Customizing Planets to Control Weight.

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A lecture given in the analysis seminar in the Department of Mathematics, University of South Carolina.

The question answered in this note was posed to me by Dominik Gothe, an undergraduate here at the University of South Carolina with the agreeable habit of asking interesting mathematical questions motivated by physics. I was unable give him much help. The question was brought up during an afternoon tea in the mathematics department generating a good deal of discussion. A graduate student, Paisa Seeluangsawat, became interested and came up with a solution. When Dominik was e-mailed this solution he replied with a solution he had found. Below is my version of a combination of the solutions of Dominik and Paisa. Any mistakes are mine and not theirs.

Given a region K in the Euclidean space $\mathbf{R}^3$ assume that mass is distributed over K with constant density ρ. If a particle of unit mass is at the origin and assuming the Newtonian inverse square law for gravitational attraction, the gravitational force of K on the particle is the vector

$$\mathbf{F}(K) := \iiint_K \frac{G\rho\,\hat{\mathbf{r}}}{|\mathbf{r}|^2}\,dV(\mathbf{r}) = G\rho \iiint_K \frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}\,dV(\mathbf{r})$$

where $\mathbf{r}$ is the position vector, $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$, dV is volume measure, and G is the gravitational constant.

Problem 1. *If the mass, M_0, of K is fixed, then what shape for K maximizes $|\mathbf{F}(K)|$?*

A restatement is: What is the shape of a planet (of mass M_0 and uniform density ρ) that maximizes the weight of someone standing at the origin?

As scaling will simplify several of our calculations, we record here that if $\lambda > 0$ and $\lambda K := \{\lambda\mathbf{r} : \mathbf{r} \in K\}$ is the dilate of K by λ, then a change of variable $\mathbf{r} \mapsto \lambda\mathbf{r}$ in the integral formula above for $\mathbf{F}(K)$ shows

$$\mathbf{F}(\lambda K) = \lambda \mathbf{F}(K). \tag{1}$$

As $G\rho > 0$, Problem 1 is equivalent to

Problem 2. *If the volume of K is fixed to be $V_0 := M_0/\rho$, that what is the shape of K that maximizes $|\mathbf{P}(K)|$ where $\mathbf{P}(K)$ is*

$$\mathbf{P}(K) := \iiint_K \frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}\,dV(\mathbf{r}).$$

This shows that the value of the gravitational constant G does not effect the optimal shape.

The functional $\mathbf{P}(K)$ is not easy to work with as it is vector valued. The following shows $\mathbf{P}(K)$ can be replaced by a scalar valued functional.

Lemma 1. *Let* $\mathbf{e}$ *be a unit vector and* E *a region that maximizes the inner product* $\mathbf{e}\cdot\mathbf{P}(K)$ *over all regions with* $V(K) = V_0$*, then* E *maximizes* $|\mathbf{P}(K)|$ *over all regions with* $V(K) = V_0$*.*

Remark. Conversely if E maximizes $|\mathbf{P}(K)|$ over all domains with $V(K) = V_0$ and $\mathbf{e} = |\mathbf{P}(E)|^{-1}\mathbf{P}(E)$, then E maximizes $e\cdot\mathbf{P}(K)$ over all domains with $V(K) = V_0$.

Proof. Towards a contradiction, assume that E does not maximize $|\mathbf{P}(K)|$ over regions with $V(K) = V_0$. Then there is a region E' with $V(E') = V_0$ and $|\mathbf{P}(E')| > |\mathbf{P}(E)|$ and $\mathbf{e}$ and $\mathbf{P}(E')$ do not point in the same direction, for if they did it would lead to the contradiction $\mathbf{e}\cdot\mathbf{P}(E') = |\mathbf{P}(E')| > |\mathbf{P}(E')| \geq \mathbf{e}\cdot\mathbf{P}(E')$. Choose a rotation, g, about the origin with so that $g\mathbf{P}(E')$ points in the same direction as $\mathbf{e}$. Then

$$\mathbf{P}(gE') = g\mathbf{P}(E')$$

(which is clear on geometric and/or physical grounds and can also be verified by the anal retentive by a change of variable in the integral defining $\mathbf{P}(E')$). This implies $|\mathbf{P}(gE')| = |\mathbf{P}(E')|$. As rotations preserve volumes, the region gE' has volume V_0 and as $\mathbf{e}$ and $\mathbf{P}(gE')$ point in the same direction

$$\mathbf{e}\cdot\mathbf{P}(gE) = |\mathbf{P}(gE')| = |\mathbf{P}(E')| > |\mathbf{P}(E)| \geq \mathbf{e}\cdot\mathbf{P}(E),$$

which contradicts the maximizing property of E. □

Proposition 2. *Let* $h\colon \mathbf{R}^3 \to \mathbf{R}$*. Assume that there is a constant* c *such that the set*

$$E_c := \{\mathbf{r}\in\mathbf{R}^2 : h(\mathbf{r}) \geq c\}$$

has volume V_0*. Then for any other region* E *with* $V(E) = V_0$

$$\iiint_E h(\mathbf{r})\,dV(\mathbf{r}) \leq \iiint_{E_c} h(\mathbf{r})\,dV(\mathbf{r})$$

and equality holds if and only if $E = E_c$ *up to sets of measure zero. (That is* $V(E \setminus E_c) = 0$ *and* $V(E_c \setminus E) = 0$*.)*

Proof. We will show the contrapositive: If E and E_c do not differ by only sets of measure zero and $V(E) = V(E_c)$, then $\iiint_E h\,dV < \iiint_{E_c} h\,dV$. Thus assume that E and E_c do not differ by only sets of measure zero and that $V(E) = V(E_c)$. Then $V(E \setminus E_c) = V(E_c \setminus E) > 0$. Therefore

$$\begin{aligned}
\iiint_E h(\mathbf{r})\,dV(\mathbf{r}) &= \iiint_{E\cap E_c} h(\mathbf{r})\,dV(\mathbf{r}) + \iiint_{E\setminus E_c} h(\mathbf{r})\,dV(\mathbf{r}) \\
&< \iiint_{E\cap E_c} h(\mathbf{r})\,dV(\mathbf{r}) + \iiint_{E\setminus E_c} c\,dV(\mathbf{r}) \\
&= \iiint_{E\cap E_c} h(\mathbf{r})\,dV(\mathbf{r}) + cV(E \setminus E_c) \\
&= \iiint_{E\cap E_c} h(\mathbf{r})\,dV(\mathbf{r}) + cV(E_c \setminus E) \\
&\le \iiint_{E\cap E_c} h(\mathbf{r})\,dV(\mathbf{r}) + \iiint_{E_c\setminus E} h(\mathbf{r})\,dV(\mathbf{r}) \\
&= \iiint_{E_c} h(\mathbf{r})\,dV(\mathbf{r}).
\end{aligned}$$

□

Let $\mathbf{e} = (1, 0, 0)$ be the unit vector in the direction of the positive x-axis. Then the position vector and the unit vector in its direction are given in the standard coordinates as

$$\mathbf{r} = (x, y, z), \qquad \hat{\mathbf{r}} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

Therefore

$$\mathbf{P}(E) = \iiint_E \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}\,dx\,dy\,dz.$$

From Lemma 1 we see that it is enough to find a domain E that maximizes

$$\mathbf{e} \cdot \mathbf{P}(E) = \iiint_E \frac{x}{(x^2 + y^2 + z^2)^{3/2}}\,dx\,dy\,dz$$

subject to the constraint $V(E) = V_0$. In light of Proposition 2 (with $h = x/(x^2 + y^2 + z^2)^{3/2}$) to find the maximizer, which will be unique up to sets of measure zero, it is enough to find a constant, c, such that

$$E_c := \left\{ (x, y, z) : \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \ge c \right\}$$

has volume V_0. Note that, using the change of variables $u = \sqrt{c}\,x$, $v = \sqrt{c}\,y$, and $w = \sqrt{c}\,z$

$$\begin{aligned}
E_c &= \left\{ (x, y, z) : \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \ge c \right\} \\
&= \left\{ (x, y, z) : \frac{\sqrt{c}\,x}{((\sqrt{c}\,x)^2 + (\sqrt{c}\,y)^2 + (\sqrt{c}\,z)^2)^{3/2}} \ge 1 \right\} \\
&= \left\{ \left(\frac{u}{\sqrt{c}}, \frac{v}{\sqrt{c}}, \frac{w}{\sqrt{c}} \right) : \frac{u}{(u^2 + v^2 + w^2)^{3/2}} \ge 1 \right\}
\end{aligned}$$

which is just the dilate of E_1 by a factor of $1/\sqrt{c}$. That is

$$E_c = \frac{1}{\sqrt{c}} E_1.$$

Thus for $c > 0$

$$V(E_c) = c^{-3/2} V(E_1).$$

So it is enough to compute $V(E_1)$. This is defined by

$$x \geq (x^2 + y^2 + z^2)^{3/2}.$$

Raising this to the 2/3 power and rearranging

$$y^2 + z^2 \leq x^{2/3} - x^2 = x^{2/3}(1 - x^{4/3}).$$

Taking square roots gives

$$\sqrt{y^2 + z^2} \leq x^{1/3}\sqrt{1 - x^{4/3}}.$$

This is the region formed by revolving the curve $y = x^{1/3}\sqrt{1 - x^{4/3}}$ with $0 \leq x \leq 1$ about the x-axis.[1] The volume is

$$V(E_1) = \pi \int_0^1 y^2\,dx = \pi \int_0^1 (x^{2/3} - x^2)\,dx = \frac{4\pi}{15}$$

and so

$$V(E_c) = \frac{4\pi c^{-3/2}}{15}.$$

To compute $|\mathbf{P}(E_1)|$ let $f(x) = x^{1/3}\sqrt{1 - x^{4/3}}$,

$$\begin{aligned}
|\mathbf{P}(E_1)| = \mathbf{e}\cdot\mathbf{P}(E_1) &= \iiint_{E_1} \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz \\
&= \int_0^1 x \iint_{y^2+z^2\leq f(x)^2} \frac{dy\,dz}{(x^2+y^2+z^2)^{3/2}}\,dx \\
&= \int_0^1 x \int_0^{2\pi}\int_0^{f(x)} \frac{r\,dr\,d\theta}{(x^2+r^2)^{3/2}}\,dx \\
&= 2\pi \int_0^1 x \int_0^{f(x)} \frac{r\,dr}{(x^2+r^2)^{3/2}}\,dx \\
&= 2\pi \int_0^1 \left(\frac{-1}{\sqrt{r^2+f(x)^2}} \right)\Bigg|_{r=0}^{f(x)} dx \\
&= 2\pi \int_0^1 x \left(\frac{1}{x} - \frac{1}{\sqrt{f(x)^2+x^2}} \right) dx.
\end{aligned}$$

Substituting $f(x)^2 = x^{2/3} - x^2$ this becomes

$$|\mathbf{P}(E_1)| = 2\pi \int_0^1 x \left(\frac{1}{x} - \frac{1}{x^{1/3}} \right) dx = \frac{4\pi}{5}.$$

[1]In polar coordinates this is $r = \sqrt{\cos\theta}$ with $-\pi/2 \leq \theta \leq \pi/2$.

Using $E_c = (1/\sqrt{c})E_1$ and the scaling property (1) (which also holds for $\mathbf{P}$ as it is a constant multiple of $\mathbf{F}$)

$$|\mathbf{P}(E_c)| = \frac{1}{\sqrt{c}}|\mathbf{P}(E_1)| = \frac{4\pi}{5\sqrt{c}}.$$

We now find the radius of the ball with the same volume as E_1. If r_1 is this radius then

$$\frac{4\pi}{3}r_1^3 = \frac{4\pi}{15}.$$

Solving for r_1:

$$r_1 = \frac{1}{\sqrt[3]{5}}.$$

The ball, B, with this radius that has the same unit normal as E_1 at the origin has center $(r_1, 0, 0)$ and is defined by

$$(x - r_1)^2 + y^2 + z^2 \le r_1^2$$

that is

$$\sqrt{y^2 + z^2} \le \sqrt{r_1^2 - (x - r_1)^2} = \sqrt{2r_1 x - x^2},$$

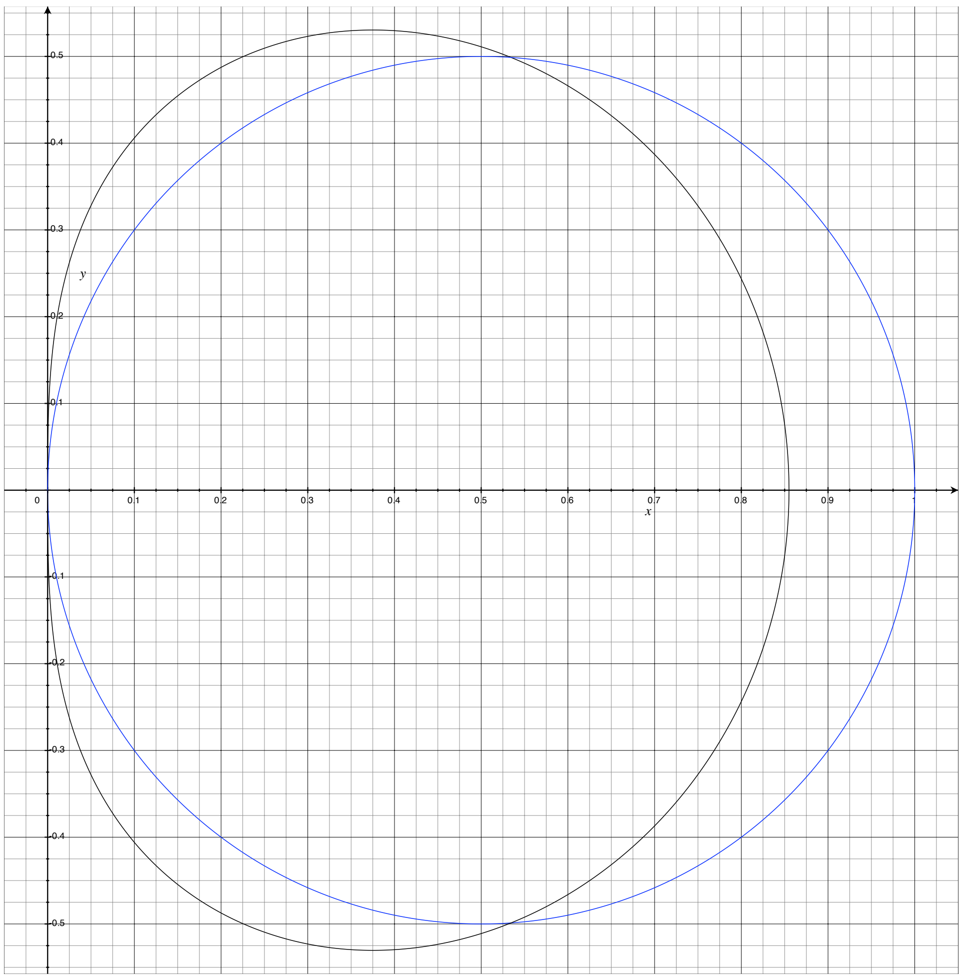
which is the body defined by rotating $y = g(x) := \sqrt{2r_1 - x^2}$ with $0 \le x \le 2r_1$ about the x-axis.

$$\begin{aligned}
|\mathbf{P}(B)| = \mathbf{e}\cdot\mathbf{P}(B) &= \iiint_B \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz \\
&= \int_0^{2r_1} x \iint_{y^2+z^2\le g(x)^2} \frac{dy\,dz}{(x^2+y^2+z^2)^{3/2}}\,dx \\
&= \int_0^{2r_1} x \int_0^{2\pi}\int_0^{g(x)} \frac{r\,dr\,d\theta}{(x^2+r^2)^{3/2}}\,dx \\
&= 2\pi\int_0^{2r_1} x \int_0^{g(x)} \frac{r\,dr}{(x^2+r^2)^{3/2}}\,dx \\
&= 2\pi\int_0^{2r_1} x \left(\frac{1}{x} - \frac{1}{\sqrt{g(x)^2+x^2}}\right) dx \\
&= 2\pi\int_0^{2r_1} x \left(\frac{1}{x} - \frac{1}{\sqrt{2r_1 x}}\right) dx \\
&= 2\pi\frac{2}{3}r_1 \\
&= \frac{4\pi}{3\sqrt[3]{5}}.
\end{aligned}$$

Thus the ratio of the force on the particle by E_1 and the force by a spherical planet of equal mass and density is

$$\frac{|\mathbf{F}(E_1)|}{|\mathbf{F}(B)|} = \frac{|\mathbf{P}(E_1)|}{|\mathbf{P}(B)|} = \frac{4\pi}{5}\,\frac{3\sqrt[3]{5}}{4\pi} = \frac{3}{\sqrt[3]{25}} \approx 1.025985568.$$

So the custom planet only increases the weight on someone on the surface of a spherical planet of the same mass and density by a bit less than 2.6%.



The figure shows the profile curves for two bodies of revolution about the x-axis. One is for a spherical planet of diameter 1 and uniform density, the other is a planet of the same volume and same uniform density (and thus the same mass) that maximizes force on a particle at the origin over all planets with the same mass and uniform dersity. The ratio of the sizes of the forces is $3/\sqrt[3]{25} \approx 1.025985568$.