Mathematics 700 Test #1 Name: Solution Key

Show your work to get credit. An answer with no work will not get credit.

1. (15 Points) Define the following:
   (a) Linear independence.
   The vectors $v_1, \dots, v_m$ in the vector space V are linearly independent iff the only scalars $c_1, \dots, c_m \in \mathbb{F}$ with $c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0$ are $c_1 = c_2 = \cdots = c_m = 0$.
   (b) The span of a subset S of a vector space V.
   The span of S is the set of all linear combinations formed form elements of S.
   (c) The vector space V is direct sum of its subspaces U and W.
   The V is direct sum of its subspaces U and W (written $V = U \oplus W$) iff $V = U + W$ and $U \cap W = \{0\}$. $\square$
2. (10 Points) Find (no proof required) a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ so that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

First solution: We first write the vector $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ as a linear combination of the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$.

$$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\right) = (x-y)\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix}.$$

Now use that we know the values of T on the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$:

$$\begin{aligned} T\begin{bmatrix}x\\y\end{bmatrix} &= T\left((x-y)\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix}\right) = (x-y)T\begin{bmatrix}1\\0\end{bmatrix} + yT\begin{bmatrix}1\\1\end{bmatrix} \\ &= (x-y)\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}x\\2x-2y\\2x-3y\end{bmatrix}. \end{aligned}$$

$\square$

Second Solution: We look for T as being given by a matrix:

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Then the conditions on T yeild

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\\e\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\\e+f\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

This leads to the equations $a = 1$, $c = 2$, $e = 3$, $a+b = 1$, $c+d = 0$, and $e+f = 0$. This gives the values of a, c, and e. Then it is easy to see that $b = 0$, $d = -2$, and $f = -3$. Therefore

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-2y \\ 2x-3y \end{bmatrix}.$$

□

3. (10 Points) Let v_1, v_2, v_3 be linearly independent vectors in a vector space V. Then show that the vectors $v_1, 2v_1 + v_2, 3v_1 + 2v_2 + v_3$ are also linearly independent.
Solution: Let $c_1, c_2, c_3 \in \mathbb{F}$ be scalars so that

$$c_1 v_1 + c_2(2v_1 + v_2) + c_3(3v_1 + 2v_2 + v_3) = 0.$$

To finish we need to show that $c_1 = c_2 = c_3 = 0$. Regrouping gives

$$(c_1 + 2c_2 + 3c_3)v_1 + (c_2 + 2c_3)v_2 + c_3 v_3 = 0. \tag{1}$$

Because v_1, v_2, v_3 are linearly independent (and if you did not say very explicitly say this, you lost most of the points on the problem) the coefficients of v_1, v_2, v_3 vanish in (1) and therefore

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ c_3 &= 0. \end{aligned}$$

Back solving in this gives $c_1 = c_2 = c_3 = 0$, which completes the proof. □

4. (10 Points) Let $M_{2\times 2}$ be the 2 by 2 matrices over the field $\mathbb{F}$ and let

$$\mathcal{D} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F} \right\}$$

be the subspace of diagonal matrices. Show that any three dimensional subspace of $M_{2\times 2}$ contains a nonzero diagonal matrix.
Solution: First note that $\dim \mathcal{D} = 2$ (which is clear as $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $\mathcal{D}$). Let $\mathcal{V}$ be a three dimensional subspace of $M_{2\times 2}$. Then

$$\dim(\mathcal{V} \cap \mathcal{D}) = \dim(\mathcal{D}) + \dim(\mathcal{V}) - \dim(\mathcal{V} + \mathcal{D}) = 5 - \dim(\mathcal{V} + \mathcal{D}) \geq 1$$

as $\mathcal{V} + \mathcal{D} \subset M_{2\times 2}$ so that $\dim(\mathcal{V} + \mathcal{D}) \leq \dim(M_{2\times 2}) = 4$. But $\dim(\mathcal{V} \cap \mathcal{D}) \geq 1$ implies that $\mathcal{V} \cap \mathcal{D}$ contains a nonzero element, which is the desired nonzero diagonal matrix in $\mathcal{V}$. □

5. (10 Points) Find (no proof required) a basis for the set of the space of vectors $(x, y, z, w) \in \mathbb{R}^4$ that satisfy

$$\begin{aligned} x + y + z + w &= 0 \\ x + y + 2z + 3w &= 0. \end{aligned}$$

Solution: Row reducing the matrix of coefficients of the system leads to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

which implies the system is equivalent to

$$\begin{aligned} x &= -y + w \\ z &= -2w. \end{aligned}$$

Therefore the subspace in question is

$$\{(-y + w, y, -2w, w) : y, w \in \mathbb{R}\} = \{y(-1, 1, 0, 0) + w(1, 0, -2, 1) : y, w \in \mathbb{R}\}$$

so that $\{(-1,1,0,0),(1,0,-2,1)\}$ is the required basis. $\square$

6. (15 Points) Show that if $v_1,\dots,v_k$ are vectors in the vector space V and $c_1,\dots,c_k\in\mathbb{F}$ are scalars so that
$$c_1v_1+c_2v_2+\cdots+c_kv_k=0,\quad\text{and}\quad c_k\neq 0$$
then
$$\operatorname{Span}\{v_1,\dots,v_{k-1}\}=\operatorname{Span}\{v_1,\dots,v_k\}.$$
Solution: The inclusion $\operatorname{Span}\{v_1,\dots,v_{k-1}\}\subseteq\operatorname{Span}\{v_1,\dots,v_k\}$ is clear and so we are done if we can show $\operatorname{Span}\{v_1,\dots,v_k\}\subseteq\operatorname{Span}\{v_1,\dots,v_{k-1}\}$. From the relation $c_1v_1+c_2v_2+\cdots+c_kv_k=0$ and using $c_k\neq 0$ we can solve for v_k to get
$$v_k=-\frac{c_1}{c_k}v_1-\frac{c_2}{c_k}v_2-\cdots-\frac{c_{k-1}}{c_k}v_{k-1}.$$
This shows v_k is a linear combination of $\{v_1,\dots,v_{k-1}\}$. Therefore, by the definition of the span of a set of vectors, $v_k\in\operatorname{Span}\{v_1,\dots,v_{k-1}\}$. As $\{v_1,\dots,v_{k-1}\}\subseteq\operatorname{Span}\{v_1,\dots,v_{k-1}\}$ we therefore have $\{v_1,\dots,v_{k-1},v_k\}\subseteq\operatorname{Span}\{v_1,\dots,v_{k-1}\}$. Thus
$$\operatorname{Span}\{v_1,\dots,v_{k-1},v_k\}\subseteq\operatorname{Span}\operatorname{Span}\{v_1,\dots,v_{k-1}\}=\operatorname{Span}\{v_1,\dots,v_{k-1}\}$$
and we are done. $\square$

7. (15 Points) Let U and V be subspaces of a vector space so that
$$\dim U=3,\quad \dim W=4,\quad \dim(U\cap W)=2.$$
Then show directly, that is without using the theorem that $\dim(U+W)=\dim U+\dim W-\dim U\cap W$, that $\dim(U+W)=5$. (So you are being ask to prove $\dim(U+W)=\dim U+\dim W-\dim(U\cap W)$ in this special case.)
Solution: Let v_1,v_2 be a basis for $U\cap W$. This can be extended to a basis v_1,v_2,u_3 of U and to a basis v_1,v_2,w_3,w_4 of W. We now claim that $\mathcal{B}=\{v_1,v_2,u_3,w_3,w_4\}$ is a basis of $U+W$. As $\mathcal{B}$ has 5 elements this will show that $\dim(U+W)=5$. To show that $\mathcal{B}$ is a basis of $U+W$ we to show two things. First that $\operatorname{Span}\mathcal{B}=U+W$ and second that $\mathcal{B}$ is linearly independent.

To see that $\operatorname{Span}\mathcal{B}=U+W$ first note that $\{v_1,v_2,u_3\}\subset U$ and $\{w_3,w_4\}\subset W$ so that $\mathcal{B}=\{v_1,v_2,u_3,w_3,w_4\}\subset U\cup W$. Therefore
$$\operatorname{Span}(\mathcal{B})\subseteq\operatorname{Span}(U\cup W)=U+W.$$
To get set containment in the other direction let $x\in U+W$. Then $x=u+w$ where $u\in U$ and $w\in W$. As $\{v_1,v_2,u_3\}$ is a basis for U there are scalars $a_1,a_2,a_3\in\mathbb{F}$ so that $u=a_1v_1+a_2v_2+a_3u_3$. Likewise $\{v_1,v_2,w_3,w_4\}$ is a basis of W so that there are scalars $b_2,b_2,b_3,b_4\in\mathbb{F}$ so that $w=b_1v_1+b_2v_2+b_3w_3+b_4w_4$. Adding these expressions for u and w and doing a bit of regrouping gives
$$x=u+w=(a_1+b_1)v_1+(a_2+b_2)v_2+a_3u_3+b_3w_3+b_4w_4$$
so that $x\in\operatorname{Span}\in\operatorname{Span}\{v_1,v_2,u_3,w_3,w_4\}=\operatorname{Span}\mathcal{B}$. As x was an arbitrary element of $U+W$ this shows $U+W\subseteq\operatorname{Span}\mathcal{B}$ and completes the proof that $\operatorname{Span}\mathcal{B}=U+W$.

To see that $\mathcal{B}$ is linearly independent assume that there are scalars $c_1,\dots,c_5$ so that
$$c_1v_1+c_2v_2+c_3u_3+c_4w_3+c_5w_4=0.$$
We need to show that $c_1=c_2=\cdots=c_5=0$. Toward this end rewrite the last equation as
$$c_1v_1+c_2v_2+c_3u_3=-c_4w_3-c_5w_4. \tag{2}$$
Then setting $y=c_1v_1+c_2v_2+c_3u_3=-c_4w_3-c_5w_4$ we see from $y=c_1v_1+c_2v_2+c_3u_3$ that $y\in U$ and from $y=-c_4w_3-c_5w_4$ that $y\in W$. Therefore $y\in U\cap W$. Thus y can be expressed

as a linear combination of the basis elements v_1, v_2 of $U \cap W$. That is $y = d_1v_1 + d_2v_2$. Equating two of our expressions for y gives $d_1v_1 + d_2v_2 = -c_4w_3 - c_5w_4$ which can be rewritten as

$$d_1v_1 + d_2v_2 + c_4w_3 + c_5w_4$$

and as $\{v_1, v_2, w_3, w_4\}$ is a basis of W, and thus linearly independent, this implies $d_1 = d_2 = c_4 = c_5 = 0$. Using $c_4 = c_5 = 0$ in (2) gives

$$c_1v_1 + c_2v_2 + c_3u_3 = 0.$$

As $\{v_1, v_2, u_3\}$ is a basis for U this implies $c_1 = c_2 = c_3 = 0$. Thus we now have $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ which completes both the proof that $\mathcal{B}$ is linearly independent and the proof of the proposition. $\square$

8. (15 Points) Let $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3\}$ be two linearly independent sets in a vector space W such that $\mathcal{U} \cup \mathcal{W}$ is linearly independent. Then show

$$\mathrm{Span}(\mathcal{U}) \cap \mathrm{Span}(\mathcal{W}) = \{0\}.$$

Solution 1: It is clear that $\{0\} \subseteq \mathrm{Span}(\mathcal{U}) \cap \mathrm{Span}(\mathcal{W})$. Let $x \in \mathrm{Span}(\mathcal{U}) \cap \mathrm{Span}(\mathcal{V})$ then $x \in \mathrm{Span}(\mathcal{U})$ implies that $x = a_1u_1 + a_2u_2$ for some scalars $a_1, a_2 \in \mathbb{F}$. Likewise $x \in \mathrm{Span}(\mathcal{V})$ implies $x = b_1w_1 + b_2w_2 + b_3w_3$ for scalars $b_1, b_2, b_3 \in \mathbb{F}$. Setting these expressions equal to each other gives $x = a_1u_1 + a_2u_2 = b_1w_1 + b_2w_2 + b_3w_3$ which can be rewritten as

$$a_1u_1 + a_2u_2 - b_1w_1 - b_2w_2 - b_3w_3 = 0.$$

As $\mathcal{U} \cup \mathcal{W} = \{u_1, u_2, w_1, w_2, w_3\}$ is linearly independent this implies $a_1 = a_2 = b_1 - b_2 = b_3 = 0$. So $x = a_1u_1 + a_2u_2 = 0$. As x was an arbitrary element of $\mathrm{Span}(\mathcal{U}) \cap \mathrm{Span}(\mathcal{W})$ this completes the proof that $\mathrm{Span}(\mathcal{U}) \cap \mathrm{Span}(\mathcal{W}) = \{0\}$. $\square$

Remark: There are more hypothesis than needed in this problem. We only used that $\mathcal{U} \cup \mathcal{W}$ is linearly independent. (However if $\mathcal{U} \cup \mathcal{W}$ is linearly independent then its subsets $\mathcal{U}$ and $\mathcal{W}$ will each be linearly independent so it is not surprising that assuming that $\mathcal{U}$ and $\mathcal{W}$ are linearly independent is redundant.)

Solution 2: Note that $\mathcal{U} \cup \mathcal{W} = \{u_1, u_2, w_1, w_2, w_3\}$ will be a basis for $\mathrm{Span}\,\mathcal{U} + \mathrm{Span}\,\mathcal{W}$ and therefore $\dim(\mathrm{Span}\,\mathcal{U} + \mathrm{Span}\,\mathcal{W}) = 5$. Whence

$$\begin{aligned}\dim(\mathrm{Span}\,\mathcal{U} \cap \mathrm{Span}\,\mathcal{W}) &= \dim \mathrm{Span}\,\mathcal{U} + \dim \mathrm{Span}\,\mathcal{W} - \dim(\mathrm{Span}\,\mathcal{U} + \mathrm{Span}\,\mathcal{W}) \\ &= 2 + 3 - 5 = 0.\end{aligned}$$

As $\{0\}$ is the only zero dimensional subspace this implies $\mathrm{Span}\,\mathcal{U} \cap \mathrm{Span}\,\mathcal{W} = \{0\}$. $\square$