Mathematics 700 Homework
Due Friday, November 1

The following is an important part of the duality theorem of vector spaces.

Theorem 1. *Let V be a finite dimensional vector space and $S \subset V$ a non-empty subset of V. Then for $v \in V$*

$$f(v) = 0 \quad \text{for all } f \in S^{\perp} \qquad \Longrightarrow \qquad v \in \operatorname{Span}(S).$$

□

Problem 6. Prove this. Hint: The theorem on page 111 of the class notes is relevant. □

Another basic result is

Theorem 2. *Let V be a finite dimensional vector space and $S \subset V$ a non-empty subset of V. Then*

$$(S^{\perp})^{\circ} = \operatorname{Span}(S).$$

In particular if W is a subspace to V, then $(W^{\perp})^{\circ} = W$.

Problem 7. Prove this. Hint: This follows easily from Theorem 1 above. □

Frank's solution to Problem 5 on the last assignment.

We wish to show that if $f, f_1, \ldots, f_k \in V^*$ and

$$\ker(f_1) \cap \ker(f_2) \cap \cdots \cap \ker(f_k) \subseteq \ker(f)$$

then f is a linear combination of $f_1, \ldots, f_k$. A restatement would be

$$\ker(f_1) \cap \ker(f_2) \cap \cdots \cap \ker(f_k) \subseteq \ker(f) \qquad \Longrightarrow \qquad f \in \operatorname{Span}\{f_1, \ldots, f_k\}. \tag{1}$$

Thus assume that

$$\ker(f_1) \cap \ker(f_2) \cap \cdots \cap \ker(f_k) \subseteq \ker(f) \tag{2}$$

holds. From the second proposition on page 116 of the class notes, we know that for any subset $R \subset V^*$ that $R^{\circ} = \operatorname{Span}(R)^{\circ}$. Therefore

$$\operatorname{Span}\{f\}^{\circ} = \{f\}^{\circ} = \{v \in V : f(v) = 0\} = \ker(f)$$

and likewise

$$\begin{aligned}\operatorname{Span}\{f_1, \ldots, f_k\}^{\circ} &= \{f_1, \ldots, f_k\}^{\circ} \\ &= \{v \in V : f_1(v) = f_2(v) = f_k(v) = 0\} \\ &= \ker(f_1) \cap \ker(f_2) \cap \cdots \cap \ker(f_k).\end{aligned}$$

Combining these with (2) gives

$$\operatorname{Span}\{f_1, \ldots, f_k\}^{\circ} \subseteq \operatorname{Span}\{f\}^{\circ}.$$

Now the "dual form" of Problem 4a implies

$$\operatorname{Span}\{f\} \subseteq \operatorname{Span}\{f_1, \ldots, f_k\}.$$

Therefore $f \in \operatorname{Span}\{f\} \subseteq \operatorname{Span}\{f_1, \ldots, f_k\}$. *done.*