THE GEOMETRY OF SHADOW BOUNDARIES ON SURFACES IN SPACE

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1. Introduction

These are somewhat expanded notes from a talk on the geometry of shadow curves given in the differential geometry seminar at the University of South Carolina. They are meant to be readable by anyone who has has a one term course in the differential geometry of curves and surfaces, but probably fail at archiving quit that level of accessibility.

2. The Second Fundamental Form of a an Oriented Surface.

Let M^2 be an oriented surface immersed in $\mathbf{R}^3$ and let $\mathbf{n}\colon M^2 \to S^2$ be the Gauss map (unit normal field) along M^2. We denote the standard flat connection on $\mathbf{R}^3$ by $\overline{\nabla}$ and the induced connection on M^2 by ∇. If X and Y are vector fields on M^2 then these are related by

$$\overline{\nabla}_X Y = \nabla_X Y + I\!I(X,Y)\mathbf{n}$$

here $I\!I$ is the second fundamental form of M^2 which is a symmetric bilinear form. The Weingarten map (or shape operator), A, of M^2 is a linear map defined on tangent spaces by

$$A(X) := -\overline{\nabla}_X \mathbf{n}.$$

If $\langle\,,\rangle$ is the standard inner product on $\mathbf{R}^3$ then A and $I\!I$ are related by

$$\langle AX, Y\rangle = I\!I(X,Y).$$

Therefore the symmetry of $I\!I(\, , \,)$ implies that A is a self-adjoint linear map. Some idea of the geometric meaning of $I\!I(\, , \,)$ can be form the following:

Exercise 2.1. Let M^2 be a smooth surface in $\mathbf{R}^3$ then by a rigid motion we can move M^2 so that any given point is at the origin and the tangent plane to M at the origin is the x-y plane and that the normal $\mathbf{n}$ to M at the origin points upward along the z-axis. Then locally near the origin M^2 is given by $z = f(x, y)$ where $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Then show the second fundamental form of M^2 is

$$I\!I_{(0,0)} = f_{xx}(0, 0)\, dx^2 + 2f_{xy}(0, 0)\, dxdy + f_{yy}(0, 0)dy^2.$$

More generally show that if M^2 is given as a graph $z = f(x, y)$ over $U \subset \mathbf{R}^2$ the unit normal to M^2 is

$$\mathbf{n}(x, y) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

and the second fundamental form is given by

$$I\!I_{(0,0)} = \frac{f_{xx}\, dx^2 + 2f_{xy}\, dxdy + f_{yy}\, dy^2}{\sqrt{1 + f_x^2 + f_y^2}}.$$

□

We will also have occasion to work with the covariant derivatives of A and $I\!I$. If X, Y, Z are smooth vector fields on M then these are defined by

$$(\nabla_X A)(Y) := \nabla_X(A(Y)) - A(\nabla_X Y).$$

and

$$(\nabla_X I\!I)(Y, Z) := X(I\!I(Y, Z)) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

These are tensors (that is are linear in each X, Y, Z over the smooth functions) and are related by

$$(\nabla_X I\!I)(Y, Z) = \langle (\nabla_X A)(Y), Z\rangle.$$

Also

$$(\nabla_X A)(Y) = (\nabla_Y A)(X)$$

(This is the Codazzi equation) and $(\nabla_X I\!I)(Y, Z)$ is a symmetric function of all of X, Y, Z.

Exercise 2.2. As in Exercise 2.1 by a rigid motion we can move M^2 so that any given point is at the origin and the tangent plane to M at the origin is the x-y plane and that the normal $\mathbf{n}$ to M at the origin points upward along the z=axis. Then locally near the origin M^2 is given by $z = f(x, y)$ where $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Then show that at the origin the tensor $\nabla I\!I$ is given by

$$\nabla I\!I_{(0,0)} = f_{xxx}(0, 0)\, dx^3 + 3f_{xxy}(0, 0)\, dx^2dy + 3f_{xyy}(0, 0)\, dxdy^2 f_{yyy}(0, 0)\, dy^2.$$

That is at $(0,0)$

$$(\nabla_{\frac{\partial}{\partial x}} I\!I)\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big) = f_{xxx}(0,0), \qquad (\nabla_{\frac{\partial}{\partial x}} I\!I)\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big) = f_{xxy}(0,0),$$
$$(\nabla_{\frac{\partial}{\partial x}} I\!I)\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big) = f_{xyy}(0,0), \qquad (\nabla_{\frac{\partial}{\partial y}} I\!I)\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big) = f_{yyy}(0,0).$$

Therefore if M^2 is expressed locally as a graph $z = f(x,y)$ as above, then $I\!I$ $\nabla I\!I$ give the second and third order terms in the Taylor expanson of M^2. $\square$

3. The Light Source at an Infinite Distance.

For any unit vector e in $\mathbf{R}^3$ define a function $f_e\colon M^2 \to \mathbf{R}$ by

$$f_e(x) := \langle \mathbf{n}(x), e\rangle. \tag{3.1}$$

If we think of a light source at a very great distance from M^2 then near M^2 the light rays can be assume parallel (for example light form the sun is generally treated as being parallel). If e is a unit vector parallel to the light rays and pointing toward the light source then the set $\{x \in M^2 : f)e(x) > 0\}$ can be thought of as the part of M^2 that is illuminated. (This is not perfect model as it there may be parts of the surface that are illuminated under this definition, but which are in really in the shadow of some other part of the surface.)

The the ***shadow boundary*** of M^2 in the direction of e is the set defined by

$$S_e := \{x \in M : \langle \mathbf{n}(x), e\rangle = 0\}.$$

That is S_e is the zero set of f_e. We wish to understand the basic geometry of S_e. That is for what e is S_e a smooth curve and if S_e is not smooth is it at least a graph in the sense of combinatorics, that is S_e is a finite set of points connected by smooth curves. The main tool will be the implicit function theorem. We first compute the derivative of f_e. For a vector X tangent to M we have

$$df_e(X) = X\langle \mathbf{n}, e\rangle = \langle \overline{\nabla}_X \mathbf{n}, e\rangle + \langle \mathbf{n}, \overline{\nabla}_X e\rangle = -\langle A(X), e\rangle \tag{3.2}$$

as $\overline{\nabla}_X e = 0$. This leads to

Proposition 3.1. *At a point x in the shadow boundary S_e the vector e is tangent to M and the derivative of f_e is given by*

$$(df_e)_x(X) = -\langle X, Ae\rangle = -I\!I(X,e). \tag{3.3}$$

Therefore df_e vanishes at the point x of S_e if and only if $Ae = 0$. At points x of M^2 where $Ae \neq 0$ the set S_e is a smooth curve in a neighborhood of x.

Proof. At points $x \in S_e$ we have $\langle \mathbf{n}(x), e\rangle = 0$ and therefore $e \in T(M)_x$. Using the formula (3.2) and that A is self-adjoint we see that (3.3) holds. That $(df_e)_x = 0$ if and only if $Ae = 0$ is now clear. If $Ae \neq 0$ at $x \in S_e$ then the zero set of f_e is a smooth curve near x by the implicit function theorem. $\square$

Definition 3.2. Two vectors $X, Y \in T(M)_x$ tangent to a surface at the same point are ***conjugate*** iff $I\!I(X, Y) = 0$. That is vectors are conjugate iff they are orthogonal with respect to the second fundamental form. □

Remark 3.3. If $x \in S_e$ and $Ae \neq 0$ so that S_e is a smooth curve near x then let X be tangent to S_e at x. Then $df_e(X) = -I\!I(X, e) = 0$. This implies that at its smooth points the shadow boundary is orthogonal, with respect to the second fundamental form, to the direction of the light source. That is X and e are conjugate. We will see below this is also true for light sources at a finite distance. Thus at a point x in a surface M by shining a light on M from directions e tangent to M at x we can see the vectors X conjugate to e as being the tangents to the shadow boundary S_e at x. This give quite a bit of information about the second fundamental form because of:

Exercise 3.4. Let $b_1(\,,\,)$ and $b_2(\,,\,)$ be two symmetric bilinear form on a two dimensional vector space V. Assume that there are three pairwise linear independent vectors $v_1, v_2, v_3 \in V$ so that

$$\{X \in V : b_1(X, v_i) = 0\} = \{X \in V : b_2(X, v_i) = 0\} \quad \text{for} \quad i = 1, 2, 3.$$

Then show there is a nonzero constant λ so that $b_2 = \lambda b_1$. □

Therefore three different shadow boundaries through a point allows one to "see" the second fundamental form up to a constant multiple. Some thought, and drawing some pictures, shows that in if $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is an affine map and S_e is a shadow boundary on M then φS_e will be a shadow boundary on the image φM. This shows that that conformal class of the second fundamental form is an affine invariant of $M^2 \subset \mathbf{R}^3$. We will see a generalization (cf. Remark 4.3) of this below. □

On a surface with a Riemannian metric let K denote the Gauss curvature. On an immersed surface in $\mathbf{R}^3$ this is given by $K = \det(A)$ where A is the Weingarten map.

Proposition 3.5. *In the space of immersions of a compact oriented surface into $\mathbf{R}^3$ let $\mathcal{K}$ be the set of immersions so that*

$$K = 0 \quad \textit{implies} \quad dK_x \neq 0.$$

Then $\mathcal{K}$ is an open dense set in the C^2 in the space of immersions. For all surfaces M^2 in $\mathcal{K}$ the set $\{K = 0\}$ is a smooth curve in M.

Proof. This follows Thom's Jet Transversality Theorem [1, Thm 4.9 p. 54]. □

Proposition 3.6. *If $M \in \mathcal{K}$ (with $\mathcal{K}$ as defined above) then M has no planar points. That is there are no points x where $A_x = 0$. More precisely let $c = \{x \in M : K(x) = 0\}$. Then if $X \in T(M)_x$ is a vector that is not tangent to c then*

$$A_x \neq 0 \quad \textit{and} \quad \nabla_X A \neq 0.$$

Proof. Let $x \in c$. Then there is a oriented orthonormal moving frame e_1, e_2 defined in a neighborhood x so that for all $Y \in T(M)_x$ we have $(\nabla_Y e_1)(x) = (\nabla_Y e_2)(x) = 0$. If $X \in T(M)_x$ and X is not tangent to c then as c is defined by $K = 0$ we have that $dK(X) \neq 0$. This implies that at the point x we have

$$\begin{aligned} 0 \neq dK(X) e_1 \wedge e_2 &= \nabla_X (K e_1 \wedge e_2) \\ &= ((\nabla_X A) e_1) \wedge (A e_2)) + (A e_2) \wedge ((\nabla_X A) e_2). \end{aligned}$$

Now $A_x \neq 0$ and $\nabla_X A \neq 0$ follows at once. □

For a smooth surface M^2 immersed in $\mathbf{R}^3$ recall that we classify the points $x \in M$ as follows:

- ⇒ The point x is ***elliptic*** iff $K(x) > 0$. This means that $I\!I(\, ,)$ is either positive or negative definite. Near such a point the M is locally convex in the sense that if locally lies on one side of its tangent plane. The ellipsoids $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ are examples of surfaces with all points elliptic.
- ⇒ The point x is ***hyperbolic*** iff $K(x) < 0$. In this case $I\!I(\, ,)$ is non-degenerate and indefinite. In this case M is locally a saddle surface near x. The surface $z = xy$ and the hyperboloid of one sheet $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ are example of surfaces with all points hyperbolic.
- ⇒ The point x is ***parabolic*** iff $K(x) = 0$ but $I\!I_x(\, ,) \neq 0$. In this case A_x has and one zero eigenvalue and one nonzero eigenvalue. The cylinder $x^2/a^2 \pm y^2/b^2 = 1$ is a surface with all points parabolic.
- ⇒ The point x is ***planar*** iff $I\!I_x(\, ,) = 0$. The surface $z = xy(x - y)$ has a planer point at the origin. A connected surface with all its points planar is an open subset of a plane.

The set of all parabolic points of M^2 is call the ***parabolic locus*** of M^2. If c is the parabolic locus of M^2 then Propositions 3.7 and 3.6 taken together imply that if $M^2 \in \mathcal{K}$ that the parabolic locus in M^2 is defined by $\{K = 0\}$ and that this is a smooth curve in M.

Now let $M^2 \in \mathcal{K}$ and let $S(M)$ be the unit sphere bundle of M. Then for each $x \in M^2$ with $K(x) = 0$ there are exactly two unit vectors $u_1, u_2 \in T(M)_x$ so that $Au_1 = Au_2 = 0$. (This is because $A \neq 0$ and $\det A = 0$ implies A has rank one.) Let

$$\mathcal{B} := \{(x, u) \in S(M) : K(x) = 0, u \in T(M)_x, \text{ and } Au = 0\}. \tag{3.4}$$

The as $\{K = 0\}$ is a smooth curve and $\mathcal{B}$ double covers $\{K = 0\}$ the curve $\{K = 0\}$ we have that $\mathcal{B}$ is a smooth curve in $S(M)$. There is a natural smooth map $g \colon \mathcal{B} \to S^2$ given by

$$g(x, u) = u.$$

Proposition 3.7. *With notation as above if $M^2 \in \mathcal{K}$ and $e \in S^2 \setminus g[\mathcal{B}]$ then the shadow boundary S_e is a smooth imbedded one dimensional submanifold of M^2. (Note that $g[\mathcal{B}]$ is a smooth image of a smooth curve so that $g[\mathcal{B}]$ not* *only has two dimensional measure zero, but is a closed set of finite length, i.e. one dimensional Hausdorff measure. Thus for almost all e the shadow boundary S_e will be smooth.)*

Proof. Let $x \in S_e$. By Proposition 3.1 it is enough to show that for all $x \in S_e$ that $A_x e \neq 0$. But if $A_x e = 0$ then $(x, e) \in \mathcal{B}$ which would in turn imply that $e \in g[\mathcal{B}]$ which is not the case. $\square$

Definition 3.8. Let $\mathcal{V}$ be a finite set of points on the surface. Then by a ***generalized graph on*** M ***with vertices*** $\mathcal{V}$ we mean a subset c of M so that $\mathcal{V} \subseteq c$ and $c \setminus \mathcal{V}$ is a smooth embedded curve in $M \setminus \mathcal{V}$.

Proposition 3.9. *Let $M^2 \in \mathcal{K}$ and let $e \in S^2$ so that the set*

$$\mathcal{V} := \{x \in M^2 : A_x e = 0\}$$

is finite. Then the shadow boundary S_e is a generalized graph on $\mathcal{V}$. In particular this implies that in some generalized sense S_e is a piecewise smooth curve.

Proof. This follows at once from Proposition 3.1. $\square$

We compute the Hessian of f_e. Let X, Y be smooth vector fields on M^2 and $f \colon M \to \mathbf{R}$ a smooth function. Then the ***Hessian*** of $D^2 f$ is defined by

$$D^2 f(X, Y) := X df(Y) - df(\nabla_X Y).$$

As a function of X this is clearly linear over the smooth functions. Using the the connection ∇ is symmetric, that is $\nabla_X Y - \nabla_Y X = [X, Y]$, we have

$$\begin{aligned} D^2 f(X, Y) - D^2 f(Y, X) &= X df(Y) - Y df(X) - df(\nabla_X Y - \nabla_Y X) \\ &= [X, Y] f - df([X, Y]) \\ &= [X, Y] f - [X, Y] f = 0. \end{aligned}$$

Therefore $D^2 f(X, Y)$ is symmetric in X and Y. Thus $D^2 f(X, Y)$ is also linear as a function of Y over the smooth functions. Thus $D^2 f(,)$ is a tensor. Note the at a critical point x of f we have $df_x(\nabla_X Y) = 0$ and thus

$$D^2 f(X, Y) = X df(Y) = XY f \quad \text{at points } x \text{ where } df_x = 0. \tag{3.5}$$

Exercise 3.10. Let x, y be local coordinates on M whose domain contains the point p. Let $f \colon M \to \mathbf{R}$ be a smooth function function that has a critical point at p. Then use (3.5) to show that at p the Hessian is

$$D^2 f = f_{xx}\, dx^2 + 2 f_{xy}\, dx dy + f_{yy}\, dy^2$$

at the point p. Therefore at critical points the Hessian is just the usual Hessian we know and love from advanced calculus. (At points other than critical points the local coordinate form of the Hessian will involve the Christoffel symbols Γ^k_{ij} of the connection.) $\square$

Definition 3.11. Let $f\colon M^2 \to \mathbf{R}$ be a smooth function. Then a critical point p of f is ***non-degenerate*** iff the Hessian $D^2 f_p(\,,\,)$ is a non-degenerate bilinear form. The function f is a ***Morse function*** iff is a all the critical points of f are non-degenerate.

Morse functions are particular nice because of the following result due to M. Morse (which also holds in higher dimensions).

Proposition 3.12 (The Morse Lemma)**.** *Let p be a non-degenerate critical point of $f\colon M^2 \to \mathbf{R}$. Then there is a local coordinate system x, y centered at p so that*

(1) *If the Hessian $D^2 f_p(\,,\,)$ is positive definite then*
$$f = f(p) + x^2 + y^2$$
in a neighborhood of p

(2) *If the Hessian $D^2 f_p(\,,\,)$ is negative definite then*
$$f = f(p) - x^2 - y^2$$
in a neighborhood of p

(3) *If the Hessian $D^2 f_p(\,,\,)$ is indefinite then*
$$f = f(p) + x^2 - y^2 = f(p) + (x+y)(x-y)$$
in a neighborhood of p

Proof. See [1, Thm 6.9 p. 65]. $\square$

We now commute the Hessian of the functions f_e given by (3.1).

$$\begin{aligned}
D^2 f_e(X,Y) &= X df_e(Y) - df_e(\nabla_X Y) \\
&= -(X\langle AY, e\rangle - \langle A(\nabla_X Y), e\rangle) \\
&= -(\langle \overline{\nabla}_X (AY), e\rangle + \langle AY, \overline{\nabla}_X e\rangle - \langle A(\nabla_X Y), e\rangle) \\
&= -(\langle \nabla_X (AY) + I\!I(X,Y)\mathbf{n}, e\rangle - \langle A(\nabla_X Y), e\rangle) \\
&= -(\langle \nabla_X (AY) - A(\nabla_X Y) e\rangle + \langle I\!I(X,Y)\mathbf{n}, e\rangle) \\
&= -\langle (\nabla_X A)(Y), e\rangle - I\!I(X,Y)\langle \mathbf{n}, e\rangle \qquad (3.6)
\end{aligned}$$

This leads to

Proposition 3.13. *Let p be a point where $f_e(p) = 0$ (so that p is on the shadow boundary S_e). Assume that p is a critical point of f_e, then the Hessian of f_e is given by*

$$D^2 f_e(X,Y) = -\langle (\nabla_X A)(Y), e\rangle = -(\nabla_e I\!I)(X,Y). \qquad (3.7)$$

Therefore the point p is a non-degenerate critical point if and only if the symmetric bilinear form $(\nabla_e I\!I)(X,Y)$ is non-degenerate.

Proof. At a point of where $f_e = 0$ we have $\langle \mathbf{n}, e\rangle = 0$ and so (3.6) reduces to (3.7). $\square$

As a concrete example let M^2 be the graph of $z = cx^2/2 + q(x,y)$ where $q(x,y)$ is a function that vanishes to third order at the origin. That is $q(0,0) = q_x(0,0) = q_y(0,0) = q_{xx}(0,0) = q_{xy}(0,0) = q_{yy}(0,0)$. Then the second fundamental form of M^2 at the origin is

$$I\!I = c dx^2.$$

and so if $c \neq 0$ then the origin is a parabolic point of M^2. To see what the parabolic locus looks like near the origin recall that for a graph $z = \varphi(x,y)$ the Gauss curvature is given by

$$K(x,y) = \frac{\varphi_{xx}\varphi_{yy}^2 - \varphi_{yy}^2}{(1+\varphi_x^2+\varphi_y^2)^2}.$$

See for example [2, p. 95]. In our case this becomes

$$K = \frac{q_{yy}(c+q_{xx}) - q_{xy}^2}{(1+(cx+q_x)^2+q_y^2)^2}.$$

As q vanishes to order three the function q_x and q_y vanish to order two and q_{xx}, q_{xy}, and q_{yy} vanish to order one. Thus we see that

$$K = cq_{yy} + O(2)$$

where $O(2)$ represents terms that vanish to order at least two. Thus if $c =\neq 0$ and either $q_{yyx}(0,0) \neq 0$ or $q_{yyy}(0,0) \neq 0$ (that is $(dq_{yy})_{(0,0)} \neq 0$) then we have that $K(0,0) = 0$ and $dK_{(0,0)} \neq 0$.

Let e be in the direction of the positive y axis then a calculation shows that

$$f_e = \frac{-q_y}{\sqrt{1+(cx+q_x)^2+(q_y)^2}}.$$

(This is just the y-component of the normal.) As q vanishes to order 3 q_y vanishes to order 2 and so the origin is a critical point of f_e. Then by messy calculation we

$$(D^2 f_e)_{(0,0)} = -(q_{xxy}\, dx^2 + 2q_{xyy}\, dxdy + q_{yyy}\, dy^2).$$

(I confess to having used Maple to do this calculation.)

This can be summarized as

Lemma 3.14. *If $e = (0,1,0)$ and q vanishes to third order at the origin and*

(1) $c \neq 0$,
(2) $q_{yyx}^2(0,0) + q_{yyy}^2 \neq 0$,
(3) $q_{xxy}(0,0)q_{yyy}(0,0) - q_{xyy}^2(0,0) \neq 0$.

Then

(1) $K(0,0) = 0$,
(2) $dK_{(0,0)} \neq 0$,
(3) $Ae = 0$
(4) *The function f_e has a non-degenerate critical point at the origin.*

The conditions on functions q vanishing to order three at the origin are open and dense in the C^3 topology on such functions.

Definition 3.15. Let $\mathcal{N}$ be the set of all compact oriented surfaces M^2 immersed in $\mathbf{R}^3$ so that

(1) $M^2 \in \mathcal{K}$
(2) The number of points p in the parabolic locus $\{K = 0\}$ of M^2 with $\nabla_e I\!I(X, Y)$ a degenerate quadratic form (where e is a unit vector with $A_p e = 0$) is finite. □

Proposition 3.16. *The set $\mathcal{N}$ is open and dense in the C^3 topology on the space of immersions. If $M^2 \in \mathcal{N}$ then for all but finitely many vectors e we have that either*

(1) 0 *is a regular value of f_e so that the shadow boundary S_e is a smooth curve, or*
(2) *any critical point p of f_e on the shadow boundary S_e is a non-degenerate critical point and thus by the Morse Lemma 3.12 we have a norm form for f_e near p (and this also a normal form for the shadow boundary S_e near p).*

Proof. That $\mathcal{N}$ is open and dense in the C^3 topology follows from the calculations of Lemma 3.14 and Thom's Jet Transversality Theorem [1, Thm 4.9 p. 54].

If M^2 is in $\mathcal{N}$ and let $x_1, \dots, x_n$ be the finite number of points x on the parabolic locus $c := \{K = 0\}$ where $(\nabla_v I\!I)_x(\ ,\ )$ is degenerate (here v is a unit vector so that $A_x v = 0$). Let $v_1, \dots, v_n$ be unit vectors so that v_i is tangent to M at x_i and $A_{x_i} v_i = 0$. Then if e is a unit vector so that f_e has a degenerate critical point x on the shadow boundary S_e then by Proposition 3.1 the point x is on $\{K = 0\}$ and $A_x e = 0$. By Proposition 3.13 as the Hessian of f_e is degenerate at x we have that $x = x_i$ for some i and that $e = \pm v_i$. Therefore the only directions where f_e has a degenerate critical point on S_e is when e is one of the $2n$ vectors $\pm v_1, \dots, \pm v_n$. This completes the proof. □

4. Light Sources at a Finite Distance.

Let $a \in \mathbf{R}^3$ and for a smooth oriented surface M^2 in $\mathbf{R}^3$ define $F_a \colon M^2 \to \mathbf{R}^3$ by

$$F_a(x) := \langle \mathbf{n}(x), a - x \rangle.$$

Then the set $\{x \in M^2 : F_a(x) > 0\}$ can be thought of as the set of points on M^2 that are illuminated be a light source at a. In this case we define the ***shadow boundary*** by

$$S_a := \{x : F_a(x) = 0\}.$$

As in the case of a light source coming from infinity we wish to investigate when S_a is a well behaved subset of M^2. We first compute the derivative of

F_a. Let X be tangent to M^2. Then

$$\begin{aligned} dF_a(X) = X\langle \mathbf{n}, a-x\rangle &= \langle \overline{\nabla}_X \mathbf{n}, a-x\rangle + \langle \mathbf{n}, \overline{\nabla}_X(a-x)\rangle \\ &= \langle -AX, a-x\rangle + \langle \mathbf{n}, -X\rangle \\ &= \langle AX, x-a\rangle \end{aligned}$$

where we have used that $\overline{\nabla}_X a = 0$ and $\overline{\nabla}_X x = X$. This leads to

Proposition 4.1. *Let $p \in S_a$. Then*

$$(dF_a)_p(X) = \langle X, A(p-a)\rangle = I\!I(X, p-a\rangle.$$

Therefore at a point p of the shadow boundary S_a we have $(dF_a)_p = 0$ if and only if $A(p-a) = 0$. If $A(p-a) \neq 0$ then S_a is a smooth curve near p.

Exercise 4.2. Give a proof of this along the lines of Proposition 3.1. $\square$

Remark 4.3. This shows that at smooth points p of S_a the tangent space to S_a is defined by $\{X \in T(M)_p : I\!I(X, p-a) = 0\}$. Therefore, just as in the case of the light source at infinity, the tangent to the shadow boundary and the direction of the light source are conjugate directions. Then as in Remark 3.3 this can be used to show that the conformal class of the second fundamental form is invariant under projective transformations. $\square$

Define the class of immersed surfaces $\mathcal{K}$ as above and let $\mathcal{B}$ be defined by (3.4). Then given $M^2 \in \mathcal{K}$ let $\mathcal{L}$ be

$$\mathcal{L} := \{p + tu : (p,u) \in \mathcal{B}, t \in \mathbf{R}\}.$$

This is just the union of the lines $t \mapsto p + tu$ with $(p,u) \in \mathcal{B}$. As $\mathcal{B}$ is a smooth curve the set $\mathcal{L}$ is a ruled surface in $\mathbf{R}^3$ (which in general will have a non-empty singular set).

Proposition 4.4. *Let $M^2 \in \mathcal{M}$. Then for all $a \in \mathbf{R}^3 \setminus (\mathcal{L} \cup M^2)$ the shadow boundary S_a is a smooth embedded curve in M^2. Thus for almost all $a \in \mathbf{R}^3$ the shadow boundary S_a is a smooth curve.*

Exercise 4.5. Prove this Proposition. Hint: Modify the proof of Proposition 3.7. $\square$

Proposition 4.6. *Let $M^2 \in \mathcal{K}$ and let $a \in \mathbf{R}^3\setminus$ so that the set*

$$\mathcal{V} := \{p \in M^2 : A_p(p-a) = 0\}$$

is finite. Then the shadow boundary S_e is a generalized graph on $\mathcal{V}$.

Exercise 4.7. Prove this. Hint: See the proof of Proposition 3.9 $\square$

Exercise 4.8. Show the Hessian of $F_a \colon M^2 \to \mathbf{R}^3$ is given by

$$(D^2F_a)_p(X,Y) = -I\!I(X,Y)F_a(p) + (\nabla_{p-a})(X,Y) + I\!I(X,Y)$$

for any $p \in M^2$. If $p \in S_a = \{F_a = 0\}$ show this simplifies to

$$(D^2F_a)_p(X,Y) = (\nabla_{p-a}I\!I)(X,Y) + I\!I(X,Y).$$

Therefore if $p \in S_a$ is a critical point of F_a (that is $A_p(p-a) = 0$) then p is a non-degenerate critical point if and only if the quadratic form

$$(X, Y) \mapsto (D^2 F_a)_p(X, Y) = (\nabla_{p-a} I\!I)(X, Y) + I\!I(X, Y)$$

is non-degenerate. □

Exercise 4.9. Let $b_1(\, , \,)$ and $b_2(\, , \,)$ be two symmetric bilinear forms on a two dimensional vector space V. Assume that b_1 has rank one and b_2 is non-degenerate. Then there is exactly two real numbers so that the quadratic form $b_1(\, , \,) - tb_2(\, , \,)$ is degenerate and one of these values if $t = 0$. (It is possible that $t = 0$ with multiplicity two and that there are no non-zero t with $tb_2 + b_1$ degenerate). Thus there is at most one non-zero value of t so that $b_1(\, , \,) + tb_2(\, , \,)$ is degenerate. Hint: Let $P(t) = \det(b_1 - tb_2)$. Then $b_1(\, , \,) - tb_2(\, , \,)$ is degenerate if and only if $P(t) = 0$. Clearly $P(0) = \det(b_1) = 0$ as b_1 has rank one.

Proposition 4.10. *Let $M^2 \in \mathcal{N}$ (where $\mathcal{N}$ is as in Definition 3.15). For any $(p, u) \in \mathcal{B}$ (with $\mathcal{B}$ given by (3.4)) let L_p be the line*

$$L_p := \{p + tu : t \in \mathbf{R}\}.$$

Then for all but finitely many points $p \in \{K = 0\}$ there are only two point $a \in L_p$ so that p is a degenerate critical point of F_a and one of these points is $a = p$.

Exercise 4.11. Prove this. Hint: Any point a on the line P_p is of the form $a = p + tu$ and so $p - a = -tu$. But then for $X, Y \in T(M)_p$ we have

$$(D^2 F_a)_p(X, Y) = (\nabla_{p-a} I\!I)(X, Y) + I\!I(X, Y) = -t(\nabla_u I\!I)_p(X, Y) + I\!I_p(X, Y).$$

The rank of $I\!I_p(\, , \,)$ is one by Proposition 3.6 and $(\nabla_u I\!I)_p(\, , \,)$ is non-degenerate for all but finitely many $p \in \{K = 0\}$ as $M^2 \in \mathcal{N}$. Now use Exercise 4.9. □

References

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