Mathematics 551 Test #1, Take Home Portion

It is all right to work together on this test, but this does not mean you are allowed to copy!

Note: In this test if α is a regular curve, we will denote the unit tangent to α by $\mathbf{t}$ and the unit normal by $\mathbf{n}$.

(1) (10 points) A pond has a convex surface. If the shore line is 500 meters long and the surface area of the pond is 10,000 meters^2, then show that it is possible for a duck to swim to a point that is at least 20 meters from any point on the shore.

(2) (10 points) This problem gives a well known and useful formula for the curvature of a regular planar curve. Let $\alpha\colon [a,b]\colon \mathbb{R}^2$ be a C^2 regular curvature. We do not assume that $\alpha(t)$ is unit speed. Then show

$$\alpha'(t) \times \alpha''(t) = |\alpha(t)|^2 \kappa(t) e_3$$

where $\times$ is the usual cross product from vector analysis and $e_3 = (0,0,1)$ is the vector in the direction of the positive z-axis.

(3) (20 points) Let $\alpha\colon [0,L] \to \mathbb{R}^2$ be a unit speed close curve. Let $s_0 \in (0,L)$ be the point of α that is farthest from the origin. In this problem you will show the curvature of α at satisfies $|\kappa(s_0)| \geq 1/R$ where $R = |\alpha(s_0)|$ is the maximum distance of α from the origin. To start let

$$f(s) = |\alpha(s)|^2 = \alpha(s) \cdot \alpha(s).$$

(a) The function $f(s)$ has a maximum at $s = s_0$ and therefore $f'(s_0) = 0$. Use this to show that $\alpha(s_0) \cdot \mathbf{t}(s_0) = 0$.

(b) Explain why $\alpha(s_0) = \pm R\mathbf{n}(s_0)$.

(c) Use the Frenet formulas to find a general formula for $f''(s)$ that only involves $\alpha(s)$, $\kappa(s)$ and $\mathbf{n}(s)$.

(d) As $f(s)$ has a maximum at s_0 we have by the second derivative test that $f''(s_0) \leq 0$. Use this and parts (a), (b) and (c) to show $\kappa(s_0) \geq 1/R$.

(4) (20 points) Let $\alpha\colon [0,L] \to \mathbb{R}^2$ be a unit speed curve and assume that the curvature κ_α of α is positive. Let $r > 0$ be a constant. Define a new curve $\beta\colon [0,L] \to \mathbb{R}^2$ by

$$\beta(s) = \alpha(s) - r\mathbf{n}(s).$$

(a) Draw a picture to indicate why β is "the parallel curve at a distance r from α".

(b) Show that the speed of β is $v_\beta(s) = 1 + r\kappa_\alpha(s)$.

(c) Show the length of β is

$$\text{Length}(\beta) = L + r \int_0^L \kappa_\alpha(s)\, ds$$

where L is the length of α.

(d) Find a formula $\kappa_\beta(s)$ for the curvature of β.