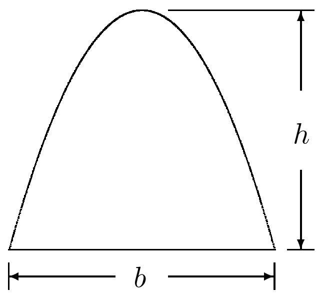
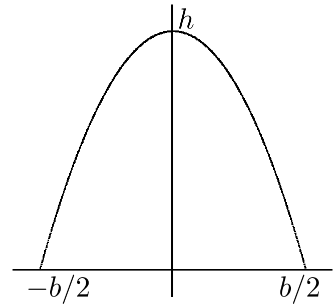
Answers to Group Project 1

1. Find a formula for area of the figure bounded by a parabolic arc and a line segment perpendicular to the axis of the parabola. The formula should be in terms of the length b of the base and the height h (see the figure labeled symmetric parabola). You should have a detailed description of why your formula holds even if you use the computer to do most of the calculations.



Symmetric Parabola



Solution: The general form of a parabola with its axis parallel to the y-axis is

$$y = ax^2 + bx + c.$$

(Where, sa usual, the x-axis is horizontal and the y-axis is vertical) We will choose a, b and c so that the top of the parabola is at the point $(0, h)$ on the y-axis and so that it cross the x axis at the points $(-b/2, 0)$ and $(b/2, 0)$ and shown on the figure on the right. The because $(0, h)$ is on the graph we have

$$a(0)^2 + b(0) + c = 0 \qquad \text{which implies} \quad c = h.$$

Now using $c = h$ and that $(-b/2, 0)$ and $(b/20, 0)$ are on the graph:

$$\begin{aligned} a(-b/2)^2 + b(-b/2) + h &= 0 \\ a(b/2)^2 + b(b/2) + h &= 0 \end{aligned}$$

Subtracting these two equations leads to $b = 0$. Adding them gives $a = -4h/b^2$. So the equation for our parabola is

$$y = \frac{-}{4}\frac{h}{b^2}x^2 + h.$$

To get the area now all we have to do is integrate this between the limits $x = -b/2$ and $x = b/2$.

$$\text{Area} = \int_{-b/2}^{b/2} \left(\frac{-}{4}\frac{h}{b^2}x^2 + h \right) dx = \left(\frac{-}{4}\frac{h}{b^2}\frac{x^3}{3} + hx \right)\Bigg|_{x=-b/2}^{b/2} = \frac{2}{3}bh.$$

That is

$$\text{Area} = \frac{2}{3}(\text{base})(\text{height}).$$

2. A projectile is shot from the top of a 200-foot high building with initial horizontal velocity $x'(0) = 10$ft/sec and initial vertical velocity of $y'(0) = 5$ft/sec. Ignore friction due to air and find the length the trajectory until it hits the ground. Note that by Newton's laws $x'' = 0$ and $y'' = -32$.

Solution: We put the coordinates so that the base of the building is at the origin. The the top edge is at the point $(0, 200)$.

Let t be the time in second since the projectile is fired. Let $x(t)$ be the horizontal distance the projectile has traveled and $y(t)$ the vertical distance it has traveled in feet. What we are given can then be expressed as the following initial value problems:

$$\begin{aligned} x''(t) &= 0, & x(0) &= 0, & x'(0) &= 10, \\ y''(t) &= -32, & y(0) &= 200, & y'(0) &= 5. \end{aligned}$$

The equation $x'' = (x')' = 0$ implies that $x'(t)$ is constant, say $x'(t) = c$. But this $x'(0) = 10$ implies $c = 10$. Thus $x'(t) = 10$. The antiderivative of this is $x(t) = 10t + c_1$ for anther constant c_1. Now $x(0) = 0$ implies $x(t) = 10t$. Likewise $y'' = (y')' = -32$ implies $y'(t) = -32t + c_2$ for a constant c_2 and $y'(0) = 5$ implies $c_2 = 5$. Thus $y'(t) = -32t + 10$. Taking anther antiderivative gives $y(t) = -16t^2 + 10t^c_2$. Using $y(0) = 200$ gives $c_3 = 200$. Thus we now know x and y as a function

$$x(t) = 10t \tag{1}$$
$$y(t) = -16t^2 + 5t + 200 \tag{2}$$

We next find out how long after the projectile that it hits the ground. As in our set up the ground level is defined by $y = 0$ we solve the equation $y(t) = -16^t + 5t + 200 = 0$ for t. This has two solutions, one positive and one negative. Only the positive one is relevant and using Maple we find it is

$$t_0 = 3.695234892$$

Putting this into the formula for $x(t)$ gives that the projectile hits at a distance of

$$x_0 = x(t_0) = 10t_0 = 36.95234892$$

This is not the final answer. It is only the distance from the base of the building where the projectile lands. What we are trying to do is find the length that the projectile has traveled during its trip.

For the graphs of functions $y = f(x)$ we have the formula length

$$\text{Length} = \int_a^b \sqrt{1 + (y')^2}\, dx = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

To get our function into this form we solve for t is (1) to get $t = x/10$. We then substitute this into (2) to get y as a function $f(x)$ of x.

$$y = f(x) = -16\left(\frac{x}{10}\right)^2 + 5\left(\frac{x}{10}\right) + 200 = -\frac{4}{25}x^2 + \frac{1}{2}x + 200.$$

In our problem x varies between $a = 0$ and $b = x_0 = 36.95234892$. Thus we can use Maple to compute

$$\text{Length} = \int_0^{x_0} \sqrt{1 + f'(x)^2} = 207.6739681$$

which is the answer.

Solution: a. When I first saw this problem my guess was that if the radius of the sphere was large enough the volume of the napkin holder would become very large. As I used to be in the habit of thinking very large I did the following thought experiment: Let $h = 1$foot take the sphere to have the same radius as the Earth (that is 2,963miles = 6,380km) and bore it out so that all that is left is band around the equator one foot wide. Then, dispute the fact that the strip is very thin it is $2\pi(2963) = 12450.13$ miles long so that its volume seemed to me to have to be larger that that of a ball of radius 2 with a hole bored in it.

b. Let R be the radius of the sphere and consider the sphere of radius R centered at the origin in the x-y plane. This circle has equation $x^2+y^2 = r^2$. The volume we are interested in obtained by rotating the shaded region in the figure about the y axis. We are only interested in the parts of the circle between $y = h/2$ and $y = -h/2$. But putting $y = h/2$ into the equation $x^2+y^2 = R^2$ we can solve for x and get that this line intersects the circle at the points $(\sqrt{R^2-(h/2)^2}, h/2)$ and $(-\sqrt{R^2-(h/2)^2}, h/2)$. Likewise the line $y = -h/2$ intersects $x^2+y^2 = R^2$ at the points $(\pm\sqrt{R^2-(h/2)^2}, -h/2)$. This means that the radius of the cylinder that we are removing from the ball is

$$r_{in} = \sqrt{R^2-(h/2)^2}.$$

More generally the line through the point $(0, y)$ and parallel to the x-axis will intersect our circle $x^2+y^2 = R^2$ at the point $(\sqrt{R^2-y^2}, y)$ and $(-\sqrt{R^2-y^2}, y)$ (the x coordinates of these points were found by solving for x in the equation $x^2+y^2 = R^2$). Therefore the plane perpendicular to the y-axis at the point $(0, y, 0)$ intersects out napkin holder in the region between to concentric circles the inner radius being r_{in} as above and the outer radius being

$$r_{out}(y) = \sqrt{R^2-y^2}.$$

The area of this region is found by subtracting the area on the inner circle form that of the outer circle. That is

$$A(y) = \pi(r_{out}(y)^2 - r_{in}^2) = \pi(R^2 - y^2 - (R^2 - (h/2))) = \pi(h^2/4 - y^2).$$

(Note that something surprising happens and the R cancel out.) Now we can find the volume of the napkin holder by integrating up these areas:

$$\text{Volume} = \int_{-h/2}^{h/2} \pi \left(\frac{h^2}{4} - y^2 \right) dy = \pi \left(\frac{h^2}{4} y - \frac{y^3}{3} \right) \Bigg|_{y=-h/2}^{h/2} = \frac{\pi h^3}{6}.$$

This shows that the volume is independent of the radius R is the sphere and so my original guess that very large radii.

In the case of $h = 1$ foot and $R = 2963$ then a calculation shows that the thickness of the band is 2.6213×10^{-8}cm. The diameter of an atom is just about 1.0×10^{-8}cm so that if I had done my original experiment the resulting strip would only have the thickness of about two and half atoms.