Mathematics 700 Homework
Due Wednesday, November 16

Problem 1. Prove the Cayley-Hamilton theorem as outlined in Problem 28 of the notes. $\square$

Problem 2. Show that the determinant of a triangular matrix is the product of its diagonal elements. $\square$

Problem 3. Let A be a block triangular matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are both square. Show that $\det(A) = \det(A_{11})\det(A_{22})$. $\square$

Problem 4. Let $A \in M_{n\times n}(R)$ where R is a commutative ring. Show for $c \in R$ that $\det(cA) = c^n \det(A)$. HINT: Write $A = [A_1, A_2, \ldots, A_n]$ in terms of its columns. Then $cA = [cA_1, cA_2, \ldots, cA_n]$. Now use the n linearity of det. $\square$

Problem 5. Let A be a 5×5 real matrix which is skew symmetric. That is $A^t = -A$ where A^t is the transpose of A. Then show that $\det(A) = 0$. Can you think of a generalization of this? $\square$

Problem 6. Let $A \in M_{n\times n}(R)$ where R is a commutative ring. Then show that in the characteristic polynomial $\operatorname{char}_A(x) = \det(xI_n - A)$ that the coefficient of x^{n-1} is $-\operatorname{tr}(A)$, where $\operatorname{tr}(A)$ the trace of A, and that the constant term is $(-1)^n \det(A)$. $\square$

Problem 7. Let $\lambda_1, \ldots, \lambda_n \in \mathbf{F}$ be n scalars. Then define the ***elementary symmetric functions*** $\sigma_1, \ldots, \sigma_n$ in $\lambda_1, \ldots, \lambda_n$ by

$$\begin{aligned}
\sigma_1 &= \sum_{1\le i\le n} \lambda_i \\
\sigma_2 &= \sum_{1\le i<j\le n} \lambda_i\lambda_j \\
\sigma_3 &= \sum_{1\le i<j<k\le n} \lambda_i\lambda_j\lambda_k \\
\vdots &= \quad \vdots \\
\sigma_k &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \\
\vdots &= \quad \vdots \\
\sigma_n &= \lambda_1\lambda_2\cdots\lambda_n.
\end{aligned}$$

When $n = 4$ the elementary symmetric function in $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are

$$\begin{aligned} \sigma_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ \sigma_2 &= \lambda_3\,\lambda_4 + \lambda_2\,\lambda_4 + \lambda_2\,\lambda_3 + \lambda_1\,\lambda_4 + \lambda_1\,\lambda_3 + \lambda_1\,\lambda_2 \\ \sigma_3 &= \lambda_2\,\lambda_3\,\lambda_4 + \lambda_1\,\lambda_3\,\lambda_4 + \lambda_1\,\lambda_2\,\lambda_4 + \lambda_1\,\lambda_2\,\lambda_3 \\ \sigma_4 &= \lambda_1\,\lambda_2\,\lambda_3\,\lambda_4. \end{aligned}$$

(a) Show that if $p(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$ then the expanded form of $p(x)$ is

$$p(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \sigma_3 x^{n-3} + \cdots + (-1)^n \sigma_n.$$

Thus the elementary symmetric functions are basically (up to some signs) the coefficients of the monic polynomial with roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. HINT: It is fine by me if you do this in the case when $n = 4$ and then explain how it generalizes. (If you find this problem hard, it is likely because at some point in your schooling a miss-guided teacher taught you to multiply binomials using the "foil" method. You should track this teacher down and explain to them, forcefully, that they have damaged your ability to do serious mathematics and that they should stop all farther "foiling" in their classrooms.)

(b) Let $T\colon V \to V$ be a linear operator on a finite dimensional vector space with $\dim V = n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the elementary symmetric functions in $\lambda_1, \ldots, \lambda_n$. Then show that the characteristic polynomial of T is

$$\operatorname{char}_A(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \sigma_3 x^{n-3} + \cdots + (-1)^n \sigma_n.$$

□

Problem 8. Let $T\colon V \to V$ be a linear operator on a finite dimensional vector space with $\dim V = n$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T. Then show that the trace and determinant of T are given by

$$\operatorname{tr}(T) = \lambda_1 + \cdots + \lambda_n \qquad \det(A) = \lambda_1\lambda_2\cdots\lambda_n.$$

That is the trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues. □

Problem 9. Let $A = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$, where D and G are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$. (Cf. Problem 4 on the January 1984 Admission to candidacy exam.) □