Introduction to Series

Review: the microscope equation. Recall if x and y are related by $y = f(x)$ then the microscope equation at $x = a$ is

$$\Delta y \approx f'(a)\Delta x$$

where $\Delta x = (x - a)$ and $\Delta y = (y - f(a))$. For small values of Δx this approximation is quite good.

1. Write the microscope equation for the function $y = e^x$ at the point $x = 0$.

2. Use your answer to the last question to fill in the following table accurate to eight decimal places

x	True value of e^x	Approx. value of e^x	Error	Relative error
$.5$				
$-.5$				
$.1$				
$-.1$				
$.01$				
$-.01$				

We now want to get better approximations, what might be thought of as "super microscope equations", but instead are referred to by the more boring name of Taylor series. The idea is as follows: To approximate $f(x)$ at $x = a$ find the polynomial $P(x)$ of degree n that has its first n derivative equal to those of $f(x)$ at $x = a$ (where you get to choose the value of n). That is to say we want

$$P(a) = f(a), \quad P'(a) = f'(a), \quad P''(a) = f''(a), \ldots \quad P^{(n)}(a) = f^{(n)}(a)$$

This gives an approximation $f(x) \approx P(x)$. Rather than give general formulas here is an example that shows what is going on.

3. Let $P(x) = a_0 + a_1 x + a_2 x^2$ be a polynomial of degree 2 so that if $f(x) = e^x$ then $P(0) = f(0)$, $P'(0) = f'(0)$ and $P''(0) = f''(0)$. This gives an approximation

$$e^x \approx P(x) = a_0 + a_1 x + a_2 x^2 = \underline{\qquad} + \underline{\qquad} x + \underline{\qquad} x^2.$$

(you fill in the blanks.)

4. Using the approximation from the last problem fill in the following table to eight decimal places.

x	True value of e^x	Approx. value of e^x	Error	Relative error
$.5$				
$-.5$				
$.1$				
$-.1$				
$.01$				
$-.01$				

5. This problem is just like problem 3 except that we now want a "third order" approximation, that is a polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ of degree 3 so that if $f(x) = e^x$ then $P(0) = f(0)$, $P'(0) = f'(0)$, $P''(0) = f''(0)$ and $P'''(0) = f'''(0)$.This gives an an approximation

$$e^x \approx P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \underline{\qquad} + \underline{\qquad} x + \underline{\qquad} x^2 \underline{\qquad} x^3.$$

(you fill in the blanks.)

6. Using the approximation from the last problem fill in the following table to eight decimal places.

x	True value of e^x	Approx. value of e^x	Error	Relative error
1				
-1				
.5				
$-.5$				
.1				
$-.1$				
.01				
$-.01$				

7. If you are going to approximate a general function $f(x)$ to third order then, as in Problem 5 you want a polynomial $P(a) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ so that $P(0) = f(0)$, $P'(0) = f'(0)$, $P''(0) = f''(0)$ and $P'''(0) = f'''(0)$. Find a general formula for the coefficients a_0, a_1, a_2 and a_3.