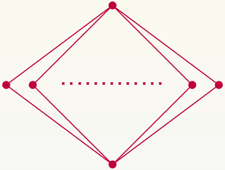
Searching for Diamonds



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Celebrating the 70th birthday of
Prof. G.O.H. Katona
EuroComb, Budapest, August 31, 2011

For a poset P, we consider how large a family $\mathcal{F}$ of subsets of $[n] := \{1, \ldots, n\}$ we may have in the Boolean Lattice $\mathcal{B}_n : (2^{[n]}, \subseteq)$ containing no (weak) subposet P.

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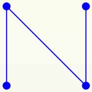
Example



For the poset $P = \mathcal{N}$, $\mathcal{F} \not\supset$ means $\mathcal{F}$ contains no 4 subsets A, B, C, D such that

$$A \subset B,\ C \subset B,\ C \subset D$$

.

Note that but $A \subset C$ is allowed: The subposet does not have to be *induced*, e.g., $\mathcal{F} \not\supset \Rightarrow \mathcal{F} \not\supset$

Given a finite poset P, we are interested in determining or estimating $\boxed{\mathrm{La}(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]}, P \not\subset \mathcal{F}\}.}$

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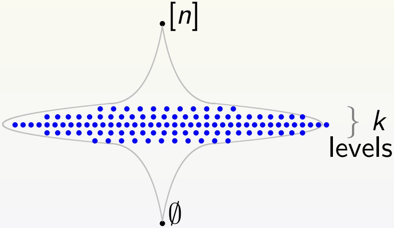
For many posets, $\mathrm{La}(n, P)$ is exactly equal to the sum of middle k binomial coefficients, denoted by $\color{red}{\Sigma(n, k)}$.

Moreover, the largest families may be $\color{red}{\mathcal{B}(n, k)}$, the families of subsets of middle k sizes.



Foundational results: Let $\mathcal{P}_k$ denote the k-element chain (path poset).

Theorem (Sperner, 1928)

For all n,

$$\mathrm{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

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Theorem (Erdős, 1945)

For general k and n,

$$\mathrm{La}(n, \mathcal{P}_k) = \Sigma(n, k-1),$$

and the extremal families are $\mathcal{B}(n, k-1)$.

Foundational results: Let $\mathcal{V}_r$ denote the poset of r elements above a single element.

Theorem (Katona-Tarján, 1981)

As $n \to \infty$,

$$\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}(n, \mathcal{V}_2) \leq \left(1 + \frac{2}{n}\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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Theorem (Thanh 1998, DeBonis-Katona, 2007)

For general r, as $n \to \infty$,

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \frac{\mathrm{La}(n, \mathcal{V}_r)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \left(1 + 2\frac{r-1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

More results for small posets: Let B denote the Butterfly poset with two elements each above two other elements. Let $\mathcal{N}$ denote the four-element poset shaped like an N.

Theorem (DeBonis-Katona-Swanepoel, 2005)

For all $n \geq 3$

$$\mathrm{La}(n, B) = \Sigma(n, 2),$$

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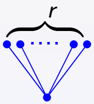
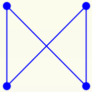
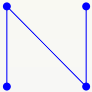
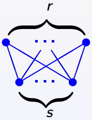
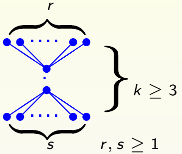
Excluded subposet P	$\mathrm{La}(n, P)$	
$\mathcal{P}_2$ 	$\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[Sperner, 1928]
Path $\mathcal{P}_k$, $k \geq 2$ 	$\Sigma(n, k-1)$ $\sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[P. Erdős, 1945]
r-fork $\mathcal{V}_r$ 	$\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$	[Katona-Tarján, 1981] [DeBonis-Katona 2007]

Excluded subposet P	$\mathrm{La}(n, P)$	
Butterfly B 	$\Sigma(n,2)$ $\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[DeBonis-Katona-Swanepoel, 2005]
$\mathcal{N}$ 	$\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$	[G.-Katona, 2008]
$\mathcal{K}_{r,s}(r,s \geq 2)$  $r, s \geq 2$	$\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[De Bonis-Katona, 2007]

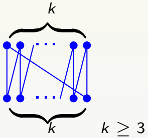
Excluded subposet P — $\mathrm{La}(n, P)$

Batons, $\mathcal{P}_k(s,t)$



$\sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$ — [G.-Lu, 2009]

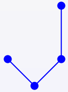
Crowns $\mathcal{O}_{2k}$



k even: $\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$

k odd: $\leq (1 + \frac{1}{\sqrt{2}})\binom{n}{\lfloor \frac{n}{2} \rfloor}$ — [G.-Lu, 2009]

$\mathcal{J}$



$\Sigma(n, 2)$

$\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$ — [Li, 2009]

Asymptotic behavior of $\mathrm{La}(n, P)$

Definition

$\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.

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Moreover, Saks and Winkler (2008) observed what $\pi(P)$ is in known cases, leading to the stronger

Conjecture (G.-Lu, 2009)

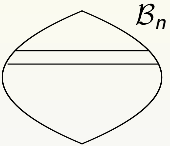
For all P, $\pi(P) = e(P)$, where

Definition

$e(P) :=$ max m *such that for all n, $P \not\subset \mathcal{B}(n, m)$.*

Example: Butterfly B

For all n, $\mathcal{B}(n,2) \not\supset \bowtie \Rightarrow e(\bowtie) = 2$,



while $\mathrm{La}(n, \bowtie) = \Sigma(n,2) \Rightarrow \pi(\bowtie) = 2$.

$\pi(P)$ and Height

Definition

The height $h(P)$ is the maximum size of any chain in P.

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Theorem (G.-Lu, 2009)

Let T *be a height 2 poset which is a tree (as a graph) of order* t*, then*

$$\frac{\mathrm{La}(n,T)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1 + \frac{16t}{n} + O\left(\frac{1}{n\sqrt{n \log n}}\right).$$

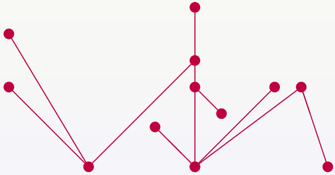
$\pi(P)$ and Height

The Forbidden Tree Theorem

Theorem (Bukh, 2010)

Let T be a poset such that the Hasse diagram is a tree. Then

$$\pi(T) = e(T) = h(T) - 1.$$



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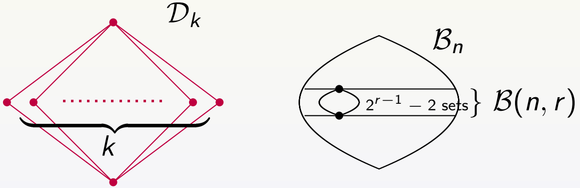
What about taller posets P?

For P of height 3 $\pi(P)$ cannot be bounded:

Example (Jiang, Lu) k-diamond poset $\mathcal{D}_k$

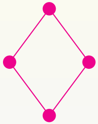


$\mathcal{B}(n, r) \not\supset \mathcal{D}_k$ for $k = 2^{r-1} - 1$, so $\pi(\mathcal{D}_k) \geq r$ if it exists.

On the Diamond $\mathcal{D}_2$

Problem

Despite considerable effort it remains open to determine the value $\pi(\mathcal{D}_2)$ or even to show it exists!



Easy bounds:
$\Sigma(n,2) \leq \mathrm{La}(n,\mathcal{D}_2) \leq \Sigma(n,3)$
$\Rightarrow 2 \leq \pi(\mathcal{D}_2) \leq 3$

The conjectured value of $\pi(\mathcal{D}_2)$ is its lower bound, $e(\mathcal{D}_2) = 2$.
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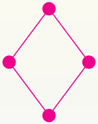

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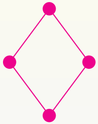

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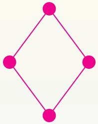

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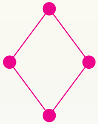

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The D_2 Diamond Theorem

Theorem

As $n \to \infty$,

$$\boxed{\Sigma(n,2) \le \mathrm{La}(n,\mathcal{D}_2) \le \left(2\frac{3}{11} + o_n(1)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.}$$

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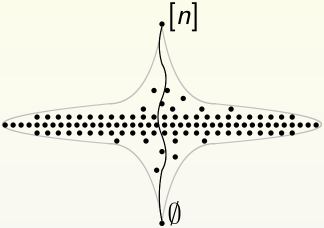
We prove this and most of our other results by considering, for a P-free family $\mathcal{F}$ of subsets of $[n]$, the average number of times a random full (maximal) chain in the Boolean lattice $\mathcal{B}_n$ meets $\mathcal{F}$, called the *Lubell function*.

Lubell Function

A *full chain* $\mathcal{C}$ in $\mathcal{B}_n$ is a collection of $n+1$ subsets as follows:

$$\emptyset \subset \{a_1\} \subset \cdots \subset \{a_1, \ldots, a_n\}.$$



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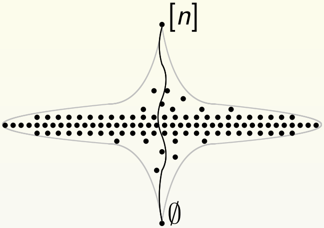

Definitions

Let $\mathcal{C} = \mathcal{C}_n$ be the set of full chains in $\mathcal{B}_n$.

For $\mathcal{F} \subset 2^{[n]}$, the height $h(\mathcal{F}) := \max_{C \in \mathcal{C}} |\mathcal{F} \cap C|$.

The Lubell function $\bar{h}(\mathcal{F}) := \mathrm{ave}_{C \in \mathcal{C}} |\mathcal{F} \cap C|$.

Lubell Function

Lemma

Let $\mathcal{F}$ be a collection of subsets of $[n]$.

1. We have

$$\bar{h}(\mathcal{F}) = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.$$

2. If $\bar{h}(\mathcal{F}) \leq m$, for some real number $m > 0$, then

$$|\mathcal{F}| \leq m \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It means that the Lubell function provides an upper bound on $|\mathcal{F}| / \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Lubell Function

Lemma

(ctd.) Let $\mathcal{F}$ be a collection of subsets of $[n]$.

3. If $\overline{h}(\mathcal{F}) \leq m$, for some **integer** *$m > 0$, then*

$$|\mathcal{F}| \leq \Sigma(n, m),$$

and equality holds if and only if

(1) $\mathcal{F} = \mathcal{B}(n, m)$ when $n + m$ is **odd***, or*

(2) $\mathcal{F} = \mathcal{B}(n, m-1)$ together with any $\binom{n}{(n+m)/2}$ subsets of sizes $(n \pm m)/2$ when $n + m$ is **even***.*

Lubell Function

Let $\lambda_n(P)$ be $\max \bar{h}(\mathcal{F})$ over all P-free families $\mathcal{F} \subset 2^{[n]}$. Then we have

$$\Sigma(n, e(P)) \leq \mathrm{La}(n, P) \leq \lambda_n(P)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We study $\lambda_n(P)$ and use it to investigate the $\pi(P) = e(P)$ conjecture for many posets.

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We study $\lambda_n(P)$ and use it to investigate the $\pi(P) = e(P)$ conjecture for many posets.
Asymptotics: Recall the limit $\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$. Let $\lambda(P) := \lim\limits_{n\to\infty} \lambda_n(P)$.

$$\boxed{e(P) \leq \pi(P) \leq \lambda(P),}$$

if both limits exist.

Note on $\mathcal{D}_2$-free Families

The limit $\pi(\mathcal{D}_2)$ is shown to be < 2.3, if it exists, by proving that the maximum Lubell values $\lambda_n(\mathcal{D}_2)$ are nonincreasing for $n \geq 4$ and by investigating their values for $n \leq 12$.

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However, there are known families of subsets with Lubell function values $\rightarrow 2.25$ as $n \rightarrow \infty$. Hence, $\lambda(\mathcal{D}_2)$ exists, and is at least 2.25, which is a barrier for this approach to showing $\pi(\mathcal{D}_2) = 2$.

Uniformly L-bounded Posets

For many posets we can use the Lubell function to completely determine $\mathrm{La}(n, P)$ and the extremal families.

Proposition

For a poset P satisfying $\lambda_n(P) \le e(P)$ for all n, we have

$$\boxed{\mathrm{La}(n, P) = \Sigma(n, e(P))} \text{ for all } n.$$

If $\mathcal{F}$ is a P-free family of the largest size, then

$$\boxed{\mathcal{F} = \mathcal{B}(n, e(P)).}$$

We say posets that satisfy the inequality above are *uniformly L-bounded*.

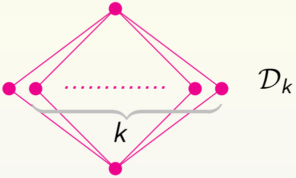
The k-Diamond Theorem

Theorem

The k-diamond posets $\mathcal{D}_k$ satisfy

$$\boxed{\lambda_n(P) \leq e(P)}$$

for all n, if k is an integer in the interval $[2^{m-1} - 1, 2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor} - 1]$ for any integer $m \geq 2$.



This means the posets $\mathcal{D}_k$ are uniformly L-bounded for $k = 1, 3, 4, 7, 8, 9, \ldots$. Consequently, for most values of k, D_k satisfies the $\pi = e$ conjecture, and, moreover, we know the largest D_k-families for all values of n.

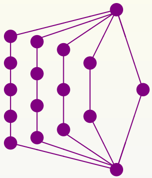
The Harp Theorem

Theorem

The harp posets $\mathcal{H}(\ell_1, \ldots, \ell_k)$ satisfy

$$\boxed{\lambda_n(P) \leq e(P)}$$

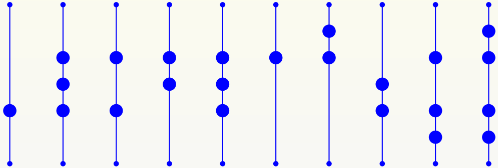
for all n, if $\ell_1 > \cdots > \ell_k \geq 3$.



Hence, harps with distinct path lengths are uniformly L-bounded and satisfy the $\pi = e$ conjecture.

Proof Sketch: The Partition Method

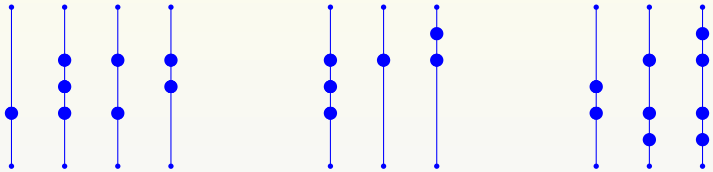
The Lubell function $\bar{h}(\mathcal{F})$ is equal to the average number of times a full chain intersects the family $\mathcal{F}$.



One of the key ideas (due to Li) involves splitting up the collection $\mathcal{C}_n$ of full chains into blocks that have a nice property, and computing the average on each block. Then $\bar{h}(\mathcal{F})$ is at most the maximum of those averages.

Proof Sketch: The Partition Method

The Lubell function $\bar{h}(\mathcal{F})$ is equal to the average number of times a full chain intersects the family $\mathcal{F}$.



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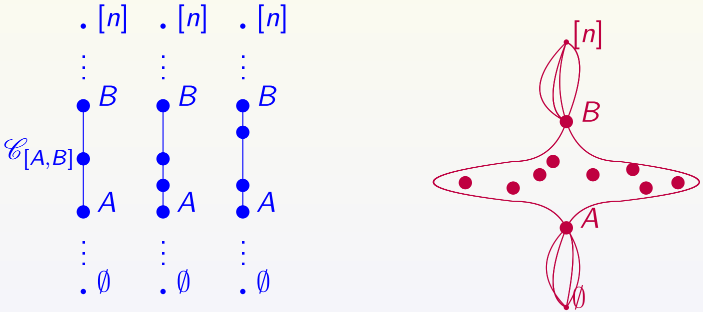
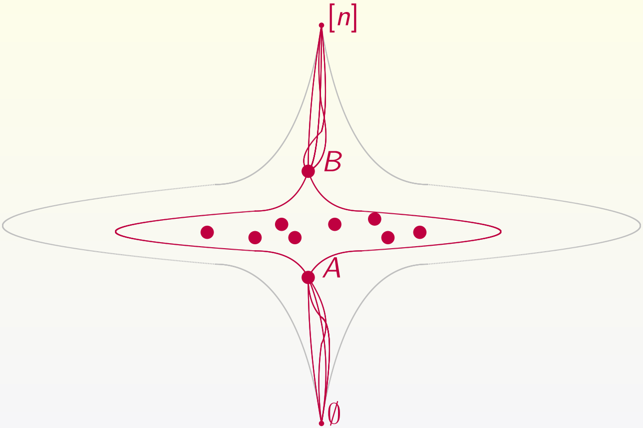
Proof Sketch: The Partition Method

Min-Max Partition

The block $\mathscr{C}_{[A,B]}$ consists of full chains with $\min \mathcal{F} \cap \mathcal{C} = A$ and $\max \mathcal{F} \cap \mathcal{C} = B$.





Compute $\operatorname{ave}_{\mathcal{C}\in\mathscr{C}_{[A,B]}}|\mathcal{F}\cap\mathcal{C}|$ for each block $\mathscr{C}_{[A,B]}$. $\square$

More on the Lubell Function

Recall that $e(P) \leq \pi(P) \leq \lambda(P)$ when the limits $\pi(P)$ and $\lambda(P)$ both exist. For a uniformly L-bounded poset P,
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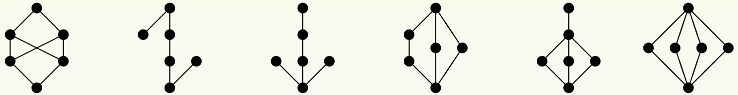
The Butterfly B is not uniformly L-bounded (since $\lambda_2 = 3 > e$), though $\mathrm{La}(n, B) = \Sigma(n, 2)$ for all $n \geq 3$.
The diamond $\mathcal{D}_2$ is not uniformly L-bounded, though many diamonds $\mathcal{D}_k$ and harps are.
Still, it can be proven that $\lambda(P)$ exists whenever P is a diamond $\mathcal{D}_k$ or a harp $\mathcal{H}(\ell_1, ..., \ell_k)$.

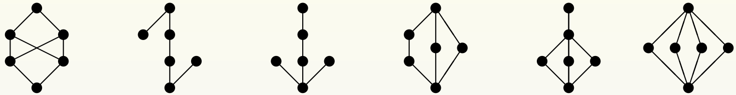
More on the Lubell Function

More uniformly L-bounded posets



More on the Lubell Function

More uniformly L-bounded posets

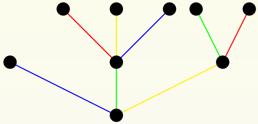


Definition

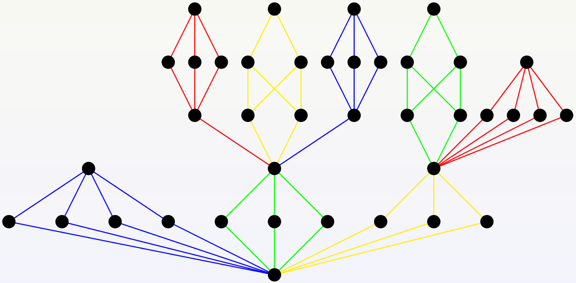
Suppose posets $P_1, \ldots, P_k$ are uniformly L-bounded with 0 and 1. A blow-up of a rooted tree T on k edges has each edge replaced by a P_i.

Constructions

Theorem (Li, 2011)
If P is a blow-up of a rooted tree T, then $\pi(P) = e(P)$.
If the tree is a path, then P is uniformly L-bounded.



A blow-up of the rooted tree above:



Forbidding Induced Subposets

Less is known for this problem:

Definition

We say P *is an* *induced* *subposet of* Q*, written* $P \subset^* Q$ *if there exists an injection* $f : P \to Q$ *such that for all* $x, y \in P$*,* $x \leq y$ *iff* $f(x) \leq f(y)$*. We define* $\mathrm{La}^*(n, P)$ *to be the largest size of a family of subsets of* $[n]$ *that contains no induced subposet* P*.*

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Theorem (Carroll-Katona, 2008)

As $n \to \infty$,

$$\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}^*(n, \mathcal{V}_2) \leq \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Forbidding Induced Subposets

Extending Bukh's Forbidden Tree Theorem:

Theorem (Boehnlein-Jiang, 2011)

For every tree poset T,

$$\mathrm{La}^*(n, T) \sim (h(T) - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}, \mathrm{as} n \to \infty.$$

Future Research

Problem

Determine for the diamond $\mathcal{D}_2$ whether $\pi(\mathcal{D}_2)$ exists and equals 2. The current best upper bound is $2.2727\ldots$.

Problem

Determine for the crown $\mathcal{O}_6$ whether $\pi(\mathcal{O}_6)$ exists and equals 1. The current best upper bound is $1.707\ldots$.

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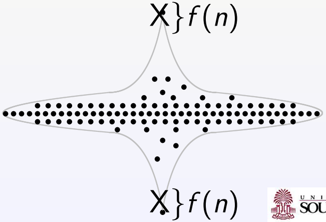
A possible way to tackle it:

Compute the maximum value of $\bar{h}(\mathcal{F})$ over all P-free families $\mathcal{F}$ such that every $F \in \mathcal{F}$ satisfies the condition $f(n) \leq |F| \leq n - f(n)$.



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Prove that for the diamond poset $\mathcal{D}_2$, the limiting Lubell function value $\lambda(\mathcal{D}_2)$, which exists, equals its lower bound of 2.25.

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Prove that $\lambda(P)$ exists for general P.

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Prove that $\lambda(P)$ exists for general P.

Problem

Provide insight into why

- $\mathrm{La}(n, P)$ *behaves very nicely for some posets, equalling* $\Sigma(n, e(P))$ *for all* $n \geq n_o$ *(such as the butterfly* B *and the diamonds* $\mathcal{D}_k$ *for most values of* k*);*
- *Is more complicated, but behaves well asymptotically (such as* $\mathcal{V}_2$*); or*
- *Continues to resist asymptotic determination (such as* $\mathcal{D}_2$ *and* $\mathcal{O}_6$*).*

Prof. Gyula O. H. Katona Through History.
Rényi Institute photo

Lecturing in Cochin, India, 2010 (Katona is on the left).

Lecturing in Columbia, SC, 2007

Reacting to Southern food? Columbia, SC, 2007

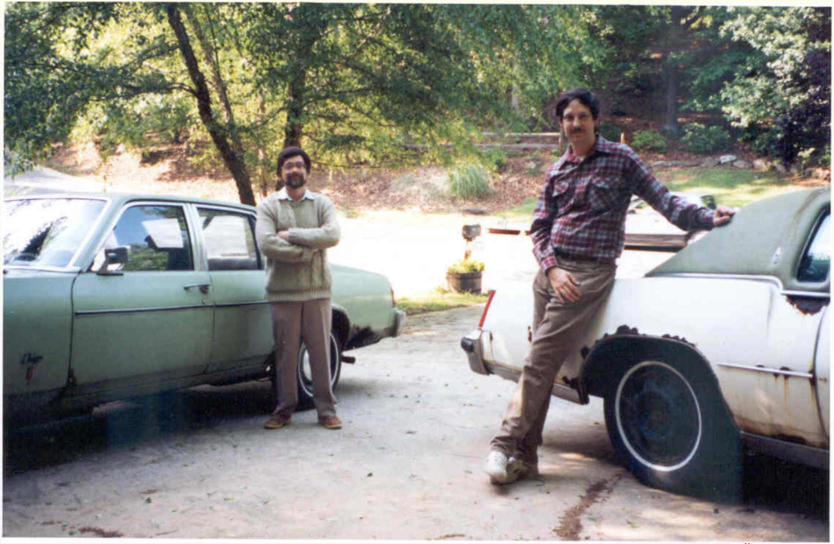
Let us go back in time. Here is a page about him circa 2005.

KATONA GYULA 1941-ben, Budapesten született. Szülei 1945-ben utcai harcok során elhunytak, nagynénje és nagyanyja nevelte. 1958-ban az Országos Tanulmányi Versenyen első díjat nyert, 1959-ben részt vett az első Nemzetközi Matematikai Olimpián. Az ELTE matematikus szakán szerzett diplomát 1964-ben. 1966-tól az MTA Matematikai Kutatóintézetében dolgozik (ma Rényi Alfréd Matematikai Kutatóintézet), 1996 és 2005 között mint igazgató. Fő kutatási területei a kombinatorika és annak alkalmazásai (valószínűség-számítás, adatbázisok elmélete, kriptográfia). Tudományos közleményeinek száma mintegy 110. 1964 óta tanít is az ELTÉ-n. 1995-ben választották a Magyar Tudományos Akadémia levelező, 2001-ben rendes tagjává. Göttingenben, Moszkvában, és 8 amerikai egyetemen töltött összesen 12 félévet mint vendégprofesszor. 14 (többségében nemzetközi) folyóirat szerkesztőbizottsági tagja. Neves tanítványai közül Füredi Zoltán az MTA levelező tagja, Frankl Péter (Japán) pedig külső tagja. Fiai, Katona Gyula és Katona Zsolt, szintén matematikusok.

(BOJTÁR OTTÓ

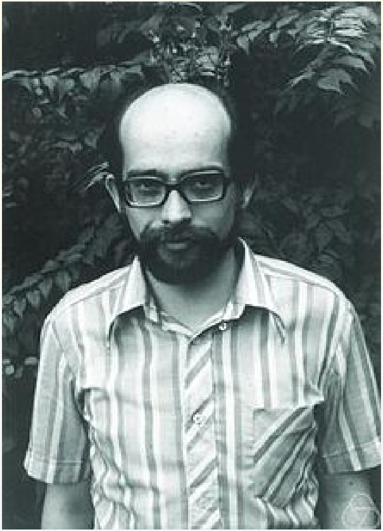
Az egyik asztaltársaság egyértelműen – És az sem, hogy ki a kezdő, de ne *lepő gyorsasággal nagyon ne*

My photo with Katona on his 1989 visit to Columbia.



Family photo.

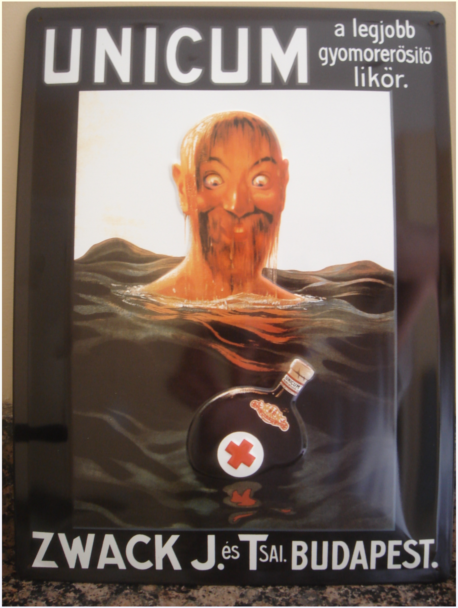
Katona in Erlangen in 1975, the year I met him at MIT.



Katona in 1957 in school.

Katona in Prehistory, when humans came down from trees.

He has even penetrated old Hungarian advertising.



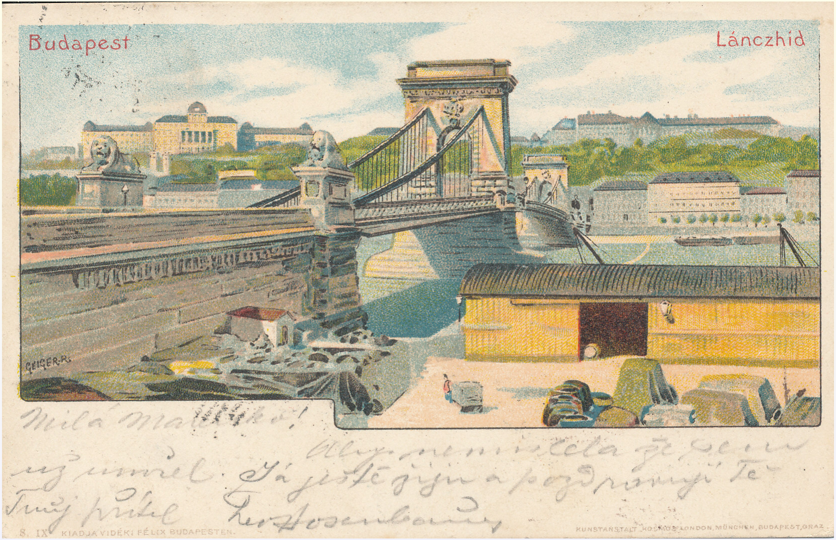
Look closer.

Look even closer!

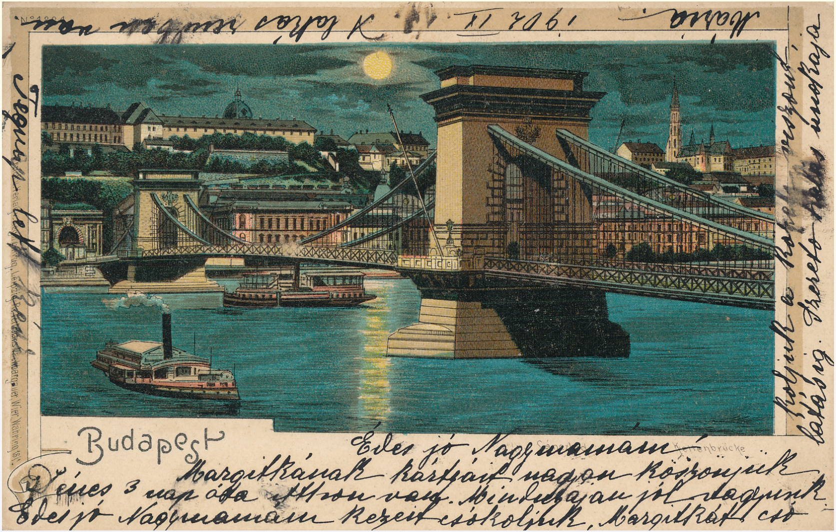

Happy Birthday!
Happy Birthday


Thank You

Lánchid postcard, sent 1899



Lánchid postcard, sent 1902



View of Duna (Danube) from Buda side, ca. 1910

Hungarian Academy of Sciences, ca. 1910