ESTIMATES ON THE GRAPHING RADIUS OF SUBMANIFOLDS AND THE INRADIUS OF DOMAINS

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Abstract. Let M be a submanifold of $\mathbf{R}^n$ be a complete immersed submanifold of $\mathbf{R}^n$ so that the principle curvatures in all normal directions are ≤ 1. Then any open geodesic ball of radius π in M is embedded in $\mathbf{R}^n$. Moreover M graphs over any of its tangent planes on a closed ball (in the tangent plane) of radius 1. Several related estimates are given and used to prove inradius estimates for domains $D \subset \mathbf{R}^n$ whose boundaries have all principle curvatures in the interval $[-1, 1]$. For example the theorem of Lagunov any such domain with ∂D connected must have inradius at least $2/\sqrt{3} - 1 \approx .15470038\ldots$ is proven.

Contents

1. Introduction

This notes are an attempt to extend some of the results of [5] to higher dimensions and in particular to prove the conjecture that, with the notation below, that if $D \subset \mathbf{R}^n$ is a bounded domain which is starlike with respect to the origin and so that all principle curvatures λ_i of ∂D satisfy $|\lambda_1| \leq 1$, then the inradius of D is ≥ 1. Not much progress was made on this conjecture, and all the results here are in papers of Alexander and Bishop [2, 1], Howard and Treibergs [5], Lagunov [6, 7, 8], Lagunov-Fyet [9], and Pestov and Iomin [10]. Farther references can be found in these papers.

2. Curves of Bounded Curvature

Let $c(t)$ be a unit speed curvature in $\mathbf{R}^n$ with curvature ≤ 1. That is

$$|c'(t)| = 1 \tag{2.1}$$

and

$$|c''(t)| \leq 1. \tag{2.2}$$

Choose coordinates so that $c(0) = 0$ and $c'(0) = e_1$ (where $e_1, \dots, e_n$ is the standard orthonormal basis of $\mathbf{R}^n$). Write $c(t)$ as

$$c(t) = (x(t), y(t)) \in \mathbf{R} \times \mathbf{R}^{n-1}. \tag{2.3}$$

Where $x(t) \in \mathbf{R}$ and $y(t) \in \mathbf{R}^{n-1}$ and

$$c(0) = (x(0), y(0)) = (0, 0) \tag{2.4}$$

$$c'(0) = (x'(0), y'(0)) = (1, 0) \tag{2.5}$$

$$|c'(t)| = x'(t)^2 + |y'(t)|^2 = 1 \tag{2.6}$$

$$|c''(t)| = x''(t)^2 + |y''(t)|^2 \leq 1 \tag{2.7}$$

Proposition 2.1. *With notation as above we have for* $0 \leq t \leq \pi$

$$x'(t) \geq \cos(t) \tag{2.8}$$

$$|y'(t)| \leq \sin(t) \tag{2.9}$$

$$x(t) \geq \sin(t) \tag{2.10}$$

$$y(t) \leq 1 - \cos(t) \tag{2.11}$$

If equality holds at $t = t_0$ *in any one of these then for some unit vector* $u \perp e_1$

$$c(t) = \cos(t)e_1 + \sin(t)u \tag{2.12}$$

holds for $0 \leq t \leq t_0$. *That is the segment of the curve between* 0 *and* t_0 *is a segment of a great circle of a sphere tangent to* e_1 *at the origin.*

Proof. Because $x'(t)^2 + |y'(t)|^2 = 1$ there is a continuous function $\theta(t)$ defined on the real line with $\theta(0) = 0$ and

$$x'(t) = \cos(\theta(t)), \qquad |y'(t)| = \sin(\theta(t)) \tag{2.13}$$

Note that because of the norm on y' the function need not be smooth, however it is Lipschitz and therefore differentiable almost everywhere and the usual rules of calculus work almost everywhere. Define a plane curve γ by

$$\gamma(t) = (\cos(\theta(t)), \sin(\theta(t))) = (x'(t), |y'(t)|) \tag{2.14}$$

Whence at all points where $y' \neq 0$

$$\gamma'(t) = (-\sin(\theta(t)), \cos(\theta(t)))\theta'(t) = (x''(t), \frac{\langle y'(t), y''(t)\rangle}{|y'(t)|}) \tag{2.15}$$

and so at these points we can use (2.7)

(2.16)
$$|\gamma'(t)|^2 = (\theta'(t))^2 = (x''(t))^2 + \frac{\langle y'(t), y''(t)\rangle^2}{|y'(t)|^2} \leq (x''(t))^2 + |y''(t)|^2 \leq 1$$

At points where both y' and y'' vanish it is not hard to see that $\theta' = 0$. This leaves the points where $y' = 0$, but $y'' \neq 0$. At these points the derivative θ' does not exist and as θ' exists almost every where this is a set of measure zero. Therefore we have

$$|\theta'(t)| \leq 1 \tag{2.17}$$

almost everywhere. This can be integrated to give

$$\theta(t) \leq t \tag{2.18}$$

for all $t \geq 0$. This implies

$$x'(t) = \cos(\theta(t)) \geq \cos(t) \qquad \text{for} \qquad 0 \leq \pi \tag{2.19}$$

and if equality holds at $t = t_0$ then $x'(t) = \cos(t)$ on all of $[0, t_0]$. This prove the first part of the proposition.

For the second part note that

$$|y'|^2 = 1 - (x')^2 \geq 1 - \cos^2(t) = \sin^2(t). \tag{2.20}$$

The last two parts now follow by integration. □

Proposition 2.2. *If c is a curve in $\mathbf{R}^n$ with curvature ≤ 1 and length $\leq 2\pi$, then c is imbedded unless it has length 2π and is a standard circle. Moreover the endpoints of c satisfy the estimate (2.21) below.*

Proof. Let L be the length of c, and choose a parameterization $c : [-L/2, L/2] \to \mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$ with $c(0) = (0, 0)$ and $c'(0) = (1, 0)$. With the notation as above we get $c(t) = (x(t), y(t))$ will satisfy

$$|c(L/2) - c(-L/2)| \geq |x(L/2) - x(-L/2)| \geq 2\sin(L/2) \tag{2.21}$$

□

Proposition 2.3. *If a curve c of length $L \leq \pi$ has curvature ≤ 1 and is tangent to a unit sphere at one of its end points, then c is disjoint from the interior of the sphere. If some point of the curve other that the end point is on the sphere, then the part of the curve between this point and the endpoint of tangency is a part of a great circle of the sphere.*

Proof. We use the notation of the first proposition. Then $c(0) = (0, 0)$ and $c'(0) = (1, 0)$. Therefore the center of the sphere is at a point $(0, u)$ for some unit vector $u \in \mathbf{R}^{n-1}$. Thus

$$
\begin{aligned}
|c(t)-(0,u)|^2 &= x(t)^2 + |u-y(t)|^2 \\
&\geq x(t)^2 + (|u|-|y(t)|)^2 \\
&\geq \sin^2(t) + (1-1+\cos(t))^2 \\
&= 1 \qquad (2.22)
\end{aligned}
$$

which is exactly the statement of the proposition. □

3. Submanifolds of Bounded Curvature

Let $\mathcal{K} = \mathcal{K}(m,n)$ be the class m dimensional immersed submanifolds of $\mathbf{R}^n$ that have all principle curvatures in all normal directions ≤ 1. To be a little more precise let $\overline{\nabla}$ be the usual flat connection on $\mathbf{R}^n$, and for vector fields X, Y tangent to a submanifolds M^m of $\mathbf{R}^n$ split $\overline{\nabla}_X Y$ into its tangent part $\nabla_X Y$ and the part $I\!I(X,Y)$ normal to TM. Then ∇ is the connection on M^m and $I\!I$ is the second fundamental form of M^m in $\mathbf{R}^n$. The Gauss equation is then

$$\overline{\nabla}_X Y = \nabla_X Y + I\!I(X,Y). \tag{3.1}$$

With this notation the submanifold M^m is in $\mathcal{K}$ if and only if

$$|I\!I(X,Y)| \leq |X||Y| \tag{3.2}$$

for all vectors tangent to M^m. Note then for a curve c in M^m this leads to the relation

$$c'' = \overline{\nabla}_{c'} c' = \nabla_{c'} c' + I\!I(c',c') \tag{3.3}$$

A unit speed curve in M^m is a geodesic if and only if $\nabla_{c'} c' = 0$ so we have

Proposition 3.1. *If c is a unit speed geodesic in $M^m \in \mathcal{K}$ then the curvature of c as a space curve is ≤ 1.*

If X and Y are orthogonal unit vectors tangent to M^m at some point and $K(X,Y)$ is the sectional curvature of the two plane spanned by X and Y then the Gauss curvature equation is

$$K(X,Y) = \langle I\!I(X,X), I\!I(Y,Y)\rangle - |I\!I(X,Y)|^2 \tag{3.4}$$

This leads to the following:

Proposition 3.2. *The sectional curvatures of a manifold M in $\mathcal{K}$satisfy*

$$-2 \leq K_M \leq 1 \tag{3.5}$$

Remark 3.3. These estimates are sharp. The upper is exact on the unit sphere. The Lower bound is sharp on complex curves in $\mathbf{C}^2 = \mathbf{R}^4$ at points where the Gauss curvature is -2. This second observation is due to Anton Schep. □

Proof. Let P be a two plane tangent to M and let X, Y be an orthonormal basis of P. Then from the Gauss curvature equation

$$\begin{aligned} K(X,Y) &= \langle I\!I(X,X), I\!I(Y,Y)\rangle - |I\!I(X,Y)|^2 \\ &\le \langle I\!I(X,X), I\!I(Y,Y)\rangle \\ &\le |I\!I(X,X)||I\!I(Y,Y)| \\ &\le |X|^2|Y|^2 \\ &\le 1 \end{aligned} \tag{3.6}$$

which proves the upper bound.

To prove the lower bound first note that

$$4|I\!I(X,Y)| = |I\!I(X+Y,X+Y) - I\!I(X-Y,X-Y)| \tag{3.7}$$

$$\le |X+Y|^2 + |X-Y|^2 = 2+2 = 4 \tag{3.8}$$

and so $|I\!I(X,Y)| \le 1$

$$\begin{aligned} K(X,Y) &= \langle I\!I(X,X), I\!I(Y,Y)\rangle - |I\!I(X,Y)|^2 \\ &\ge -|I\!I(X,X)||I\!I(Y,Y)| - 1^2 \\ &\ge -1-1 = -2 \end{aligned} \tag{3.9}$$

□

Proposition 3.4. *If M^m is in $\mathcal{K}$ and the geodesic ball of radius π about $x_0 \in M^m$ is complete, then the injectivity radius of M^m at x_0 is $\ge \pi$.*

Proof. We argue that the cut locus $\mathcal{C}$ of x_0 in M^m is disjoint from $B(x_0, \pi)$. Any point x of $\mathcal{C}$ is either a focal point or there are two or more minimizing geodesics form x_0 to x. If x is a focal point then by proposition 3.2 the sectional curvature is ≤ 1 and so by a standard comparison theorem the distance from x to x_0 is at least π and thus x is not in $B(x_0, \pi)$. Let x be the point of $\mathcal{C}$ closest to x_0, and assume (toward a contradiction) that x is in $B(x_0, \pi)$. Then by a theorem of Klingenberg (cf. [3, lemma 5.6 p. 95] there is a a geodesic segment γ starting and ending at x_0 that has x as a midpoint. However the length of this is less than 2π and by proposition 3.1 the curvature of γ as a space curve is ≤ 1. This contradicts proposition 2.2 that any such curve must be imbedded. □

Proposition 3.5. (First Schur Lemma for Submanifolds) *Let $M^m \in \mathcal{K}$ and let $x_0 \in M^m$. Assume that the closed geodesic ball of radius π about x_0 in M^m is complete. Then the open ball $B(x_0, \pi)$ in M^m is an imbedded submanifold of $\mathbf{R}^n$. If a unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$ is tangent to M^m at x_0 then the ball $B(x_0, \pi)$ is disjoint from the interior of this sphere. If some point x of $B(x_0, \pi)$ is on $\mathbf{S}^{n-1}$ then the geodesic segment $\overline{x_0 x}$ of M^m also lies on $\mathbf{S}^{n-1}$ as a geodesic of $\mathbf{S}^{n-1}$.*

Remark 3.6. The proof of this given here (based on proposition 3.4) is due to Lars Andersson. I know of no other proof, however it is the type of thing that one expects to be well known to a certain school of Russian mathematicians. □

Remark 3.7. In this result we mean that $B(x_0, \pi)$ is imbedded in the strong sense that the map $\exp : TM_{x_0} \to M^m \subset \mathbf{R}^n$ is an injective immersion on the open ball of radius π in TM_{x_0}. □

Proof. This follows from the the last proposition, propositions 2.1 and 2.3 and that geodesics of M^m have curvature ≤ 1. □

Proposition 3.8. (Schur Lemma on the Graphing Radius) *Let $M^m \subset \mathbf{R}^n$ be an immersed submanifold in the class $\mathcal{K}$. Assume that M^m is tangent to $\mathbf{R}^m$ at the origin, and that the closed geodesic ball $\overline{B}(0,\pi)$ of radius π about 0 is complete. Then there is a smooth function $f : \overline{B}^{\mathbf{R}^m}(0,1) \to \mathbf{R}^{n-m}$ so that*

$$\operatorname{Graph}(f) := \{(x, f(x)) : x \in \overline{B}^{\mathbf{R}^m}(0,1)\} \subseteq \overline{B}^{M^m}(0,\pi) \subseteq M^m \tag{3.10}$$

That is if M^m is in $\mathcal{K}$ then it is a graph over each of is tangent spaces on a disk of radius at least one.

Proof. The proof will only be give in the case of hypersurfaces. I know of no proof in the general case, but believe the ideas here, with some extra notational problems, should handle the higher codimension case also.

We can assume that M^m is tangent to $\mathbf{R}^m$ at the origin. (Where we have written $\mathbf{R}^n = \mathbf{R}^{m+1} = \mathbf{R}^m \times \{0\}$.) Then there is a neighborhood U of 0 so that M^m is a graph over U. That is there is a smooth function $f : U \to \mathbf{R}$ with

$$f(0) = 0, \qquad df_0 = 0 \tag{3.11}$$

and so that

$$F(x) := (x, f(x)) \tag{3.12}$$

is a parameterization of M^m near 0. We want to show that U can be taken to include the open unit ball of $\mathbf{R}^m$ centered at the origin.

Recall the basic imbedding invariants of M^m are give in this parameterization as follows: $dF = (dx, df)$ and so the induced metric on M^m is given by

$$g = \langle dF, dF \rangle = |dx|^2 + df^2$$

where $|X|$ is the usual Euclidean length of a vector $X \in \mathbf{R}^m$. The unit normal is

$$\eta(x) = \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}. \tag{3.13}$$

The Hessian of F is given by

$$D^2 F = (0, D^2 f) \tag{3.14}$$

and so the second fundamental form of the parameterization is

$$I\!I = \langle D^2 f, \eta \rangle = \frac{D^2 f}{\sqrt{1 + |\nabla f|^2}} \tag{3.15}$$

As M^m is of class $\mathcal{K}$ we have the estimate $|I\!I(X,Y)| \le g(X,X)^{1/2} g(Y,Y)^{1/2}$, whence,

$$\begin{aligned}
\frac{|D^2 f(X,Y)|}{\sqrt{1+|\nabla f|^2}} &\le g(X,X)^{\frac{1}{2}} g(Y,Y)^{\frac{1}{2}} \\
&= (|X|^2 + df(X)^2)^{\frac{1}{2}} (|Y|^2 + df(Y)^2)^{\frac{1}{2}} \\
&\le ((1+|\nabla f|^2)|X|^2)^{\frac{1}{2}} ((1+|\nabla f|^2)^{\frac{1}{2}}) \\
&= (1+|\nabla f|^2)|X||Y| \qquad (3.16)
\end{aligned}$$

Let $c(t)$ be a smooth curve in U and $e_1, \dots, e_m$ an orthonormal basis for $\mathbf{R}^m$. Then

$$\begin{aligned}
\frac{d}{dt}|df_{c(t)}|^2 &= \frac{d}{dt} \sum df_{c(t)}(e_i) df_{c(t)}(e_i) \\
&= 2 \sum df_{c(t)}(e_i) D^2 f(c'(t), e_i) \\
&= 2 D^2_{c(t)} \left(c'(t), \sum df_{c(t)}(e_i) e_i \right) \\
&= D^2 f(c'(t), \nabla f) \qquad (3.17)
\end{aligned}$$

Which can be used to show that at points where $df_{c(t)} \ne 0$ that

$$\frac{d}{dt}|df_{c(t)}| = D^2 f\left(c'(t), \frac{\nabla f}{|\nabla f|} \right) \qquad (3.18)$$

In what follows we will use this formula along curves where it may be that $df_{c(t)} = 0$ at some points. When we do this in will always be the case that the result of the computation is going to be integrated. As the function $t \mapsto |df_{c(t)}|$ is Lipschitz it will always be differentiable almost everywhere, and the formula

$$\int_a^b \frac{d}{dt}|df_{c(t)}|\, dt = |df_{c(b)}| - |df_{c(a)}| \qquad (3.19)$$

will hold. This will be enough to get the estimates we need.

Now consider the function $\Phi : \mathbf{R} \to \mathbf{R}$ given by

$$\Phi(s) = \frac{s}{\sqrt{1+s^2}} \quad \text{so that} \quad \Phi'(s) = \frac{1}{(1+s^2)^{\frac{3}{2}}} \qquad (3.20)$$

Let $u \in \mathbf{R}^m$ be a unit vector and define a curve in $\mathbf{R}^n$ by $c(t) = tu$. Then $c'(t) = u$ and so we can use (3.16) and (3.18)

$$\begin{aligned}
\left| \frac{d}{dt} \Phi(|df_{c(t)}|) \right| &= \left| \Phi' |df_{c(t)}| D^2 f\left(u, \frac{\nabla f}{|\nabla f|} \right) \right| \\
&= \frac{|D^2 f(u, |\nabla f|^{-1} \nabla f)|}{(1+|\nabla f|^2)^{\frac{3}{2}}} \\
&\le 1 \qquad (3.21)
\end{aligned}$$

where at the last step we have used that u and $|\nabla f|^{-1}\nabla f|$ are both unit vectors.

But $\Phi(0) = 0$ so (3.21) can be integrated to give

$$|\Phi(df_{c(t)})| = \frac{|df_{c(t)}|}{\sqrt{1 + |df_{c(t)}|}} \leq |t| = |c(t)| \tag{3.22}$$

Setting $x = c(t)$, this can be solved for $|df_x|$ to give the inequality

$$|df_x| \leq \frac{|x|}{\sqrt{1 - |x|^2}}. \tag{3.23}$$

This estimate will hold on any ball $B(0, r)$ with $r < 1$ provided that $B(0, r)$ is contained in U. However note that if this holds on the closed ball of radius $r < 1$, then the tangent plane to M^m at the points over the boundary of $B(0, r)$ do not have vertical tangent spaces as $|df|$ is finite at these points. Thus the implicit function theorem lets us write M^m as a graph over a larger ball, and as long as the radius of this larger ball is less than one the estimate (3.23) will hold. Thus a straight forward analytic continuation argument shows that M^m is a graph over all of $B(0, 1)$.

4. Applications to Gradient Estimates and Reverse Isoperimetric Inequalities

We now give some applications of the results of the last section.

Proposition 4.1. (Gradient Estimates for Hypersurfaces) *Let D be a bounded domain in $\mathbf{R}^n$ and assume that the boundary $M := \partial D$ is of class $\mathcal{K}$. Define a function $\rho : M \to [0, \infty)$ by*

$$\rho(x) = |x|^2 \tag{4.1}$$

and let $\nabla\rho$ be the gradient of ρ with respect to the induced metric on M.

1. *If D contains the ball $B(0, r)$ then*

$$|\nabla\rho|^2 \leq (\rho - r^2)((r + 2)^2 - \rho) \qquad \text{whenever } \rho \leq r^2 + 2r \tag{4.2}$$

2. *If D is contained in $B(0, R)$ with $R > 2$, then*

$$|\nabla\rho|^2 \leq (R^2 - \rho)(\rho - (R - 2)^2) \qquad \text{whenever } \rho \geq R^2 - 2R \tag{4.3}$$

Proof. To prove the first of these note that if (4.2) is false at a point x then both the unit spheres tangent to $M = \partial D$ at x interest the interior of the ball $B(0, r)$. (The picture is exactly the same as Figure 1 "Crash to earth" in the paper *A reverse isoperimetric inequality, stability and extremal theorems for plane curves with bounded curvature*). Now use the Schur lemma on the graphing radius to write $M = \partial D$ as a graph over its tangent space at the point x. As both the unit spheres tangent to M at x interest $B(0, r)$ and M is graph over the unit disk of the tangent space to M at these points that is sandwiched between these two spheres (by the first Schur lemma) this implies that M must also interest $B(0, r)$, a contradiction.

The proof of the second inequality is similar. If (4.3) is false at the point x, then both the unit spheres tangent to M intersect the complement of

$B(0,R)$. Then the Schur lemma on the graphing radius will imply that $M=\partial D$ also intersects the complement of $B(0,R)$. □

Remark 4.2. This result is true for higher codimension submanifolds of $\mathbf{R}^n$ in the following form: If M^m is a compact submanifold of $\mathbf{R}^n$ of class $\mathcal{K}$ and M^m is disjoint from the ball $B(0,r)$ (resp. contained in $B(0,R)$) then the inequality (4.2) (resp. inequality (4.3)) holds (where ρ is defined as before). The proof is exactly as above. □

Corollary 4.3. *If D is a domain in $\mathbf{R}^n$ with boundary of class $\mathcal{K}$ and for some $R<3$ we have*

$$B(0,1)\subseteq D\subseteq B(0,R), \tag{4.4}$$

then D is starlike with respect to the origin.

Proof. Let $M=\partial D$. If D is not starlike with respect to the origin then there is a point x of M where the (affine) tangent space TM_x passes through the origin. At such a point the gradient of the restriction to M of the function $x\mapsto|x|$ has norm one. This implies that the function $\rho(x)=|x|^2$ satisfies

$$|\nabla\rho|^2=4\rho. \tag{4.5}$$

But if $x\in M$ then either $|x|<\sqrt{3}$ (*i.e.* $\rho(x)<3$) in which case (4.2) with $r=1$ implies

$$|\nabla\rho|^2\leq(\rho-1)(9-\rho)<4\rho \qquad \text{when } \rho<3 \tag{4.6}$$

or $\sqrt{3}\leq|x|\leq R$ (*i.e.* $3\leq\rho(x)\leq R^2$) then by (4.3) (and using both $R<3$ and $3\leq\rho$)

$$|\nabla\rho|^2\leq(R^2-\rho)(\rho-(R-2)^2)<4\rho. \tag{4.7}$$

This means that it is impossible for (4.5) to hold and therefore D must be starlike with respect to the origin. □

Proposition 4.4. (Reverse Isoperimetric Inequality) *Let D be a bounded domain in $\mathbf{R}^n$ whose boundary is of class $\mathcal{K}$. Let A be the surface area of ∂D and V the volume of D. If D is starlike with respect to the origin then the inequality*

$$A\leq nV \tag{4.8}$$

holds. There are also sharp lower bounds for A and V

$$A\geq A(\mathbf{S}^{n-1}) \qquad V\geq V(B(0,1)) \tag{4.9}$$

Equality in any one of these inequalities implies D is a ball of radius one.

Proof. Let H be the mean curvature of ∂D with respect to the outward unit normal $\mathbf{n}$, and let $p(x)=\langle x,\mathbf{n}\rangle$ be the support function of ∂M. Then the Minkowski formulas

$$A=-\int_{\partial D}Hp\,dA, \qquad nV=\int_{\partial D}p\,dA \tag{4.10}$$

hold. As D is starlike the function p is positive, and as ∂D is in $\mathcal{K}$ the mean curvature satisfies $|H| \le 1$. Therefore

$$A = -\int_{\partial D} Hp\,dA \le \int_{\partial D} p\,dA = nV. \tag{4.11}$$

If equality holds then $H \equiv -1$, which implies that ∂D is a sphere of radius one.

The isoperimetric inequality in $\mathbf{R}^n$ is

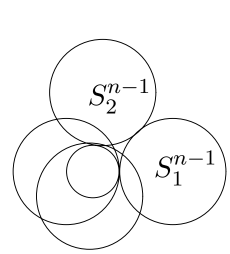
$$A(\mathbf{S}^{n-1})^n V(D)^{n-1} \le V(B(0,1))^{n-1}(A(\partial D))^n \tag{4.12}$$

Using the relation $A(\mathbf{S}^{n-1}) = nV(B(0,1))$ and the inequalities (4.8) and (4.12) leads at once to (4.9). If equality holds in (4.9) then it must also hold in (4.8) and so again D must be a ball of radius one. □

5. Applications to Inradius Estimates

Proposition 5.1. *Let D be a domain in $\mathbf{R}^n$ with boundary of class $\mathcal{K}$. Assume that the interior of the ball $B(0,r)$ is contained in D, and that $\partial B(0,r)$ is tangent to ∂D at the points x_1 and x_2. If $r < 1$ then the unit spheres tangent to D at x_1 and x_2 are disjoint.*

Proof. Let $\mathbf{S}_i^{n-1}$ be the unit sphere tangent to D at x_i. Assume toward a contradiction that these spheres intersect. Then in the figure to the right let the small circle represent the ball $B(0,r)$ and let $\mathbf{S}_1^{n-1}$ and $\mathbf{S}_2^{n-1}$ be as shown. Then by the the first Schur lemma (proposition 3.5) the boundary ∂D must stay between the sphere $\mathbf{S}_i^{n-1}$ and the sphere tangent to the boundary ∂D at x_i. Then using the lemma on the graphing radius (proposition 3.8) we see that the two "branches" of ∂D, the one tangent to $\mathbf{S}_1^{n-1}$ and the piece tangent to $\mathbf{S}_2^{n-1}$ must interest. Let c be the minimizing geodesic segment in ∂D from x_1 to x_2. Then the length of c is less that π, but both ends of c are on the sphere $\partial B(0,r)$. By the results of section 2 this is impossible. This contradiction completes the proof. □

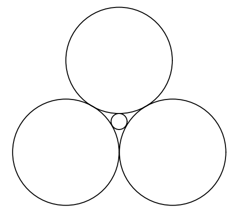


We record the following well known results:

Proposition 5.2. *If D is a domain in $\mathbf{R}^n$ then the cut point of $x \in \partial D$ is the point y along the inner normal to ∂D where x stops being the of ∂D closest to y. The locus $\mathcal{C}$ of D is the is the set all cuts points of D.*

1. *Every point x in the cut locus is either a focal point of ∂D or there are two or more minimizing segments form x to ∂D. If x is not a focal point then the number of minimizing geodesics to the boundary is finite.*
2. *For a bounded domain D the cut locus $\mathcal{C}$ is a strong deformation retract of D.*

3. *The map that takes a point y on the boundary to the cut point on the normal to ∂D at y is continuous.* □



Define R_1 to be the radius of the circle at the center of an equilateral triangle with sides of length 2 and which is tangent to the three unit circles at the vertices of this triangle. Then $R_1 = 2/\sqrt{3} - 1 \approx .15470038$

Theorem 5.3 (Alexander-Bishop [1], Lagunov [6, 7, 8], Lagunov-Fyet [9])**.** *Let D be a connected domain in $\mathbf{R}^n$ with boundary ∂D of class $\mathcal{K}$. Assume that the inradius of D is less that R_1 as defined above. Then the cut locus $\mathcal{C}$ of D is a smooth hypersurface of $\mathbf{R}^n$ and the natural map from ∂D to $\mathcal{C}$ is a double cover of $\mathcal{C}$. There are the two cases*

1. *The boundary ∂D has two components. Then D is homeomorphic to the product $\mathcal{C} \times [0, 1]$*
2. *The boundary ∂D is connected. Then there is a double cover $\widehat{D}$ of D that is diffeomorphic to $\mathcal{C} \times [0, 1]$.*

And in fact this latter case can never arise so that D is homeomorphic to a product $\mathcal{C} \times [0, 1]$. (The reason the result is stated with the extra case is that in the case of manifolds with boundary (which is what Alexander and Bishop consider) this case does come up.)

Proof. Let $x \in \mathcal{C}$. Then x can not be a focal point of ∂D as ∂D is of class $\mathcal{K}$ and therefore any focal point is at a distance of at least one from ∂D, contradicting that the inradius of D is less that $R_1 < 1$.

We now claim that there are exactly 2 minimizing segments from x to ∂D. There are at least two by proposition 5.2 If there were 3 or more such segments then let $r = \text{dist}\,(x, \partial D)$ y_1, y_2, and y_3 be three points of ∂D that can be connected to x by minimizing segments. Then $\partial B(x, r)$ is tangent to ∂D at these three points. By proposition 5.1 the spheres $\mathbf{S}_i^{n-1}$ tangent to $\partial B(x, r)$ at the y_i's are disjoint. But this is impossible as $r < R_1$.

Now for x in $\mathcal{C}$ let y_1 and y_2 be the two point that are at a distance of $r = \text{dist}\,(x, \partial D)$ from x. Let U_i be a very small open piece of ∂D containing y_i, and let $\rho_i(z) = \text{dist}\,(z, U_i)$. Then the functions ρ_1 and ρ_2 are smooth in a neighborhood of x. Also the cut locus is local defined near x by the equations $f := \rho_2 - \rho_1 = 0$. The gradient of f is $\nabla f = \nabla \rho_2 - \nabla \rho_1$ and at x this is just the difference of the two unit vectors $u_i := (x - y_i)/|x - y_i|$. This is now zero unless $y_1 = y_2$, which is not the case. Therefore $\nabla f(x) \neq 0$ and so by the implicit function this implies that the set $\mathcal{C} = \{f = 0\}$ is smooth near x. As x was any point of $\mathcal{C}$ we see that $\mathcal{C}$ is a smooth hypersurface.

This implicit function theorem argument also shows that the map that sends $y \in \partial D$ to the point of $\mathcal{C}$ that is on the normal to ∂D at y is a local diffeomorphism (this uses that the inradius is less that the focal distance). As this map is exactly two to one it must be a covering map. This ∂D is a double cover of $\mathcal{C}$.

Now at each point $x \in \mathcal{C}$ there are two segments connecting x with the boundary of D. Call the union of these two segments L_x. Then L_x is homeomorphic to the segment $[-1, 1]$, with the point x on L_x corresponding to the point 0 of $[-1, 1]$. More over D is a disjoint union of the L_x's and if $x_1 \neq x_2$ then the segments L_{x_1} and L_{x_2} are disjoint. Therefore D is homeomorphic to a unit ball (*i.e.* a unit segment) bundle of a line bundle over $\mathcal{C}$. This implies that the two cases given are the only ones that can come occur.

Finally we show that the case of ∂D connected can never occur. In all case we have that $\mathcal{C}$ is a connected smooth imbedded hypersurface of $\mathbf{R}^n$. Therefore by a standard result of differential topology $\mathbf{R}^n \setminus \mathcal{C}$ into exactly two connected components (cf. [4, §4.4 pp. 103–108]) Moreover for each $x \in \mathcal{C}$ the segment L_x intersects $\mathcal{C}$ exactly once and this intersection is transverse so the endpoints of L_x are in different connected components of $\mathbf{R}^n \setminus \mathcal{C}$. However these endpoints are on ∂D and ∂D is disjoint from $\mathcal{C}$ so that if ∂D where connected we would have the contradiction that the endpoints of L_x were in the same connected component of $\mathbf{R}^n \setminus \mathcal{C}$. This completes the proof. □

Corollary 5.4 (Lagunov [6, 7])**.** *Let $D \subset \mathbf{R}^n$ be a connected domain with $\partial D \in \mathcal{K}$ and so that either ∂D is connected or ∂D has three or more connected components. Then the inradius of D is at least $R_1 = 2/\sqrt{3} - 1 \approx .15470038\ldots$.*

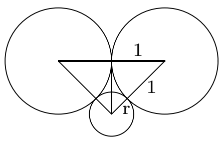
Proof. This follows from the last theorem. □

Let x be in the cut locus $\mathcal{C}$ of D. We say that x has **order** k if and only if x is not a focal point of ∂D and there are exactly k minimizing segments connecting x to ∂D. Note by the the first part of proposition 5.2 any non-focal point has a finite order ≥ 2.

Let R_2 be the radius of the sphere at the center of an equilateral tetrahedron with sides of length two that is tangent to the four unit spheres centered on the vertices of the tetrahedron. Then $R_2 = \sqrt{3/2} - 1 \approx .2247448714$.

Proposition 5.5 (Lagunov [6, 7, 8], Lagunov-Fyet [9])**.** *Let D be a domain in $\mathbf{R}^n$ with boundary of class $\mathcal{K}$. If the inradius of D is less than R_2, then every point of the cut locus $\mathcal{C}$ of D is of order ≤ 3.*

Proof. If x in the cut locus $\mathcal{C}$ of D has order four or more, then there are four points $y_i \in \partial D$ that can be joined to x by minimizing segments. Let $r = \text{dist }(x, \partial D)$. Then the ball $B(x, r)$ is tangent to ∂D at the points y_i. By proposition 5.1 the spheres tangent to $B(x, r)$ are disjoint. But this implies that $r \geq R_2$. This contradiction completes the proof. □

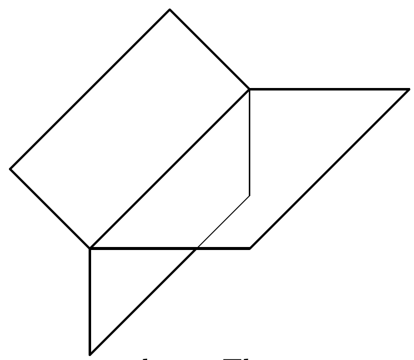


Let $S(x_i, 1)$, $i = 1, 2$ be two unit spheres tangent to the ball $B(0, r)$ of radius one centered at the origin. If the two spheres $S(x_i, 1)$ are disjoint then the angle θ between x_1 and x_2 satisfies

$$\sin\frac{\theta}{2} \geq \frac{1}{1+r} \tag{5.1}$$

as can easily be seen from the figure.

Theorem 5.6 (Lagunov [6, 7, 8], Lagunov-Fyet [9])**.** *Let D be a domain in $\mathbf{R}^n$ with boundary of class $\mathcal{K}$ and inradius less that R_2 defined above. Then the cut locus $\mathcal{C}$ of D is the disjoint union of the set $\mathcal{C}_2$ of cut points of order and the set $\mathcal{C}_3$ of points order three. The set $\mathcal{C}_2$ is a smooth imbedded hypersurface of $\mathbf{R}^n$. The set $\mathcal{C}_3$ a smooth compact submanifold of $\mathbf{R}^n$ of dimension $n-2$. Near any point $x \in \mathcal{C}_3$ the cut locus $\mathcal{C}$ has a neighborhood U in $\mathbf{R}^n$ so that $\mathcal{C} \cap U$ is homeomorphic to a union of three closed half $\mathbf{R}^{n-1}$'s glued together along a common $\mathbf{R}^{n-2}$.*



Before going on with the proof we give more detail about the normal form of the cut locus near the "singular" set of cut points of order three.

Let $x_1, x_2, \ldots, x_n$ be the standard coordinates on $\mathbf{R}^n$. Let $H_1 = \{x_1 = 0,\ x_2 \geq 0\}$, $H_2 = \{x_2 = 0,\ x_1 \geq 0\}$ and $H_3 = \{x_2 - x_1 = 0,\ x_1 \leq 0\}$. Then near cut points of order three $\mathcal{C}$ locally looks like $H_1 \cup H_2 \cup H_3$ near the origin. Thus $\mathcal{C}_3$ looks locally like the subspace $\{x_1 = x_2 = 0\}$.

Proof. By proposition 5.5 every cut point is of order at most three. That the set $\mathcal{C}_2$ is a smooth hypersurface in $\mathbf{R}^n$ follows exactly as in the proof of proposition 5.3. Now let $x \in \mathcal{C}_3$. By a translation we can assume that $x = 0$ is the origin of $\mathbf{R}^n$. Then let y_1, y_2, and y_3 be the three points on ∂D that can be connected to $x = 0$ by a minimizing geodesic, and let $r = \text{dist}\,(0, \partial D) = |y_i|$. As in the proof of theorem 5.3 choose a very small open piece P_i of ∂D near y_i and let $\rho_i(z) := \text{dist}\,(z, P_i)$. Then each function ρ_i is smooth in an neighborhood of 0 and $\nabla\rho_i(0) = -y_i/r$. Let

$$f_1 = \rho_2 - \rho_3, \qquad f_2 = \rho_1 - \rho_3, \qquad f_3 = \rho_1 - \rho_2. \tag{5.2}$$

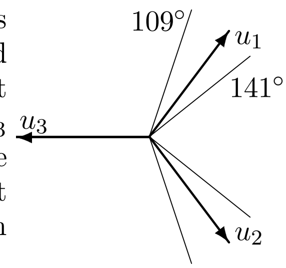
Then the set $\mathcal{C}_3$ is defined near x by the equations $f_1 = f_2 = f_3 = 0$ (as these equations are equivalent to $\rho_1 = \rho_2 = \rho_3$, that is the set of point that have three minimizing geodesics connecting them to the boundary.) But $f_3 = f_2 - f_1$ so these equations can be replace by the pair of equations $f_1 = 0$, $f_2 = 0$. Therefore if we can show that the gradients of f_1 and f_2 are independent at x then the implicit function theorem will imply that the set $\mathcal{C}_3$ is a smooth codimension two submanifold of $\mathbf{R}^n$. Let $u_i := y_i/r$. Then at x

$$\nabla f_1 = u_3 - u_2, \qquad \nabla f_2 = u_3 - u_2 \tag{5.3}$$

If these two vectors are independent, then the vectors u_1, u_2, and u_3 are dependent and thus all lie in some two dimensional plane. By the inequality 5.1 the angle θ between any two of these vectors satisfies

$$\theta \geq 2\sin^{-1}\left(\frac{1}{1+R_2}\right) > 109^\circ \tag{5.4}$$

But this implies that the figure formed by u_1, u_2, and u_3 is rather rigid. The angle between the vectors u_1 and u_3 must lay between approximately 109° and $360^\circ - 2\cdot 109^\circ = 141^\circ$. As the vectors u_i are unit vectors this implies that the vectors u_1-u_3 and u_2-u_3 are linearly independent. This is equivalent to the independence of the vectors ∇f_1 and ∇f_2 at the point $x=0$. This completes the proof that $\mathcal{C}_3$ is a smooth codimension two submanifold of $\mathbf{R}^n$.



Near the point $0 = x$ the part of $\mathcal{C}$ not in $\mathcal{C}_3$ is in one of the three sets

$$\begin{aligned}
\mathcal{S}_1 &= \{\rho_2=\rho_3<\rho_1\} = \{f_1=0,\ f_2<0\}\\
\mathcal{S}_2 &= \{\rho_1=\rho_3<\rho_2\} = \{f_2=0,\ f_1<0\}\\
\mathcal{S}_3 &= \{\rho_1=\rho_2<\rho_3\} = \{f_3=0,\ f_1>0\} = \{f_2-f_1=0,\ f_1>0\}.
\end{aligned}$$

But ∇f_1 and ∇f_2 are linearly independent at the point $x=0$ and therefore they can be completed to a coordinate system $f_1, f_2, h_3, \dots h_n$ centered at the point $x=0$. This shows not only that locally near x that $\mathcal{C}$ is homeomorphic to the set described above, but that it is equivalent it the strong sense that for any point $x\in\mathcal{C}_3$ there is a local diffeomorphism ϕ of $\mathbf{R}^n$ that moves a neighborhood of x go a neighborhood of the origin and so that $\phi[\mathcal{C}]$ coincides with the model set described above. □

As an application of the last theorem we compute the Euler characteristic $\chi(\mathcal{C}_3)$ of the singular set of the cut locus $\mathcal{C}$ of a domain D with boundary D of class $\mathcal{K}$ and with inradius as defined above. Let $f:\partial D\to\mathcal{C}$ be the natural map, that is it sends a point y on the boundary to the point of $\mathcal{C}$ where the normal geodesic to ∂D intersects $\mathcal{C}$. Choose a smooth triangulation of the submanifold $\mathcal{C}_3$ and extend it to a triangulation of $\mathcal{C}$. Then (using that the map f is locally one-to-one) we can lift the triangulation of $\mathcal{C}$ to one of ∂D. (In doing this we may have to refine the triangulation of $\mathcal{C}$ so that each cell is inside a set where we can find a local inverse to f on this cell (this is *not* saying that we can find a local inverse to f near all points of $\mathcal{C}$, but only that on cells of a properly chosen triangulation there will be an inverse)).

Let c_i denote the number of i-dimensional cells in the triangulation of ∂D. Then

$$\chi(\partial D) = \sum (-1)^i c_i \tag{5.5}$$

For each cell σ of this triangulation either $f[\sigma]\subset\mathcal{C}_3$ or $f[\sigma]$ is disjoint from $\mathcal{C}_3$. (For this to be true I guess we are working with open cells.) Let c_i^3 be the number of i-dimensional cells in ∂D that map into $\mathcal{C}_3$ and let c_i^2 be the number of i-dimensional cells that have image under f disjoint from $\mathcal{C}_3$. Note that by the definition of $\mathcal{C}_3$ each point $x \in \mathcal{C}_3$ has exactly three preimages under f and by thus the number of i-dimensional cells in $\mathcal{C}_3$ is $(1/3)c_i^3$ and thus

$$\chi(\mathcal{C}_3) = \frac{1}{3}\sum(-1)^i c_i^3. \tag{5.6}$$

Each point of $\mathcal{C}_2 := \mathcal{C}\backslash\mathcal{C}_3$ has exactly two preimages under f and so the total number of i-dimensional cells in $\mathcal{C}$ is $c_i^2/2 + c_i^3/3$ and thus the Euler characteristic of $\mathcal{C}$ is

$$\chi(D) = \chi(\mathcal{C}) = \frac{1}{2}\sum(-1)^i c_i^2 + \frac{1}{3}\sum(-1)^i c_i^3 \tag{5.7}$$

where $\chi(D) = \chi(\mathcal{C})$ as $\mathcal{C}$ is a deformation retract of D. Finally there is the obvious relation

$$c_i = c_i^2 + c_i^3 \tag{5.8}$$

The equations (5.5), (5.6), (5.7), and (5.8) imply:

Proposition 5.7. *With the notation above*

$$2\chi(D) = \chi(\partial D) - \chi(\mathcal{C}_3) \tag{5.9}$$

□

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