MATH 142 **Worksheet #2** **Spring, 1995**

Vectors and parametric equations

1. Let $\mathbf{v} = (3,4)$, $\mathbf{w} = (-2,6)$, and $\mathbf{u} = (-1,-3)$. Compute:
   a. $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, and $|\mathbf{v}|$.
   b. a vector of length one (a unit vector) that points in the direction of $\mathbf{v}$.
   c. a vector with twice the magnitude of $\mathbf{v}$, but pointing in the opposite direction.
   d. two unit vectors perpendicular to $\mathbf{v}$.
   e. the point Q if $\mathbf{w}$ represents the displacement vector $\overrightarrow{PQ}$ and P is the point $(-1,-1)$.
   f. the scalars r and s so that $\mathbf{w} = r\mathbf{v} + s\mathbf{u}$. Illustrate with a sketch of all the vectors involved.
   g. $\mathbf{u}\cdot\mathbf{w}$, $\mathbf{u}\cdot\mathbf{v}$, and $\mathbf{v}\cdot\mathbf{w}$.

2. Suppose $\mathbf{A} = 2\hat{\imath} - 3\hat{\jmath}$ and $\mathbf{B} = 3\hat{\imath} + 4\hat{\jmath}$.
   a. Write a unit vector in the direction of $\mathbf{A}$.
   b. If Q is the point $(6,4)$, and $\mathbf{A}$ represents the displacement vector $\overrightarrow{PQ}$, find the coordinates of the point P.
   c. Express $5\mathbf{A} - 3\mathbf{B}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. Sketch all the vectors involved.

3. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are as above, and a particle moves so that its position vector is $\mathbf{r}(t) = \mathbf{u} + t\mathbf{v}$, where is the particle at time $t=0$, $t=1$, $t=3/2$, $t=-1$?
   a. What is its velocity vector $\mathbf{v}(t)$ and speed (the scalar $|\mathbf{v}(t)|$)?
   b. Write the parametric equations for the motion.
   c. Describe the path of the motion by making a sketch, and by finding an equation in just the variables x and y (that is, eliminate the parameter t from the equations for x and y).
   d. Give parametric equations that describe motion along the same path, but in the opposite direction.

4. A particle moves so that $\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} = (\cos t, \sin t)$.
   a. Plot the particle's position for at least 8 values of t in the interval $0 \le t \le 2\pi$.
   b. Describe the path of the particle, including its equation in the variables x and y only.
   c. Compute the velocity vector $\mathbf{v}(t)$, speed ($|\mathbf{v}(t)|$), and acceleration vector $\mathbf{a}(t) = \mathbf{v}'(t)$. Show $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ on a sketch of the path for several different times t. If the tails of $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are placed at the tip of $\mathbf{r}(t)$, what do you notice about their geometric relationships to one another? Can you explain physically why $\mathbf{a}(t)$ points the way it does?
   c. Give parametric equations for the motion of a particle that is moving along the same path, but in the opposite direction.
   d. What is the curve parameterized by $x(t) = a\cos t$, $y(t) = a\sin t$, if a is a positive constant?

5. A particle moves so that $x(t) = t^3$ and $y(t) = t$.
   a. Give the position vector, velocity vector, and speed at time t.
   b. Describe the path of the motion; give its equation in x and y.
   c. Give parametric equations for the motion of a particle that is moving along the same path, but in the opposite direction.
   d. Suppose the particle is moving along the same path, but $y(t) = e^t$. What is $x(t)$ in this case? How does this motion differ from the one described in part (a) of this problem?

6. Suppose a curve that is given parametrically by $(x(t), y(t))$ is locally linear, so the derivative $\frac{dy}{dx}$ exists everywhere.
   a. If $\frac{dy}{dx} = \infty$ at a point, what does this tell you about the appearance of the curve near that point? (Look back to volume I of the text.)
   b. Express $\frac{dy}{dt}$ in terms of $\frac{dy}{dx}$ and $\frac{dx}{dt}$. Where does this relationship come from?
   c. Express $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Use your formula to compute $\frac{dy}{dx}$ in terms of t for the curves of problems (3), (4), and (5).
   d. Use your formula to convert $\sqrt{1 + (\frac{dy}{dx})^2}\, dx$ to $\sqrt{(\quad)^2 + (\quad)^2}\, dt$.
   e. Find the length of the path of motion in problem (3) from $t = -1$ to $t = 2$.
   f. Derive the formula for the circumference of the circle $x^2 + y^2 = a^2$.

7. Use `curves.ms` to plot the curve $x(t) = t - \sin t$, $y(t) = 1 - \cos t$ for $-2\pi \le t \le 2\pi$. Compute the length of this curve. Suggestion: do it for just one bump, and then use symmetry. The trig identity $\sin^2 \frac{t}{2} = \frac{1}{2}(1 - \cos t)$ could be useful. Can you imagine how this curve could be produced in real life?

8. a. A moth is flying towards a lamp (conveniently located at $(0, 0)$), but its guidance system is a little bit strange: it always flies at an angle of $45°$ to the line of sight of the lamp, and it always keeps the lamp on its right hand (wing?) side. Describe the path of this moth to its doom.
   b. If a particle moves so that $\mathbf{r}(t) = (e^{-t/5} \sin t, e^{-t/5} \cos t)$, compute $\mathbf{v}(t)$, and the angle between $\mathbf{r}(t)$ and $\mathbf{v}(t)$ (hints for using Maple to do these calculations are in `curves.ms`). What must the angle between $-\mathbf{r}(t)$ and $\mathbf{v}(t)$ therefore be equal to? How are parts (a) and (b) of this problem related?

9. Return once again to the vectors of problem (1) and (2).
   a. Compute the angles between each two of the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and also between $\mathbf{A}$ and $\mathbf{B}$.
   b. Compute the work done if a force given by $\mathbf{w}$ acts on an object through a displacement given by $\mathbf{v}$.
   c. Compute the work done if a force given by $\mathbf{A}$ acts on an object through a displacement given by $\mathbf{B}$.