Solution to Problem 6 Page 45

The *Jourkowski mapping* is defined by $w = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Show that

(a) $J(z) = J\left(\frac{1}{z}\right)$.

This is a straight forward chase through the definition of $J(z)$:

$$J\left(\frac{1}{z}\right) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{\left(\frac{1}{z}\right)}\right) = \frac{1}{2}\left(\frac{1}{z} + z\right) = J(z).$$

(b) J maps the unit circle $|z| = 1$ onto the real interval $[-1, 1]$.

The complex numbers z with $|z| = 1$ can be represented as $z = e^{i\theta}$ where $-\pi < \theta \leq \pi$. For these z we have

$$\begin{aligned} J(z) &= J(e^{i\theta}) \\ &= \frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right) \\ &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ &= \frac{1}{2}((cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)) \\ &= \frac{1}{2}(2\cos\theta) \\ &= \cos\theta + 0i. \end{aligned}$$

As θ moves over the interval $-\pi < \theta \leq \pi$ we see that $J(e^{i\theta}) = \cos\theta + 0i$ moves over the interval $-1 \leq x \leq 1$, $y = 0$.

(c) J maps the circle $|z| = r$ ($r > 0$, $r \neq 1$) onto the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1$$

which has foci at ± 1.

Let $z = x + iy$. For z with $|z| = 1$ we have $x^2 + y^2 = r^2$ Then by a calculation we did in class

$$\begin{aligned} J(z) &= \frac{1}{2}\left(x + iy + \frac{1}{x + iy}\right) = \frac{x}{2}\left(1 + \frac{1}{x^2 + y^2}\right) + \frac{y}{2}\left(1 - \frac{1}{x^2 + y^2}\right) \\ &= \frac{x}{2}\left(1 + \frac{1}{r^2}\right) + \frac{y}{2}\left(1 - \frac{1}{r^2}\right) =: u + iv \end{aligned}$$

Therefore

$$x = 2\frac{u}{\left(1 + \frac{1}{r^2}\right)} = \frac{u}{\frac{1}{2}\left(1 + \frac{1}{r^2}\right)}, \quad y = 2\frac{v}{\left(1 - \frac{1}{r^2}\right)} = \frac{v}{\frac{1}{2}\left(1 - \frac{1}{r^2}\right)}$$

Putting this in the equation $x^2 + y^2 = r^2$ gives

$$\frac{u^2}{\left[\frac{1}{2}\left(1 + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(1 - \frac{1}{r}\right)\right]^2} = r^2$$

and dividing this by r^2 gives the desired equation. (Strictly speaking we have only shown that J maps the circle *into* the ellipse where the problem ask us to show the map is also *onto*. I did not grad down for this.) As for the fact about the foci (which you did not have to show, the book was simply pointing out this fact) recall that if $0 < b < a$ then the foci of the ellipse $x^2/a^2 + y^2/b^2 = 1$ are at the points $(-c, 0)$ and $(c, 0)$ where $c = \sqrt{a^2 - b^2}$. In the case at hand

$$a = \frac{1}{2}\left(r + \frac{1}{r}\right), \qquad b = \left|\frac{1}{2}\left(r - \frac{1}{r}\right)\right|.$$

An easy calculation shows $a^2 - b^2 = 1$ so that $c = 1$.