stratification into disjoint union of finitely many smooth manifolds or *strata* by Łojasiewicz's theorem. The maximum dimension of these strata then defines $\dim(Z(f_i))$. So, since $Z(f_i) \subset Z(f)$, $\dim(Z(f_i)) \le n - 1$. Suppose towards a contradiction that $\dim(Z(f_i)) < n - 1$ for

all i. Then since each strata of $Z(f_i)$ is smooth, $\mathcal{H}^{n-1}(Z(f_i)) = 0$. This in turn yields that $\mathcal{H}^{n-1}(Z(f)) = 0$ which is not possible.

Now (9) implies that g satisfies the equation (10) below which is a type of real nullstellensatz [5, 27, 30]. Here $(g) \subset \mathcal{C}_o^\omega$ is the ideal generated by g, i.e., the collection of all germs ϕg where $\phi \in \mathcal{C}_o^\omega$. Further $I(Z(g)) \subset \mathcal{C}_o^\omega$ is the ideal of germs in $\mathcal{C}_o^\omega$ which vanish on $Z(g)$.

Lemma 5.2. *Let $g \in \mathcal{C}_o^\omega$ be irreducible, and suppose that $\dim(Z(g)) = n - 1$. Then*

$$(10) \qquad (g) = I\big(Z(g)\big).$$

Proof. See the proof of Theorem 4.5.1 in [5, p. 95]. Although that theorem is stated in the ring of polynomials, as opposed to our present setting of C_o^ω, the proof of the implication $(v) \Rightarrow (ii)$ in that result depends only on general algebraic properties of ideals in Noetherian rings and thus applies almost verbatim to the present setting. In particular, irreducibility of (g) implies that (g) is a prime ideal of height one. On the other hand $(g) \subset I(Z(g))$, and (9) implies that $I(Z(g))$ also has height one in C_o^ω. Thus it follows that $(g) = I(Z(g))$. $\qquad\square$

Now it follows that the gradient of g cannot vanish identically on $Z(g)$; because otherwise, $\partial g/\partial x_i \in I(Z(g)) = (g)$ which yields that

$$\frac{\partial g}{\partial x_i} = \phi_i g$$

for some $\phi_i \in \mathcal{C}_o^\omega$. Consequently, by the product rule, all partial derivatives of g of any order must vanish at o. But then, since g is analytic, it must vanish identically on U which is not possible by (9). So we may assume that g has a regular point in $Z(g) \subset X \cap U$. Then g must assume different signs on U, and therefore $U \smallsetminus Z(g)$ must be disconnected. This, we claim, implies that $Z(g) = X \cap U$ which would complete the proof. To establish this claim, first note that since g is a factor of f, we already have

$$Z(g) \subset X \cap U.$$

Further, by the generalized Jordan-Brouwer separation theorem for closed sets in $\mathbf{R}^n$ which are locally homeomorphic to $\mathbf{R}^{n-1}$, or more specifically Alexander duality, we may choose U so that $U \smallsetminus X$ has exactly two components, e.g., see [11, 8.15]. Using some elementary topology, one can then show that if $Z(g)$ is a proper subset of $X \cap U$, then it cannot separate U which is a contradiction. It would then follow that

$$Z(g) = X \cap U$$

which would complete the proof of Proposition 5.1. So all that remains is to check that if X separates U into a pair of components, and A is any proper subset of $X \cap U$, then A cannot separate U. To see this let U_1, U_2 be the two components of $U \smallsetminus X$, and set $V := U_1 \cup U_2 \cup A'$, where $A' := (X \cap U) \smallsetminus A$. We show that V is connected which is all we need. To see this let $W \subset V$ be a nonempty set which is both open and closed in V. Then $W \not\subset A'$, for otherwise there would be an open set $O \subset \mathbf{R}^n$ with $O \cap V = W \subset A'$. This would imply that O is disjoint from both U_1 and U_2, which is impossible since $U_1 \cup U_2 = U \smallsetminus X$ is dense in U. We

may assume then that W meets U_1. Then, since U_1 is connected, $U_1 \subset W$. Further, since W is closed, it will contain the closure of U_1 in V which is $U_1 \cup A'$. But $U_1 \cup A'$ is not open in V (for any neighborhood of a point of A' meets both U_1 and U_2). Thus $W \neq U_1 \cup A'$, and therefore it must contain a point of U_2. Then, since U_2 is connected, $U_2 \subset W$, which implies that $W = U_1 \cup U_2 \cup A = V$. So V is connected.

5.2.   Now we proceed towards proving Theorem 5.5 below which shows, via Proposition 5.1 above, that if the tangent cone of an analytic hypersurface is a hypersurface, then it is symmetric. Let $U \subset \mathbf{R}^n$ be an open neighborhood of o, $f \colon U \to \mathbf{R}$ be a $\mathcal{C}^{k \geq 1}$ function with $f(o) = 0$, and suppose that f does not vanish to order k at o. Then by Taylor's theorem

$$f(x) = h_f(x) + r_f(x)$$

where $h_f(x)$ is a nonzero homogenous polynomial of degree m, i.e.,

$$h_f(\lambda x) = \lambda^m h_f(x),$$

for every $\lambda \in \mathbf{R}$, and $r_f \colon \mathbf{R}^n \to \mathbf{R}$ is a continuous function which satisfies

$$\lim_{x \to 0} |x|^{-m} r_f(x) = 0.$$

Now recall that $Z(f) := f^{-1}(0)$, and $\widetilde{T}_o Z(f)$ denotes the symmetric tangent cone, i.e., the limit of all sequences of secant *lines* through o and $x_i \in Z(f) \setminus \{o\}$ as $x_i \to o$. Also let $Z(h_f) := h_f^{-1}(0)$ be the zero set of h_f. Then we have:

Lemma 5.3. $\widetilde{T}_o Z(f) \subset Z(h_f)$.

Proof. Suppose $v \in \widetilde{T}_o Z(f)$. Then, it follows from Lemma 2.1 that there are points $x_i \in Z(f) \setminus \{o\}$, and numbers $\lambda_i \in \mathbf{R}$ such that $\lambda_i x_i \to v$. Since $x_i \in Z(f)$,

$$0 = f(x_i) = h_f(x_i) + r_f(x_i)$$

which yields that

$$0 = |x_i|^{-m} \big(h_f(x_i) + r_f(x_i) \big) = h_f \big(|x_i|^{-1} x_i \big) + |x_i|^{-m} r_f(x_i).$$

Consequently

$$0 = \lim_{x_i \to o} h_f \big(|x_i|^{-1} x_i \big) + 0 = h_f \left(\lim_{x_i \to o} |x_i|^{-1} x_i \right).$$

But,

$$\lim_{x_i \to o} |x_i|^{-1} x_i = \lim_{x_i \to o} |\lambda_i x_i|^{-1} |\lambda_i| x_i = \pm |v|^{-1} v.$$

So we conclude

$$0 = h_f \big(|v|^{-1} v \big) = |v|^{-m} h_f(v),$$

which shows that $h_f(v) = 0$, or $v \in Z(h_f)$. $\square$

In contrast to the above lemma, in general $Z(h_f) \not\subset \widetilde{T}_o Z(f)$. Consider for instance the case where $f(x, y) = x(y^2 + x^4)$. Then $Z(f)$ is just the y-axis, while $h_f(x, y) = xy^2$, and therefore $Z(h_f)$ is both the x-axis and the y-axis. So in order to have $Z(h_f) \subset \widetilde{T}_o Z(f)$, we need additional conditions, as given for instance by the next lemma. Recall that a function $f \colon \mathbf{R}^n \to \mathbf{R}$ *changes sign* on a set $X \subset \mathbf{R}^n$, provided

that for every point $x \in X$, and open neighborhood U of x in $\mathbf{R}^n$ there are points in U where $f > 0$ and $f < 0$. Let $\widetilde{Z}(h_f) \subset Z(h_f)$ be the set of points p where h_f changes sign at p. Then we show that:

Lemma 5.4. $\widetilde{Z}(h_f) \subset \widetilde{T}_o Z(f)$.

Proof. Suppose, towards a contradiction, that there is $v \in \widetilde{Z}(h_f)$ such that $v \notin \widetilde{T}_o Z(f)$. Then it follows from Lemma 2.1 that there exists an open neighborhood U of v in $\mathbf{R}^n$ and an open ball B centered at o such that $\mathrm{cone}(U) \cap B \cap Z(f) = \{o\}$, where $\mathrm{cone}(U)$ is the set of all lines which pass through o and points of U. So we have $f \neq 0$ on $\mathrm{cone}(U) \cap B \smallsetminus \{o\}$. Consequently if we set

$$(11) \qquad f_\lambda(x) := \lambda^m f(\lambda^{-1} x),$$

then it follows that $f_\lambda \neq 0$ on $\mathrm{cone}(U) \cap B \smallsetminus \{o\}$ for $\lambda \geq 1$. But note that, by homogeneity of h_f,

$$f_\lambda(x) = \lambda^m h_f(\lambda^{-1} x) + \lambda^m r_f(\lambda^{-1} x) = h_f(x) + \lambda^m r_f(\lambda^{-1} x),$$

which yields that

$$(12) \qquad \lim_{\lambda \to \infty} f_\lambda(x) = h_f(x).$$

Furthermore, by assumption, there are points in U where $h_f > 0$ and $h_f < 0$. Consequently, for large λ, f_λ must change sign on U as well, which in turn implies that $f_\lambda = 0$ at some point of U, and we have a contradiction. $\square$

Now let $T_o Z(f) \subset \widetilde{T}_o Z(f)$ denote as usual the tangent cone of $Z(f)$, i.e., the limit of all sequences of secant *rays* (as opposed to lines) which emanate from o and pass through $x_i \in Z(f) \smallsetminus \{o\}$ as $x_i \to o$.

Theorem 5.5. *Let $U \subset \mathbf{R}^n$ be an open neighborhood of o and $f \colon U \to \mathbf{R}^n$ be a $\mathcal{C}^{k \geq 1}$ function with $f(o) = 0$ which does not vanish to order k at o. Suppose that $Z(f)$ is homeomorphic to $\mathbf{R}^{n-1}$, f changes sign on $Z(f)$, and $T_o Z(f)$ is also a hypersurface. Then*

$$\widetilde{T}_o Z(f) = \widetilde{Z}(h_f) = T_o Z(f).$$

In particular, $T_o Z(f)$ is symmetric with respect to o, i.e., $T_o Z(f) = -T_o Z(f)$.

Proof. For convenience we may assume that $U = \mathbf{R}^n$. Further note that, since $(T_o Z(f))^* = -T_o Z(f)$, we have

$$\widetilde{T}_o Z(f) = T_o Z(f) \cup -T_o Z(f).$$

So it is enough to show that $T_o Z(f) = \widetilde{Z}(h_f)$; because then $-T_o Z(f) = -\widetilde{Z}(h_f) = \widetilde{Z}(h_f)$ by homogeneity of h_f; consequently, $\widetilde{T}_o Z(f) \subset \widetilde{Z}(h_f)$, which in turn yields that $\widetilde{T}_o Z(f) = \widetilde{Z}(h_f)$ by Lemma 5.4.

By Lemma 5.4, $\widetilde{Z}(h_f) \subset T_o Z(f)$. So it remains to show that $T_o Z(f) \subset \widetilde{Z}(h_f)$, i.e., we have to check that h_f changes sign on $T_o Z(f)$. To see this, for $\lambda \in \mathbf{R}^+$, let

$Z(f)_\lambda := Z(f)_{o,\lambda}$ be the set of points λx where $x \in Z(f)$. Recall that by Lemma 2.2, for large λ we have

$$(13) \qquad Z(f)_\lambda \cap B^n(o,r) \subset \left(T_o Z(f) + \epsilon \mathbf{B}^n\right) \cap B^n(o,r).$$

Next note that, if f_λ is given by (11), then

$$f_\lambda^{-1}(0) = \{x \mid \lambda^m f(\lambda^{-1}x) = 0\} = \{x \mid f(\lambda^{-1}x) = 0\} = \{\lambda y \mid f(y) = 0\} = Z(f)_\lambda.$$

By the generalized Jordan-Brouwer separation theorem [11, Sec. 8.5], $\mathbf{R}^n \smallsetminus Z(f)_\lambda$ has precisely two components: $(\mathbf{R}^n \smallsetminus Z(f)_\lambda)^\pm$. We may suppose that $f_\lambda > 0$ on $(\mathbf{R}^n \smallsetminus Z(f)_\lambda)^+$ and $f_\lambda < 0$ on $(\mathbf{R}^n \smallsetminus Z(f)_\lambda)^-$. It follows from (13) that inside any ball $B^n(o,r)$ we have $(\mathbf{R}^n \smallsetminus Z(f)_\lambda)^\pm$ converging, with respect to the Hausdorff topology, to the components of $\mathbf{R}^n \smallsetminus T_o Z(f)$ which we denote by $(\mathbf{R}^n \smallsetminus T_o Z(f))^\pm$ respectively:

$$\left(\mathbf{R}^n \smallsetminus Z(f)_\lambda\right)^\pm \cap B^n(o,r) \longrightarrow \left(\mathbf{R}^n \smallsetminus T_o Z(f)\right)^\pm \cap B^n(o,r).$$

In particular, if $x \in (\mathbf{R}^n \smallsetminus T_o Z(f))^+$, then eventually (as λ grows large) $x \in (\mathbf{R}^n \smallsetminus Z(f)_\lambda)^+$, and thus $f_\lambda(x) > 0$. Consequently, $h_f(x) \geq 0$ by (12), and we conclude that $h_f \geq 0$ on $(\mathbf{R}^n \smallsetminus T_o Z(f))^+$. Similarly, we have $h_f \leq 0$ on $(\mathbf{R}^n \smallsetminus T_o Z(f))^-$. So, since h_f cannot vanish identically on any open set, it follows that h_f changes sign on $T_o Z(f)$ as desired. $\qquad\square$

5.3.   Theorem 5.5 together with Proposition 5.1 immediately yield:

Corollary 5.6. *Let $X \subset \mathbf{R}^n$ be a real analytic hypersurface, and suppose that $T_p X$ is also a hypersurface for all $p \in X$, then each $T_p X$ is symmetric with respect to p, i.e., $T_p X = (T_p X)^*$.* $\qquad\square$

Equipped with the last observation, we are now ready to prove the main result of this section:

Proof of Theorem 1.3. Since through each point p of X there passes a support ball B, it follows from Lemma 2.1 that each $T_p X$ must lie in a half-space whose boundary H_p is tangent to B at p. Then Corollary 5.6 implies that $T_p X \subset H_p$. But $T_p X$ is open in H, by the theorem on the invariance of domain, and $T_p X$ is closed in H since it is closed in $\mathbf{R}^n$ by Lemma 2.1. Thus $T_p X = H_p$. So we conclude that each $T_p X$ is flat. Then, by Lemma 4.1, $T_p X$ depends continuously on p, and we may apply Theorem 1.1 to conclude that X is $\mathcal{C}^1$.

Next note that if X is any convex hypersurface, then it has positive support; further, its tangent cones are automatically hypersurfaces, by Lemma 5.7 below. So by the above paragraph X is $\mathcal{C}^1$. $\qquad\square$

Lemma 5.7. *Let $K \subset \mathbf{R}^n$ be a closed convex set with interior points, and $p \in \partial K$. Then $T_p K$ is a convex set with interior points, and*

$$\partial(T_p K) = T_p(\partial K).$$

In particular $T_p(\partial K)$ is a hypersurface.

Proof. We may suppose that $p = o$. Then it follows from Lemma 2.2 that

$$T_p K = \bigcup_{\lambda \geq 0} \lambda K,$$

once we note that, since K is convex, $\lambda_1 K \subset \lambda_2 K$ whenever $\lambda_1 \leq \lambda_1$. Thus $T_p K$ is convex, and obviously has interior points since $K \subset T_p K$. This inclusion also shows that $\partial K \subset T_p K$, since tangent cones are always closed, by Lemma 2.1, and so

$$T_p(\partial K) \subset T_p(T_p K) = T_p K.$$

Now suppose towards a contradiction that $T_p(\partial K)$ contains a ray ℓ which lies in the interior of $T_p K$. Then, since $T_p K$ is convex, there exists a cone C about ℓ which lies in $T_p K$. Now since $\lambda C = C$, it follows that $C \cap B \subset K$ for some ball B centered at p. In particular, after making C smaller, we may assume that $C \cap B$ intersects ∂K only at p. Hence ℓ cannot belong to $T_p(\partial K)$ by Lemma 2.1. So we conclude that $T_p(\partial K) = \partial(T_p K)$. $\qquad \square$

5.4.   Here we show that when $n = 2$ in Theorem 1.3, it is not necessary to assume that the hypersurface X have positive support. To see this let $\Gamma := X \subset \mathbf{R}^2$ be a real analytic hypersurface or *simple curve*, i.e., suppose that each point $p \in \Gamma$ has an open neighborhood U homeomorphic to $\mathbf{R}$. Then we call the closure of each component of $U \smallsetminus \{p\}$ in U, which we denote by b_1 and b_2, a *half-branch* of Γ at p. By the "curve selection lemma" [23], which also holds for semianalytic sets [6, Prop. 2.2], there exist $\mathcal{C}^1$ curves $\gamma_i \colon [0, 1) \to b_i$ with $\gamma_i(0) = p$ such that $\gamma_i' \neq 0$; see [25, p. 956]. Thus each half-branch b_i has a well-defined tangent ray $\ell_i := \{p + t(\gamma_i)'_+(0) \mid t \geq 0\}$ emanating from p, where $(\gamma_i)'_+$ denotes the right hand derivative. It follows then from Corollary 2.3 that $T_p \Gamma = \ell_1 \cup \ell_2$. If $T_p \Gamma$ consists of only one ray, i.e., $\ell_1 = \ell_2$, then we say that Γ has a *cusp* at p. Otherwise, $T_p \Gamma$ is a simple curve. Consequently, by Corollary 5.6, $T_p \Gamma$ must be flat, i.e, $\mathrm{dir}(\ell_1) = -\mathrm{dir}(\ell_2)$ which in turn yields that $U = b_1 \cup b_2$ is $\mathcal{C}^1$. So we obtain the following special case of Theorem 1.3 for $n = 2$:

Corollary 5.8. *Let* $\Gamma \subset \mathbf{R}^2$ *be a real analytic simple curve. Then either* Γ *has a cusp, or else it is* $\mathcal{C}^1$. $\qquad \square$

When, on the other hand, $n \geq 3$ in Theorem 1.3, the positive support assumption cannot in general be abandoned (Example 7.3). Further, note that the above corollary implies that if the tangent cones of a simple curve are simple curves, then they must be flat. This again does not generalize to higher dimensions (Example 7.4). Finally we should mention that Corollary 5.8 also follows from classical resolution of singularities; see Appendix A, specifically Corollary A.3.

6. REGULARITY OF REAL ALGEBRAIC CONVEX HYPERSURFACES: PROOF OF THEOREM 1.4

6.1.   First we show that X in Theorem 1.4 is an entire graph. To this end we need to employ the notion of the *recession cone* [28] of a closed convex set $K \subset \mathbf{R}^n$ which is defined as

$$\mathrm{rc}(K) := \{x \in \mathbf{R}^n \mid K + x \subset K\}.$$

Further, let $\mathrm{nc}(K)$ be the *normal cone* [28] of K which is defined as the set of all outward normals to support hyperplane of K. The following observation is implicit in [40, Thm. 2], see also [15, Prop. 3.1].

Lemma 6.1 (Wu [40]). *If $K \subset \mathbf{R}^n$ is a closed convex set with interior points and boundary ∂K homeomorphic to $\mathbf{R}^{n-1}$, then $\mathrm{rc}(K) \cap \mathbf{S}^{n-1} \cap (-\mathrm{nc}(K)) \neq \emptyset$.* □

By assumption, the hypersurface X in Theorem 1.4 coincides with ∂K for some convex set K satisfying the hypothesis of the above lemma. We show that for any unit vector $u \in \mathrm{rc}(K) \cap (-\mathrm{nc}(K))$, ∂K is an entire graph in the direction u, i.e., it intersects every line parallel to u.

We may suppose for convenience that $u = (0, \ldots, 0, 1)$, and K lies in the half-space $x_n \geq 1$. Let $\mathbf{H}^n$ be the open half-space $x_n > 0$ and consider the projective transformation $P \colon \mathbf{H}^n \to \mathbf{R}^n$ given by

$$P(x_1, \ldots, x_n) := \left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n} \right).$$

Note that P preserves line segments, and so maps convex sets to convex sets. In particular, the closure $\overline{P(K)}$ will be a closed convex set. Next let $f \colon \mathbf{R}^n \to \mathbf{R}$ be the algebraic function with $\partial K = f^{-1}(0)$ and suppose that f has degree d. Then

$$\overline{f}(x) := (x_n)^d f(P(x)),$$

is an algebraic function on $\mathbf{R}^n$, and it is not hard to check that

(14) $\partial \overline{P(K)} = \overline{f}^{-1}(0).$

So by Theorem 1.3, $\partial \overline{P(K)}$ is $\mathcal{C}^1$, since it is a real analytic convex hypersurface.

Next note that $o \in \partial \overline{P(K)}$, since by assumption K contains the ray $(0, \ldots, 0, t)$ where $t \geq 1$. Since $\partial \overline{P(K)}$ is a $\mathcal{C}^1$ convex hypersurface supported by $\{x_n = 0\}$, it follows that each ray ℓ given by tv where v is a unit vector with $\langle v, u \rangle > 0$ and $t \geq 0$, intersects $\partial \overline{P(K)} \cap \mathbf{H}^n = \partial P(K) = P(\partial K)$ in exactly one point, see Figure 4. In particular,

$$P(\partial K) \cap \ell \neq \emptyset.$$

Now note that P^2 is the identity map on $\mathbf{H}^n$. Thus the last expression yields that

$$\partial K \cap P(\ell) = P^2(\partial K) \cap P(\ell) = P\big(P(\partial K) \cap \ell\big) \neq \emptyset.$$

It remains only to note that $P(\ell)$ is parallel to $(0, \ldots, 0, 1)$, and intersects $\{x_n = 1\}$ at the same point as does ℓ, which completes the proof that ∂K is an entire graph.

6.2. Now we show that if $X = \partial K$ is strictly convex, then its projective closure is a $\mathcal{C}^1$ hypersurface in $\mathbf{RP}^n$. First, let us recall that the standard embedding $i \colon \mathbf{R}^n \to \mathbf{RP}^n$ is given by

$$(x_1, \ldots, x_n) \overset{i}{\longmapsto} [x_1 : \cdots : x_n : 1],$$

where $[x_1 : \cdots : x_{n+1}]$ are the homogeneous coordinates of $\mathbf{RP}^n$. Then the projective closure of X is defined as $\overline{i(X)}$, i.e., the closure of $i(X)$ in $\mathbf{RP}^n$. To show that $\overline{i(X)}$
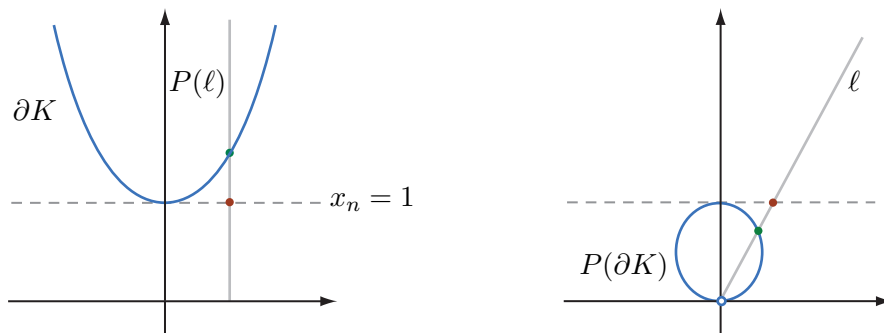
Figure 4:

is $\mathcal{C}^1$, we need to establish that the closure $\overline{P(K)}$ is compact, which in turn is due to the following basic fact:

Lemma 6.2. *Let $K \subset \mathbf{R}^n$ be a convex set. Suppose that K is contained between a pair of hyperplanes H_0 and H_1, and $H_0 \cap K$ is compact and nonempty. Then K is compact.*

Proof. First note that K is compact if and only if it does not contain any rays, or equivalently $\mathrm{rc}(K) = \emptyset$. So, if K is not compact, then there exists a ray ℓ in K which emanates from a point p of K. Let H be the hyperplane passing through p which is parallel to H_0. Then $\ell \in \mathrm{rc}(K \cap H)$. Further note that since H and H_0 are parallel, $\mathrm{rc}(H) = \mathrm{rc}(H_0)$. So

$$\mathrm{rc}(K \cap H) = \mathrm{rc}(K) \cap \mathrm{rc}(H) = \mathrm{rc}(K) \cap \mathrm{rc}(H_0) = \mathrm{rc}(K \cap H_0) \neq \emptyset,$$

which is a contradiction since $K \cap H$ is compact. $\qquad\square$

Now note that if ∂K is strictly convex, and as in the previous subsection we assume that ∂K is supported below by the hyperplane $x_n = 1$, then this hyperplane intersects ∂K and therefore K at only one point (which is compact). Consequently $\overline{P(K)}$ will intersect $\{x_n = 1\}$ only at one point as well, since P is the identity on $\{x_n = 1\}$. Thus, since $\overline{P(K)}$ is contained between $\{x_n = 0\}$ and $\{x_n = 1\}$, it follows from Lemma 6.2 that $\overline{P(K)}$ is compact. So $\overline{P(K)} \cap \{x_n = 0\}$ is compact. But recall that $\partial \overline{P(K)}$ is algebraic, which yields that so is $\overline{P(K)} \cap \{x_n = 0\}$. Consequently, $\overline{P(K)} \cap \{x_n = 0\}$ may not contain any line segments, so it must consist of only a single point since $\overline{P(K)} \cap \{x_n = 0\}$ is convex. This implies that

$$\partial \overline{P(K)} = \overline{P(\partial K)}.$$

So $\overline{P(\partial K)}$ is a $\mathcal{C}^1$ hypersurface by (14). Further, since $\overline{P(\partial K)}$ is compact, it coincides with its own projective closure. Thus, the projective closure of $\overline{P(\partial K)}$ is a $\mathcal{C}^1$ hypersurface. This yields that the projective closure of ∂K must be $\mathcal{C}^1$ as well, due

to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbf{RP}^n & \xrightarrow{\widetilde{P}} & \mathbf{RP}^n \\ \uparrow i & & \uparrow i \\ \mathbf{H}^n & \xrightarrow{P} & \mathbf{R}^n \end{array}$$

where, in terms of the homogeneous coordinates $[x_1 : \cdots : x_{n+1}]$, $\widetilde{P}$ is the map

$$[x_1 : \cdots : x_n : x_{n+1}] \xmapsto{\widetilde{P}} [x_1 : \cdots : x_{n+1} : x_n].$$

So we have, $i(\partial K) = \widetilde{P}^{-1}(i(P(\partial K)))$, which yields that $\overline{i(\partial K)} = \widetilde{P}^{-1}(\overline{i(P(\partial K))})$ because $\widetilde{P}$ is a homeomorphism. Further note that $\overline{i(P(\partial K))} = i(\overline{P(\partial K)})$, because $P(\partial K)$ is bounded, and i is a homeomorphism restricted to any compact subset of $\mathbf{R}^n$. So we conclude that

$$\overline{i(\partial K)} = \widetilde{P}^{-1}\big(i(\overline{P(\partial K)})\big)$$

which completes the proof.

7. Examples (and Counterexamples)

Here we gather a number of examples which establish the optimality of various aspects of the theorems discussed in the introduction. All curves described below may be rotated around their axis of symmetry to obtain surfaces with analogous properties.

Example 7.1. It is easy to see that without the assumption on multiplicity in Theorem 1.1, the set X may not be a hypersurface; see Figure 1, specially the middle diagram, which also demonstrates that the value of m in Theorem 1.1 is optimal. Further, the collection of hypersurfaces which make up X may not be finite, even locally. Figure 5 shows one such example. It has continuously varying
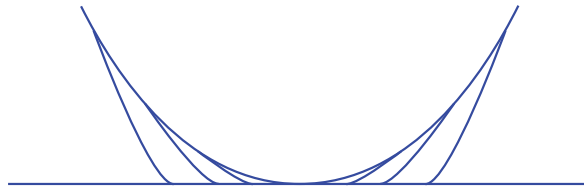

Figure 5:

flat tangent cones, but cannot be decomposed into a finite number of hypersurfaces near the neighborhood surrounding the apex of the parabola.

Example 7.2. For any given $\alpha > 0$, there is a convex real algebraic hypersurface which is not $\mathcal{C}^{1,\alpha}$. Explicit examples are given by the convex curves $y^{2n-1} = x^{2n}$, where $n = 1, 2, 3, \ldots$. These curves are $\mathcal{C}^1$ by Theorem 1.1, and are $\mathcal{C}^\infty$ everywhere except at the origin o. But they are not $\mathcal{C}^{1,\alpha}$, for $\alpha > 1/(2n-1)$, in any neighborhood of o. All these properties are shared by the projectively equivalent family of closed convex curves $(1-y)y^{2n-1} = x^{2n}$ which is depicted in Figure 6, for $n = 2, 3, 4$. Here the singular point o lies at the bottom of each curve.
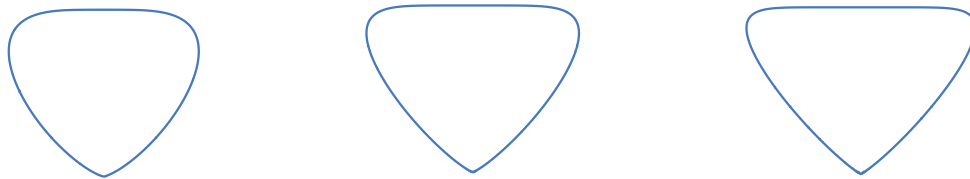
Figure 6:

Example 7.3. There are real algebraic hypersurfaces with flat tangent cones which are supported by balls (of nonuniform radii) at each point but are not $\mathcal{C}^1$, such as the sextic surface $z^3 = x^5 y + xy^5$ depicted in Figure 7a. This surface is regular in the complement of the origin. So it has support balls there. Further, one might directly check that there is even a support ball at the origin. On the other hand the surface is not $\mathcal{C}^1$, since all its tangent planes along the x and y axis are vertical, while at the origin the tangent plane is horizontal.
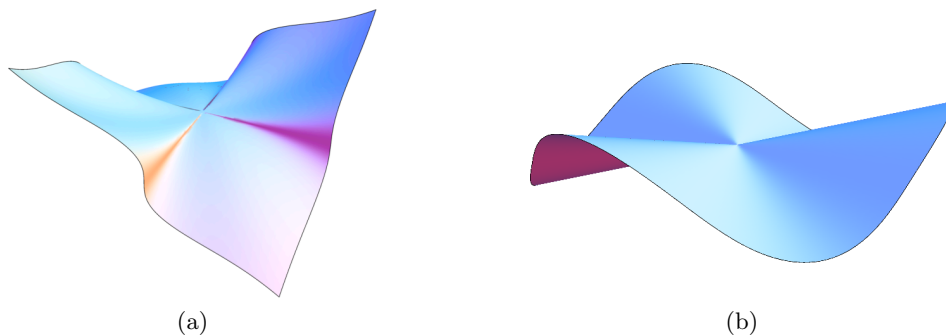

(a)　　　　　　　　　　　　　(b)

Figure 7:

Example 7.4. There are real algebraic hypersurfaces whose tangent cones are hypersurfaces but are not hyperplanes, such as the Fermat cubic $x^3 + y^3 = z^3$, see Figure 7b. All points of this surface, except the origin, are regular and therefore

the tangent cones are flat everywhere in the complement of the origin. On the other hand, the tangent cone at the origin is the surface itself, since the surface is invariant under homotheties.

Example 7.5. There are real semialgebraic convex hypersurfaces in $\mathbf{R}^n$ which are not $\mathcal{C}^1$. An example is the portion of the "Ding-dong curve" [18] given by $x^2 = y^2(1-y)$ and $y \geq 0$; see Figure 8a. Also note that this curve is projectively equivalent to $y(1-x^2) = 1$, $-1 \leq x \leq 1$, depicted in Figure 8b, which shows that there are semialgebraic strictly convex hypersurfaces which are homeomorphic to $\mathbf{R}^{n-1}$ but are not entire graphs.

(a)   (b)

Figure 8:

Example 7.6. There are real analytic convex hypersurfaces homeomorphic to $\mathbf{R}^{n-1}$ whose projective closure is not $\mathcal{C}^1$ or even differentiable; for instance, $x^2 + e^{-y} = 1$ defines an unbounded convex planar curve which is contained within the slab $-1 \leq x \leq 1$ and thus is not an entire graph, see Figure 9a.
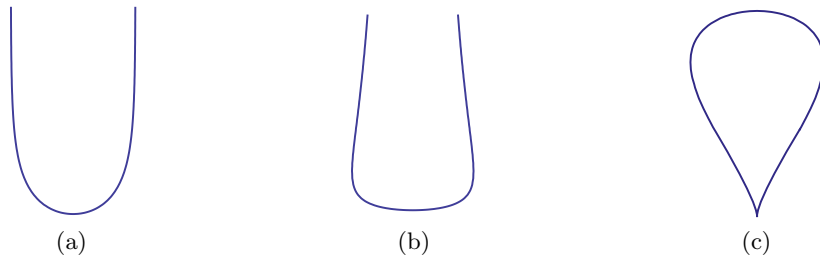

(a)   (b)   (c)

Figure 9:

Example 7.7. There are smooth real algebraic hypersurfaces which are homeomorphic to $\mathbf{R}^{n-1}$ but are not entire graphs. Consider for instance the planar curve $y(1-x^2y) = 1$ pictured in Figure 9b. This curve is projectively equivalent to the "pear-shaped quartic" or *Piriform* given by $x^2 = y^3(1-y)$, see Figure 9c.

Example 7.8. There are real algebraic convex hypersurfaces homeomorphic to $\mathbf{R}^{n-1}$ whose projective closure is not $\mathcal{C}^1$ or even a topological hypersurface. Consider for instance the parabolic cylinder $P \subset \mathbf{R}^3$ given by $2z = x^2 + 1$. This surface is projectively equivalent to the circular cylinder C given by $(z-1)^2 + x^2 = 1$, via the transformation $(x, y, z) \mapsto (x/z, y/z, 1/z)$. Translating C, we obtain C' given by $z^2 + x^2 = 1$. Now consider the projective transformation $(x, y, z) \mapsto (x/y, 1/y, z/y)$ which maps C' to the cone $z^2 + x^2 = y^2$. Since, as we discussed in Section 6.2, these projective transformations extend to diffeomorphism of $\mathbf{RP}^3$, we then conclude that that the projective closure of our original surface P had a conical singularity.

Appendix A. Another Proof for The Regularity of Real Analytic Convex Hypersurfaces

Using the classical resolution of singularities for planar curves, we describe here an alternative proof of the $\mathcal{C}^1$-regularity of real analytic convex hypersurfaces (which was a special case of Theorem 1.3). More specifically, we employ the following basic fact, c.f. [23, Lemma 3.3], which may be considered the simplest case of Hironaka's uniformization theorem [4, 19]. By a *real analytic curve* here we mean a real analytic set of dimension one.

Lemma A.1 (Newton-Puiseux [17, 23]). *Let $\Gamma \subset \mathbf{R}^2$ be a real analytic curve and $p \in \Gamma$ be a nonisolated point. Then there is an open neighborhood U of p in $\mathbf{R}^2$ such that $\Gamma \cap U = \cup_{i=1}^k \Gamma_i$ where each "branch" Γ_i is homeomorphic to $\mathbf{R}$ via a real analytic (injective) parametrization $\gamma_i \colon (-1, 1) \to \Gamma_i$.* $\qquad\square$

See [17, p. 76–77] for the proof of the above lemma in the complex case, which in turn yields the real version as described in [23, p. 29]. Other treatments of the complex case may also be found in [8, 37]. The main ingredient here is Puiseux's decomposition theorem for the germs of analytic functions, which goes back to Newton, see [21, Thm. 1.1] or [10, Sec. 2.1]. Next note that any simple planar curve which admits an analytic parametrization must be piecewise smooth, because the speed of the parametrization can vanish only on a discrete set; furthermore, the curve may not have any "corners" at these singularities:

Lemma A.2. *Let $\gamma \colon (-1, 1) \to \mathbf{R}^2$ be a nonconstant real analytic map, and set $T(t) := \gamma'(t)/\|\gamma'(t)\|$. Then $\lim_{t \to a^+} T(t) = \pm \lim_{t \to a^-} T(t)$ for all $a \in (-1, 1)$. In particular, γ has continuously turning tangent lines.*

By "tangent line" here we mean the symmetric tangent cone of the image of γ (Section 2.1). Note also that the analyticity assumption in the above lemma is essential (the curve $y = |x|$, $-1 < x < 1$, for instance, admits the well-known $\mathcal{C}^\infty$ parametrization given by $\gamma(t) := e^{-1/t^2}(t/|t|, 1)$ for $t \neq 0$, and $\gamma(0) := (0, 0)$).

Proof. We may assume $a = 0$. If $\|\gamma'(0)\| \neq 0$, then the proof immediately follows. So suppose that $\|\gamma'(0)\| = 0$. Then, by analyticity of γ', there is an integer $m > 0$ and an analytic map $\xi \colon (-\epsilon, \epsilon) \to \mathbf{R}^2$ with $\|\xi(0)\| \neq 0$ such that $\gamma'(t) = t^m \xi(t)$.

Thus

$$T(t) = \frac{t^m \xi(t)}{\|t^m \xi(t)\|} = \left(\frac{t}{|t|}\right)^m \frac{\xi(t)}{\|\xi(t)\|},$$

which in turn yields:

$$\lim_{t \to 0^+} T(t) = 1^m \frac{\xi(0)}{\|\xi(0)\|} = (-1)^{2m} \frac{\xi(0)}{\|\xi(0)\|} = (-1)^m \lim_{t \to 0^-} T(t).$$

$\square$

The last two lemmas yield the following basic fact which generalizes Corollary 5.8 obtained earlier from Theorem 1.3. Recall that a simple planar curve has a *cusp* at some point if its tangent cone there consists of a single ray (c.f. Section 5.4).

Corollary A.3. *Each branch of a real analytic curve $\Gamma \subset \mathbf{R}^2$ at a nonisolated point $p \in \Gamma$ is either $\mathcal{C}^1$ near p or has a cusp at p.* $\square$

This observation quickly shows that a convex real analytic curve $\Gamma \subset \mathbf{R}^2$ is $\mathcal{C}^1$, since by Lemma 5.7 it cannot have any cusps. This in turn yields the same result in higher dimensions, via a slicing argument, as we describe next. Let $X \subset \mathbf{R}^n$ be a real analytic convex hypersurface. There exists then a convex set $K \subset \mathbf{R}^n$ with $\partial K = X$. Thus, by Lemma 4.5, to establish the regularity of X it suffices to show that through each point p of X there passes a unique support hyperplane. Suppose, towards a contradiction, that there are two distinct support hyperplanes H_1 and H_2 passing through p. Then $L := H_1 \cap H_2$ has dimension $n-2$. Now let o be a point in the interior of K. Since $o \notin L$, there exists a (two dimensional) plane $\Pi \subset \mathbf{R}^n$ which is transversal to L at p, and passes through o. Consequently $\Gamma := X \cap \Pi = \partial(K \cap \Pi)$ is a convex real analytic planar curve, and therefore must be $\mathcal{C}^1$ (by Corollary A.3). So Γ must have exactly one support line at p, which is a contradiction since $\ell_i := H_i \cap \Pi$ are distinct support lines of Γ at p.

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