FUBINI'S THEOREM ON THE TERMWISE DIFFERENTIABILITY OF SERIES WITH MONOTONE TERMS

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1. Introduction.

A basic result in the differentiability of real valued functions of a real variable is Lebesgue's Theorem on the differentiability of monotone functions:

Theorem 1. *Let $f\colon [a,b] \to \mathbf{R}$ be a monotone increasing function. Then $f'(x)$ exists for almost all $x \in [a,b]$ and*

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

A less well known, but still fundamental, result is the Theorem of Fubini on the termwise differentiability of series with monotone terms:

Theorem 2. *Let $f_k\colon [a,b] \to \mathbf{R}$ be monotone increasing for $k = 1, 2, \ldots$ and assume that the series*

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges pointwise on $[a,b]$. Then

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

for almost all $x \in [a,b]$

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We will give a proof of this result and use it so construct some examples of strictly increasing functions with that have derivative zero almost everywhere.

2. Proof of Theorem 2.

We write

$$f(x) = \sum_{k=1}^{n} f_k(x) + R_n(x)$$

where

$$R_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$$

is the n-th remainder for the n-th partial sum $\sum_{k=1}^{n} f_k(x)$. Note that f and R_n are also monotone increasing functions and therefore are differentiable almost everywhere by Theorem 1. Let E be the set of points where all the functions f_k are f are differentiable. As each of these functions is differentiable almost everywhere, we have that $[a,b] \setminus E$ has measure zero. If $x \in E$, then the partial sum $\sum_{k=1}^{n} f_k(x)$ is differentiable at x and therefore so is $R_n'(x) = f'(x) - \sum_{k=1}^{n} f_k'(x)$ exists. Therefore, as $R_n'(x) \geq 0$,

$$f'(x) = \sum_{k=1}^{n} f_k'(x) + R_n'(x) \geq \sum_{k=1}^{n} f_k'(x)$$

for all $x \in E$. Letting $n \to \infty$ gives $f'(x) \geq \sum_{k=1}^{\infty} f_k'(x)$ for all $x \in E$ and thus

$$\sum_{k=1}^{\infty} f_k'(x) \leq f'(x) \quad a.e. \tag{2.1}$$

By Theorem 1 applied to the monotone increasing functions R_n we have

$$0 \leq \int_a^b R_n'(x)\, dx \leq R_n(b) - R_n(a) = \sum_{k=n+1}^{\infty} (f_k(b) - f_k(a)).$$

As both the series $\sum_{k=1}^{\infty} f_k(a)$ and $\sum_{k=1}^{\infty} f_k(b)$ converge the series $\sum_{k=1}^{\infty}(f_k(b) - f_k(a))$ converges and thus $\lim_{n\to\infty} \sum_{k=n+1}^{\infty}(f_k(b) - f_k(a)) = 0$. Using this in what we have just done gives

$$\lim_{n\to\infty} \int_a^b R_n'(x)\, dx = 0 \tag{2.2}$$

Also

$$\begin{aligned}\int_a^b f'(x)\,dx &= \int_a^b \left(\sum_{k=1}^{n} f_k(x)\right)' dx + \int_a^b R_n'(x)\,dx \\ &= \int_a^b \sum_{k=1}^{n} f_k'(x)\,dx + \int_a^b R_n'(x)\,dx \\ &\le \int_a^b \sum_{k=1}^{\infty} f_k'(x)\,dx + \int_a^b R_n'(x)\,dx\end{aligned}$$

Letting $n \to \infty$ and using (2.2) gives

$$\int_a^b f'(x)\,dx \le \int_a^b \sum_{k=1}^{\infty} f_k'(x)\,dx. \tag{2.3}$$

But by (2.1) $\sum_{k=1}^{\infty} f_k'(x) \le f'(x)$ almost everywhere. Therefore the only what that (2.3) can hold is if $f'(x) = \sum_{k=1}^{\infty} f_k'(x)$ a.e. This completes the proof. □

3. Some examples

Let $H(x)$ be the function

$$H(x) = \begin{cases} 0, & x \le 0; \\ 1, & 0 < x. \end{cases}$$

That is $H(x)$ is just the characteristic function of the set $(0, \infty)$. It is clearly monotone increasing, and also $H'(x) = 0$ for all $x \ne 0$. Let $\langle r_k \rangle_{k=1}^{\infty}$ be an enumeration of the rational numbers and set

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{2^k} H(x - r_k).$$

This will be a strictly increasing function which has a jump discontinuity at each rational number. However by applying Theorem 2 to intervals $[a, b]$ and letting $a \to -\infty$ and $b \to \infty$ we have

$$f'(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} H'(x - r_k) = 0$$

for almost all $x \in \mathbf{R}$.

For a somewhat more interesting example let h be the Cantor function on $[0, 1]$. This is functions that is monotone increasing on $[0, 1]$, with $h(0) = 0$, $h(1) = 1$ and is locally constant on the compliment of the Cantor set. This function is increasing and has $h'(x) = 0$ for all in the compliment of the

Cantor set. Thus $h'(x) = 0$ a.e. Let

$$g(x) = \begin{cases} 0, & x < 0; \\ h(x), & 0 \le x \le 1; \\ 1, & 1 < x. \end{cases}$$

Then g is monotone increasing and has $g'(x) = 0$ a.e. Again letting $\langle r_k \rangle_{k=1}^{\infty}$ be an enumeration of the rational numbers we set

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} g(x - r_k).$$

By Theorem 2 this will have $F'(x) = 0$ almost everywhere. It is not hard to check that F is strictly increasing. Thus we have an easy example of a continuous ***singular function***, that is a continuous strictly increasing function with $F' = 0$ a.e.