Intersecting Families with Minimum Volume

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This note is inspired by the works of G. W. Peck

Abstract

We determine the minimum volume (sum of cardinalities) of an intersecting family of subsets of an n-set, given the size of the family, by solving a simple linear program. From this we obtain a lower bound on the average size of the sets in an intersecting family. This answers a question of G. O. H. Katona, whose 60th birthday we celebrate with this result.

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Let $[n]$ denote the n-set $\{1, \ldots, n\}$. A family of subsets of $[n]$ is said to be *intersecting* if every member intersects every other. Let $\mathcal{F}$ be a nonempty intersecting family of subsets of $[n]$, and set $m = |\mathcal{F}|$. For any $A \subseteq [n]$, not both A and its complement $[n]\backslash A$ belong to $\mathcal{F}$, so we deduce the well-known fact that $m \leq 2^{n-1} = \sum_{i=1}^{n} \binom{n-1}{i-1}$. Let k be the greatest integer at most n such that $\sum_{i=1}^{k} \binom{n-1}{i-1} \leq m$. By convention, $\binom{n-1}{-1} = 0$, so that when $m = 0$, $k = 0$.

G.O.H. Katona asked [5] whether it is always true, when $m > 0$, that the average size of the sets in $\mathcal{F}$ is at least

$$\frac{\sum_{i=1}^{k} i\binom{n-1}{i-1}}{\sum_{i=1}^{k} \binom{n-1}{i-1}}.$$

This bound is the average set size when $\mathcal{F}$ consists of all subsets of $[n]$ of size at most k containing the element 1 (and $m = \sum_{i=1}^{k} \binom{n-1}{i-1}$). Katona's question is the analogue for intersecting families of the Kleitman-Milner Theorem [6] for antichains (Sperner families), and it is related to results in the paper, also in this special issue, of Bey, Engel, Katona, and Leck for intersecting antichains [2].

Katona's bound is obtained in this note by determining the intersecting families $\mathcal{F}$ of size m with minimum average set size. It is more convenient to minimize the *volume*, denoted by $v(\mathcal{F})$, which is the sum of sizes $\sum_{A\in\mathcal{F}} |A|$. The average set size in $\mathcal{F}$ is $v(\mathcal{F})/m$. Define the *profile vector* $\mathbf{p}(\mathcal{F}) = (p_0, \ldots, p_n)$, where $p_i = |\{A \in \mathcal{F} : |A| = i\}|$.

We shall derive the families of minimum volume by associating a certain linear program below with the problem and solving it by a series of simple shifts.

(1)	minimize	$\sum_{i=0}^{n} ix_i$	
(2)	subject to	$x_i \leq \binom{n-1}{i-1}$	$(0 \leq i \leq \frac{n}{2})$
(3)		$x_i + x_{n-i} \leq \binom{n}{i}$	$(0 \leq i < \frac{n}{2})$
(4)		$\sum_{i=0}^{n} x_i = m$	
(5)		$x_i \geq 0$	$(0 \leq i \leq n)$

We first claim that $\mathbf{p}(\mathcal{F})$ is feasible in the LP, taking $x_i = p_i$. Observe that the constraints (2) hold by applying the Erdős-Ko-Rado Theorem [4] to the intersecting families $\{A \in \mathcal{F} : |A| = i\}$. The constraints (3) hold since an i-subset and its complement cannot both belong to $\mathcal{F}$. Constraint (4) gives $|\mathcal{F}|$, and (5) is trivial. The objective value (1) is just the volume $v(\mathcal{F})$.

Next we solve the LP. Start with any feasible solution $\mathbf{x} = (x_0, \ldots, x_n)$. For each $i < \frac{n}{2}$, increase x_i and decrease x_{n-i} by the same amount, $\min\{x_{n-i}, \binom{n-1}{i-1} - x_i\}$. All LP constraints continue to hold after these shifts. Further, for each $i < \frac{n}{2}$ we either have $x_i < \binom{n-1}{i-1}$ and $x_{n-i} = 0$, or we have $x_i = \binom{n-1}{i-1}$ and $x_{n-i} \leq \binom{n}{i} - x_i = \binom{n}{i} - \binom{n-1}{i-1} = \binom{n-1}{i} = \binom{n-1}{n-i-1}$. Thus, along with (4) and (5), the new solution $\mathbf{x}$ satisfies these stricter conditions in place of (2) and (3):

$$(6) \qquad x_i \le \binom{n-1}{i-1} \quad (0 \le i \le n).$$

Suppose there exist indices i, j with $i < j$ such that $x_i < \binom{n-1}{i-1}$ and $x_j > 0$. Choose the least such i and the greatest such j. Now increase x_i and decrease x_j by the same amount, $\min\{x_j, \binom{n-1}{i-1} - x_i\}$. The resulting vector still satisfies (4), (5), (6).

Continue doing this until no such pair $i < j$ exists, which happens since the gaps $j - i$ get smaller with each shift. Each of the shifts above strictly decreases the objective function (1). When no further shifts are possible, let l be the largest index such that $x_l > 0$. Then x_i must be $\binom{n-1}{i-1}$ for $i < l$, and it must be 0 for $i > l$. Hence, we must be at the solution $\mathbf{y}$ with

$$y_i = \begin{cases} \binom{n-1}{i-1}, & 1 \le i \le k; \\ m - \sum_{j=1}^{k} \binom{n-1}{j-1}, & i = k+1; \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $\mathbf{y}$ is the unique optimal LP solution, since we showed that it is strictly better than any other feasible solution $\mathbf{x}$.

Next observe that $\mathbf{y}$ is the profile vector for an intersecting family of size m. Let $a \in [n]$ and consider the "star" of all subsets of $[n]$ that contain a. Take every set in the star of size at most k, along with any y_{k+1} sets of size $k+1$ in the star. Any family formed in this way for some a will be called a *cone*, and its profile is $\mathbf{y}$.

Moreover, the only intersecting families with this profile are the cones. A nonempty intersecting family $\mathcal{F}$ with profile $\mathbf{y}$ contains a singleton $\{a\}$, since $y_1 = 1$. Then all sets in $\mathcal{F}$ must contain a, and $\mathcal{F}$ lies inside a star. Because of its profile, $\mathcal{F}$ must be a cone. We have therefore proven:

Theorem. *Let $\mathcal{F}$ be an intersecting family of subsets of $[n]$ with $|\mathcal{F}| = m$. Then $v(\mathcal{F}) \ge v(\mathcal{C})$, where $\mathcal{C}$ is any cone, and equality holds if and only if $\mathcal{F}$ is a cone.*

Tossing out the sets of (largest) size $k+1$ strictly decreases the average size of the sets in a cone, so we can answer Katona's question:

Corollary. *Let $\mathcal{F}$ be a nonempty intersecting family of subsets of $[n]$. Let $m = |\mathcal{F}|$, and define k to be the greatest integer at most n such that $\sum_{i=1}^{k} \binom{n-1}{i-1} \le m$. Then the average size of the sets in $\mathcal{F}$ is at least $\frac{\sum_{i=1}^{k} i\binom{n-1}{i-1}}{\sum_{i=1}^{k} \binom{n-1}{i-1}}.$ Moreover, the bound is tight if and only if $m = \sum_{i=1}^{k} \binom{n-1}{i-1}$ and $\mathcal{F}$ is a cone of size m.*

One can avoid the shifting operations in our proof by instead formulating the dual LP and presenting its solution, call it $\mathbf{z}$. It suffices to check the feasibility of $\mathbf{y}$ and $\mathbf{z}$ and the equality of their objective values to establish their optimality. Complementary slackness can be used to prove that $\mathbf{y}$ is uniquely optimal.

Erdős (Péter), Frankl, and Katona determined the profile polytope for intersecting families of an n-set, which is the convex hull of the profile vectors (in $\mathbf{R}^{n+1}$). This means that its extreme points are known [3]. One can now add the constraint that the family size be m by restricting attention to the intersection of the polytope with the corresponding hyperplane. For the minimum volume problem, one would need to determine the extreme point of this intersection that has the minimum value of (1).

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