Mathematics 700 Homework
Due Wednesday, September 7

1. What is the span of $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$ in $\mathbf{R}^3$?
2. What is the span of $(1,0,1)$, $(0,1,1)$ and $(2,3,-3)$ in $\mathbf{R}^3$?
3. Give an example of there vectors in $\mathbf{R}^3$ so that any two are linear independent, but the set of all three is linearly dependent.
4. Let $\mathcal{P}_n$ the real polynomials of degree $\le n$. What is the dimension of $\mathcal{P}_n$. Prove your result by finding a basis of $\mathcal{P}_n$.
5. Let $z_1, z_2, z_3, z_4 \in \mathbf{C}$ be distinct complex numbers. Then when are the vectors
$$\begin{aligned} v_1 &= (1,1,1,1) \\ v_2 &= (z_1, z_2, z_3, z_4) \\ v_3 &= (z_1^2, z_2^2, z_3^2, z_4^2) \\ v_3 &= (z_1^3, z_2^3, z_3^3, z_4^3) \end{aligned}$$
linearly independent? Prove your result. (Hint: It is not hard to do this problem by direct calculation, but here is a less messy method. If $c_1, \dots, c_4 \in \mathbf{C}$ so that $c_1v_1 + c_2v_2 + c_3v_2 + c_4v_4 = 0$ then let $p(x)$ be the polynomial $p(x) = c_1 + c_2x + c_3x^2 + c_4x^3$. Then $p(z_k) = 0$ for $k = 1, \dots, 4$. How many roots does a polynomial of degree ≤ 3 have?)
6. Let $\mathcal{P}_3$ be the polynomials of degree ≤ 3 over the field $\mathbf{F}$. Let $a_1, a_2, a_3, a_4 \in \mathbf{F}$ be distinct. Let $\ell_i(x)$ for $i = 1, \dots, 4$ be the polynomials
$$\begin{aligned} \ell_1(x) &= \frac{(x-a_2)(x-a_3)(x-a_4)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}, \\ \ell_2(x) &= \frac{(x-a_1)(x-a_3)(x-a_4)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)}, \\ \ell_3(x) &= \frac{(x-a_1)(x-a_2)(x-a_4)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)}, \\ \ell_4(x) &= \frac{(x-a_1)(x-a_2)(x-a_3)}{(a_4-a_1)(a_4-a_2)(a_4-a_3)}. \end{aligned}$$
These are the ***Lagrange interpolation polynomials*** with ***nodes*** at $a_1, \dots, a_4$. (You are not responsible for this terminology, but it is standard and you will likely see it again in other classes.)
   (a) Show that $\ell_i(a_j) = \delta_{ij}$ where δ is the ***Kronecker delta function***:
$$\delta_{ij} := \begin{cases} 1, & i = j; \\ 0, & i \ne j. \end{cases}$$
   (b) Show that $\ell_1, \ell_2, \ell_3, \ell_4$ is a basis of $\mathcal{P}_3$. (Hint: To show linear independence set a linear combination of $\ell_1, \ell_2, \ell_3, \ell_4$ to 0 and evaluate this linear combination at a_1, a_2, a_3, a_4 and use that $\ell_i(a_j) = \delta_{ij}$.)
   (c) Let $b_1, b_2, b_3, b_4 \in \mathbf{F}$. Let
$$p(x) = b_1\ell_1(x) + b_2\ell_2(x) + b_3\ell_3(x) + b_4\ell_4(x).$$
   The show that $p(a_i) = b_i$ for $i = 1, \dots, 4$.

(d) OPTIONAL EXTRA CREDIT: Extend these results to $\mathcal{P}_n$ for $n \geq 1$.

7. Let $\mathcal{P}_2$ be the vector space of all real polynomials of degree ≤ 2.
   (a) Letting $\mathcal{P}_1 \subset \mathcal{P}_2$ in the natural way, show that $\mathcal{P}_1$ is a two dimensional subspace of $\mathcal{P}_2$.
   (b) For any $a \in \mathbf{R}$ show that $\mathcal{Z}(a) := \{p(x) \in \mathcal{P}_2 : p(a) = 0\}$ is a two dimensional subspace of $\mathcal{P}_2$ by giving a basis of $\mathcal{Z}(a)$.
   (c) True or False: If $\mathcal{V}$ is a two dimensional subspace of $\mathcal{P}_2$ then either $\mathcal{V} = \mathcal{P}_1$ or $\mathcal{V} = \mathcal{Z}(a)$ for some $a \in \mathbf{R}$? Prove your answer is correct.
8. Let V be a vector space with $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ subsets of V with the same finite number of elements. Assume that $\{u_1, \ldots, u_m\}$ is linearly independent and that

$$\operatorname{Span}\{u_1, \ldots, u_m\} \subseteq \operatorname{Span}\{v_1, \ldots, v_m\}.$$

   Then show $\operatorname{Span}\{u_1, \ldots, u_m\} = \operatorname{Span}\{v_1, \ldots, v_m\}$ and that $\{v_1, \ldots, v_m\}$ is linearly independent.