Mathematics 700, Test #2

Show your work to get credit. An answer with no work will not get credit.

1. Find the Smith normal form over the integers of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 8 & 10 \\ 14 & 12 \end{bmatrix}.$$

First solution: We reduce the matrix using elmentary row and column operations.

$$\begin{aligned}
\begin{bmatrix} 4 & 6 \\ 8 & 10 \\ 14 & 12 \end{bmatrix} &\cong \begin{bmatrix} 4 & 2 \\ 8 & 2 \\ 14 & -2 \end{bmatrix} && \begin{cases} C_1 \mapsto C_1 \\ C_2 \mapsto C_2 - C_1 \end{cases} \\
&\cong \begin{bmatrix} 4 & 2 \\ 0 & -2 \\ 14 & -2 \end{bmatrix} && \begin{cases} R_1 \mapsto R_1 \\ R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 \end{cases} \\
&\cong \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 14 & 0 \end{bmatrix} && \begin{cases} R_1 \mapsto R_1 + R_2 \\ R_2 \mapsto -R_2 \\ R_3 \mapsto R_3 - R_2 \end{cases} \\
&\cong \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} && \begin{cases} R_1 \mapsto R_1 \\ R_2 \mapsto R_2 \\ R_3 \mapsto R_3 - 3R_1 \end{cases} \\
&\cong \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} && \begin{cases} R_1 \mapsto R_1 - 2R_3 \\ R_2 \mapsto R_2 \\ R_3 \mapsto R_3 \end{cases} \\
&\cong \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} && \begin{cases} R_1 \mapsto R_3 \\ R_2 \mapsto R_2 \\ R_3 \mapsto R_1 \end{cases}
\end{aligned}$$

and this is the Smith normal form.

Second Solution: We know that if C is an $m \times n$ matrix with elements in a Euclidean domain and $f_1, \dots, f_r$ are the elementary divisors of C, then the product $f_1 \cdots f_k$ is the greatest common divisor of the $k \times k$ sub-determinants of C. In the case at hand if f_1 and f_2 are the elementary divisors of A then

$$f_1 = \gcd\{4, 6, 8, 10, 14, 12\} = 2$$

and

$$\begin{aligned}
f_1 f_2 &= \gcd\left\{ \det \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}, \det \begin{bmatrix} 4 & 6 \\ 14 & 12 \end{bmatrix}, \det \begin{bmatrix} 8 & 10 \\ 14 & 12 \end{bmatrix} \right\} \\
&= \gcd\{-8, -36, -44\} = 4
\end{aligned}$$

which implies that $f_2 = 2$. Therefore the Smith normal form is

$$\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

□

2. Find the invariant factors of the following matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ (with } b \neq 0).$$

Solution: Recall that the invarant factors of a square matrix M over a field $\mathbf{F}$ are, by definition, the invariant factors of matrix $xI - M$ over the Euclidean domain $\mathbf{F}[x]$.

For $xI - A = \begin{bmatrix} x-1 & 0 \\ 0 & x-1 \end{bmatrix}$ the gcd of the 1×1 sub-determinants is $x - 1$ and the gcd of the 2×2 subdeterminants is $(x-1)^2$. Thus the elmentary divsors satisfy $f_1 = x - 1$ and $f_1 f_2 = (x-1)^2$. Therefore $f_1 = f_2 = (x-1)$ are the elmentary divsors of A.

For $xI - B = \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix}$ one of the elmentents, -1, is a unit in $\mathbf{F}[x]$ so the gcd of the 1×1 sub-determinants is $f_1 = 1$. Thus $f_2 = f_1 f_2 = \det(xI - B) = (x-1)^2$. So the $f_1 = 1$ and $f_2 = (x-1)^2$ are the elmentary divsors.

For $xI - C = \begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix}$ the elment $-b \neq 0$ is a unit in $\mathbf{F}[x]$ and so the gcd of the 1×1 sub-subdeterminats is $f_1 = 1$. Therefore $f_2 = f_1 f_2 = \det(xI - C) = x^2 - (a+d)x + (ad - bc)$. □

3. Let $\mathcal{P}_1 = \mathrm{Span}\{1, x\}$ be the real polynomials of degree ≤ 1 with real coefficients and define two linear functionals $\Lambda_1, \Lambda_2 \colon \mathcal{P}_1 \to \mathbf{R}$ by

$$\Lambda_1(p) := \int_0^1 p(x)\,dx, \quad \Lambda_2(p) = \int_0^1 xp(x)\,dx.$$

Find the basis of $\mathcal{P}_1$ that is dual to $\{\Lambda_1, \Lambda_2\}$.

Solution: Let $p_1(x) = a + bx$ and $p_2(x) = c + dx$ be the basis dual to Λ_1 and Λ_2. Then by definition of dual basis

$$1 = \Lambda_1(p_1) = \int_0^1 (a + bx)\,dx = a + \frac{b}{2},$$
$$0 = \Lambda_2(p_1) = \int_0^1 x(a + bx)\,dx = \frac{a}{2} + \frac{b}{3}.$$

Solving for a and b gives $a = 4$ and $b = -6$ so that $p_1(x) = 4 - 6x$. Likewise we have

$$0 = \Lambda_1(p_2) = \int_0^1 (c + dx)\,dx = c + \frac{d}{2},$$
$$1 = \Lambda_2(p_2) = \int_0^1 x(c + dx)\,dx = \frac{c}{2} + \frac{d}{3}.$$

Solving for c and d gives $c = -6$ and $d = 12$ so that $p_2(x) = -6 + 12x$. Therefore the basis dual to $\{\Lambda_1, \Lambda_2\}$ is $\{4 - 6x, -6 + 12x\}$. □

4. Let A be an $n \times n$ matrix with real entries so that $A^t = A^{-1}$. Then show that $\det(A) = \pm 1$.

Solution: From $I = AA^{-1}$ we have $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$ so that $\det(A^{-1}) = 1/\det(A)$. Also $\det(A^t) = \det(A)$. Thus

$$\det(A) = \det(A^t) = \det(A^{-1}) = \frac{1}{\det(A)},$$

which yields $\det(A)^2 = 1$ and therefore $\det(A) = \pm 1$. □

5. If $T\colon V \to V$ is a linear operator on the vector space V that satisfies $T^2 = I$, then show that the only eigenvalues of T are 1 and -1.

Solution: Let λ be an eigenvalue and let $v \neq 0$ be an eigenvector for T. Then $Tv = \lambda v$. Therefore we have

$$T^2 v = Iv = v$$

and

$$T^2 v = TTv = T\lambda v = \lambda T v = \lambda^2 v.$$

Comparing these formulas for $T^2 v$ gives $\lambda^2 v = v$ and therefore $\lambda^2 = 1$ so that $\lambda = \pm 1$.

Remark: Let $p(x)$ be a polynomial and $T\colon V \to V$ a linear map such that $p(T) = 0$. Then any eigenvalue of T is a root of $p(x) = 0$. To see this let λ be an eigenvalue of T. Then there is a nonzero vector v so that $Tv = \lambda v$. We have shown in a homework problem that for any polynomial $q(x)$ that $q(T)v = q(\lambda)v$. Therefore using the polynomial $p(x)$ we have

$$p(\lambda)v = p(T)v = 0$$

as $p(T) = 0$. But $v \neq 0$ so this gives $p(\lambda) = 0$. The problem here was just the special case $p(x) = x^2 - 1$. □

6. Let D be an invertible $n \times n$ matrix and N a $n \times n$ matrix so that $DN = ND$ and $N^3 = 0$. Show that $D + N$ is invertible.

Solution: There are several natural ways to do this problem. Here is one closely related to ideas we have either done in class or on homework. Recall that if M is a matrix with $M^3 = 0$ then $I + M$ is invertible with $(I + M)^{-1} = I - M + M^2$. As on one of the homework assignments, this can be seen directly my noting that if $B = I - M + M^2$ then

$$B(I + M) = (I - M + M^2)(I + M) = I, \quad (I + M)B = (I + M)(I - M + M^2) = I.$$

Now write

$$D + N = D(I + D^{-1}N).$$

Then $DN = ND$ implies $ND^{-1} = D^{-1}N$ so that if $M = D^{-1}N$ can use $N^3 = 0$ to get

$$M^3 = D^{-1}ND^{-1}ND^{-1}N = (D^{-1})^3 N^3 = 0$$

Therefore $I + M = I + D^{-1}N$ is invertible. Thus $D + N = D(I + D^{-1}N)$ is a product of invertible matrices and therefore is itself invertible and we are *done*.

We can go farther and compute the inverse of $D + N$ as follows.

$$\begin{aligned}(D + N)^{-1} &= (D(I + D^{-1}N))^{-1} = (I + D^{-1}N)^{-1}D^{-1} \\ &= (I - D^{-1}N + (D^{-1}N)^2)D^{-1} = D^{-1} - D^{-2}N + D^{-3}N^2.\end{aligned}$$

7. Let A be a real 2×2 matrix so that $A^2 - 3A + 2I_2 = 0$. Show that A is similar to one of the following three matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Solution: Note that $A^2 - 3A + 2I = 0$ can be factored into

$$(A - I)(A - 2I) = 0$$

We will use that fact that for any square matrix B over a field that $B - \lambda I$ is invertible if and only if λ is not an eigenvalue of B.

Case 1: The number 1 *is not an eigenvalue of* A. Then $A - I$ is invertible and so we can multiply both sides of $(A - I)(A - 2I) = 0$ by $(A - I)^{-1}$ and conclude that $A - 2I = 0$. That is $A = 2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Case 2: The number 2 *is not an eigenvalue of* A. Then $A - 2I$ is invertible and so we can multiply both sides of $(A - I)(A - 2I) = 0$ by $(A - 2I)^{-1}$ and conclude that $A - I = 0$. That is $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Case 3: Both the numbers 1 *and* 2 *are eigenvalues of* A. Let $v_1, v_2 \in \mathbf{R}$ be the corresponding eigenvectors. That is $Av_1 = 1v_1$ and $Av_2 = 2v_2$. Then v_1 and v_2 are eigenvectors for distinct eigenvalues of A and therefore linearly independent. As $\mathbf{R}^2$ is two dimensional this implies that v_1, v_2 is a basis of $\mathbf{R}^2$. But then if P is the matrix with columns v_1 and v_1 (that is $P = [v_1, v_2]$) then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Therefore A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

So we have shown more than was required. Either $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, or $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. □

8. Let A be an $n \times n$ matrix over the reals with $\det(A) \neq 0$. Show that

$$\det(\operatorname{adj}(A)) = \det(A)^{n-1}.$$

HINT: Recall that $A \operatorname{adj}(A) = \det(A) I$.

Solution: Recall that if c is a scalar and B is an $n \times n$ matrix then $\det(cB) = c^n \det(B)$. As $\det(A)$ is a scalar this implies that $\det(\det(A)I) = \det(A)^n \det(I) = \det(A)^n$. Using this in $A \operatorname{adj}(A) = \det(A)I$ gives

$$\det(A) \det(\operatorname{adj}(A)) = \det(A \operatorname{adj}(A)) = \det(\det(A)I) = \det(A)^n.$$

As $\det(A) \neq 0$ we can cancel a $\det(A)$ off of each side of this and get $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$. □