Mathematics 700 Homework
due Wednesday, September 25

The following are problems on finding the matrices of linear maps. There are examples in Chapter 6 of the text that are relevant to these problems.

(1) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(x,y) := (x-2y, -4x+y, 7x-11y).$$

Then find the matrix of T with respect to the standard bases of each of $\mathbb{R}^2$ and $\mathbb{R}^3$. (Recall that the standard basis of $\mathbb{R}^n$ is the basis $e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $e_3 = (0,0,1,\dots,0)$ ....)

(2) With T as in the last problem find the matrix of T with respect to the bases $\mathcal{V} = \{(1,2),(3,2)\}$ of $\mathbb{R}^2$ and $\mathcal{W} := \{(1,1,1),(0,2,1),(0,0,3)\}$ of $\mathbb{R}^3$.

(3) Letting $\mathcal{P}_3$ be the real polynomials of degree ≤ 3 and using the standard basis $\mathcal{V} := \{1, x, x^2, x^3\}$ of $\mathcal{P}_3$ find the matrices, the rank, the nullity, and the trace of the following linear maps

(a) $(Tp)(x) = p(x-2)$,

(b) $(Cp)(x) = (x+1)^3 p\left(\dfrac{x-1}{x+1}\right)$,

(c) $Ap = p + p' + p'' + p''' + p''''$,

(d) $(Pp)(x) = e^{-x}\displaystyle\int_{-\infty}^{x} p(t)e^t\,dt$.

(e) $(Bp)(x) = \displaystyle\int_0^x p'(t)\,dt$

(f) $(Vp)(x) = \dfrac{d}{dx}\displaystyle\int_0^x p(t)\,dt$

(4) Let $M_{2\times 2}$ be the vector space of 2×2 matrices over the real numbers. Let

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Use for $M_{2\times 2}$ the the ordered basis basis $\mathcal{V} := \{E_{11}, E_{21}, E_{12}, E_{22}$ where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then find the matrices and the traces of the following linear maps from $M_{2\times 2}$ to itself.

(a) $LX = AX$,

(b) $RX = XA$,

(c) $CX = AX - XA$,
(d) $TX = X^t$ (the transpose of X),
(e) $SX = \frac{1}{2}(X + X^t)$ and find the rank and nullity of this map, and
(f) $GX = \frac{1}{2}(X - X^t)$ and find the rank and nullity of this map.

(5) The set of complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ is a two dimensional vector space over the real numbers $\mathbb{R}$. Using the basis $\mathcal{B} = \{1, i\}$ for this real vector space find the matrices of the following linear maps
(a) $Jz = iz$,
(b) $Cz = \overline{z}$ (where $\overline{z}$ is the complex conjugate of z),
(c) $Tz = (2 + 3i)z$,
(d) $Mz = (a + bi)z$, and
(e) $Rz = \frac{1}{2}(z + \overline{z})$.