# Mathematics 700 Homework
Due Wednesday, November 3

**Problem 1.** (a) Find the basis of $\mathbf{R}^{2*}$ dual to the basis $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ of $\mathbf{R}^2$.
(b) Find the basis of $\mathbf{R}^2$ dual to the basis $f_1 = \begin{bmatrix} 2 & 7 \end{bmatrix}$, $f_2 = \begin{bmatrix} 4 & 5 \end{bmatrix}$ of $\mathbf{R}^{2*}$.

**Problem 2.** Let $A \in M_{m\times n}(\mathbf{F})$ and $B \in M_{n\times p}$. Assume that $A$ and $B$ are partitioned into blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $A_{ij}$ is $m_i \times n_j$ and $B_{ij}$ is $n_i \times p_j$ for $1 \le i, j \le 2$. (Thus $m_1 + m_2 = m$, $n_1 + n_2 = n$ and $p_1 + p_2 = p$.) Show that we can multiply $A$ and $B$ "block at a time". That is show

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

(For example in this formula $A_{12}B_{22}$ is the $m_1 \times p_2$ matrix obtained from the matrix multiplication of the $m_1 \times n_2$ matrix $A_{12}$ with the $n_2 \times p_2$ matrix $B_{22}$.)

**Problem 3.** State the generalization of the last problem to the block product of $AB$ (with $A \in M_{m\times n}(\mathbf{F})$ and $B \in M_{n\times p}$) where $A$ and $B$ are partitioned into a larger number of blocks. You do not have to prove this (as it is just like the last problem, but messier). HINT: See Schaum's §3.7 pages 79–80.

**Problem 4.** The following are some standard facts about the relationship between how a linear operator acts on a vector space and the form of its matrix representation in an appropriate basis. This a basic part of the theory and something you should make a point of remembering for the rest of your lives. In these problems let $V$ be a finite dimensional vector space and $T\colon V \to V$ a linear operator on $V$.

(a) Recall that a subspace $W \subset V$ is ***invariant under*** $T$ iff $w \in W$ implies $Tw \in W$. Let $k = \dim W$ and $n = \dim V$. Choose a basis $\mathcal{V} = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ so that $W = \operatorname{Span}\{v_1, \dots, v_k\}$. Then show that $W$ is invariant under $T$ if and only if the matrix $A := [T]_{\mathcal{V}}$ has the block upper triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11}$ is $k \times k$ and $A_{22}$ is $(n-k) \times (n-k)$. HINT: This is easy so please don't make it hard.

(b) A basis $\mathcal{V} = \{v_1, \dots, v_n\}$ is a ***triangular basis for*** $T$ iff for $Te_k \in \operatorname{Span}\{e_1, \dots, e_k\}$ for $k = 1, \dots, n$. Show that the basis $\mathcal{V}$ is triangular for $T$ if and only if the matrix $A = [T]_{\mathcal{V}}$ is upper triangular. Thus when $n = 4$ this means that the basis

$\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ is triangular if and only if the matrix $A = [T]_{\mathcal{V}}$ of of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

(In general a matix $A = [a_{ij}]$ is upper triangular iff $a_{ij} = 0$ for $i > j$.) If you wish you can just do the proof in the case of $n = 4$.

**Problem 5.** Let $V$ and $W$ be finite dimensional vector spaces and $T\colon V \to W$ a linear map. Then show

$$\ker(T^*) = \operatorname{Image}(T)^{\perp} \quad \text{and} \quad \operatorname{Image}(T^*) = \ker(T)^{\perp}.$$

HINT: See the proofs that $\ker(T) = \operatorname{Image}(T^*)^{\circ}$ and $\operatorname{Image}(T) = \ker(T^*)^{\circ}$ on pages 155–156 of the class notes.

**Problem 6.** Let $V$ be a finite dimensional vector space and $U, W$ subspaces of $V$. Then show the following:

(a) $U \subseteq W$ if and only if $U^{\perp} \supseteq W^{\perp}$ (note the direction of the inclusion is reversed).
(b) $U = W$ if and only if $U^{\perp} = W^{\perp}$.

**Problem 7.** Let $\mathcal{P}_3$ be the polynomials of degree $\leq 3$ over the field $\mathbf{F}$. Let $a_1, a_2, a_3, a_4 \in \mathbf{F}$ be distinct points. Define linear functionals $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathcal{P}_3^*$ by

$$\varepsilon_i(p) := p(a_i).$$

(a) Show that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is a basis of $\mathcal{P}_3^*$
(b) Find the basis of $\mathcal{P}_3$ dual to $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.

HINT: This only a dressed up version of a problem we did early on. So recall facts about Lagrange interpolation polynomials, or look up Problem 6 on the homework which was due on September 7 (#2 on the class web page).

**Problem 8.** Let $V$ be a finite dimensional vector space and let $f_1, f_2, f \in V^*$ be linear functionals on $V$. Show that $f$ is a linear combination of $f_1$ and $f_2$ if and only if $\ker(f_1) \cap \ker(f_2) \subseteq \ker(f)$. **Remark:** This generalizes as follows: If $f_1, \ldots, f_k, f \in V^*$ then $f \in \operatorname{Span}\{f_1, \ldots, f_k\}$ if and only if $\bigcap_{i=1}^{k} \ker(f_i) \subseteq \ker(f)$. The proof for $k = 2$ and the general case are basically the same.

**Problem 9.** Let $V$ and $W$ be finite dimensional vector spaces. Let $f \in V^*$ be linear functional on $V$ and $w \in W$ an element of $W$. Define a map $w \otimes f\colon V \to W$ by

$$(w \otimes f)(v) = f(v)w.$$

It is not hard to see that $w \otimes f$ is linear.

(a) Show if $f \neq 0$ and $w \neq 0$ then $w \otimes f$ has rank one.
(b) Conversely show that if $T\colon V \to W$ has rank one that there is an $f \in V^*$ and a $w \in W$ so that $T = w \otimes f$. HINT: Choose $w \in \operatorname{Image} T$ with $w \neq 0$. Then as $T$ has rank one $\{w\}$ is a basis of $\operatorname{Image}(T)$. For any $v \in V$ we have $Tv \in \operatorname{Image}(T)$ and thus $Tv = cw$ for some scalar $c$. How does $c$ depend on $v$?

**Problem 10.** Finally here is one just for fun. (Well we will also be using this result latter.) Recall from calculus that if $a \in R$ has $|a| < 1$ then $1/(1-a)$ can be computed by the geometric series

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots = \sum_{k=0}^{\infty} a^k. \tag{1}$$

This has a very neat generalization to matrices. A matrix $A \in M_{n\times n}$ is ***nilpotent*** iff for some $m \geq 1$ we have $A^m = 0$. As examples you can check that all of the following are nilpotent.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 2 & -4 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -3 & 7 & 4 & 0 \end{bmatrix}$$

Show that if $A$ is nilpotent, say $A^m = 0$, then $I - A$ is invertiable and

$$(I-A)^{-1} = I + A + A^2 + \cdots + A^{m-1}.$$

Note this is exactly like putting $a = A$ in the series (1) using that $A^k = 0$ for $k \geq m$. HINT: Set $B = I + A + A^2 + \cdots + A^{m-1}$ and by direct multiplication show that $(I-A)B = B(I-A) = I$.