Theorem (Granville, Schinzel, F.): *An algorithm exists for determining if a given nonreciprocal polynomial* $f(x) \in \mathbb{Z}[x]$ *is irreducible and that runs in time*

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$$f(x) \ne \pm\, x^{\deg f} f(1/x)$$

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- **If f has a cyclotomic factor, then the algorithm will detect this and output an $m \in \mathbb{Z}^+$ with $\Phi_m(x)$ a factor.**
- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**
- **Otherwise, the algorithm outputs the complete factorization of $f(x)$ into irreducible polynomials over $\mathbb{Q}$.**

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Theorem (Granville, Schinzel, F.): *There is an algorithm that has the following property: given $f(x) = \sum_{j=0}^{r} a_j x^{d_j} \in \mathbb{Z}[x]$ of degree $n > 1$ and with $r+1$ terms, the algorithm determines if $f(x)$ has a cyclotomic factor in running time*

$$O_{r,H}\big(\log n\,(\log\log n)^2 |\log\log\log n|\big).$$

There is a cyclotomic factor of $f(x) = \sum_{j=0}^{r} a_j x^{d_j}$ if and only if $\exists$ a partition

$$\{0, 1, \ldots, r\} = J_1 \dot\cup J_2 \dot\cup \cdots \dot\cup J_s$$

such that if, for $1 \le i \le s$,

$$\sum_{j \in J_i} a_j x^{d_j} = x^{b_i} g_i(x^{e_i}), \quad M_i = \cdots$$

then there are $m_i \in M_i$ for which

$$m_0 = \prod_{p \mid m_1 \cdots m_s} \max_{1 \le i \le s} \{p^k : p^k \| m_i e_i\}$$

satisfies

$$m_0 = m_i \gcd(m_0, e_i), \quad i \in \{1, 2, \ldots, s\}.$$

- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**

We'll come back to this.

- **Otherwise, the algorithm outputs the complete factorization of $f(x)$ into irreducible polynomials over $\mathbb{Q}$.**

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}$$

f has no reciprocal factors

(other than constants)

- **Otherwise, the algorithm outputs the complete factorization of $f(x)$ into irreducible polynomials over $\mathbb{Q}$.**

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}$$

$$\begin{aligned} F &= F(x_1, x_2, \ldots, x_r) \\ &= a_r x_r + \cdots + a_1 x_1 + a_0, \end{aligned}$$

$$f(x) = F\left(x^{d_1}, x^{d_2}, \ldots, x^{d_r}\right)$$

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}, \quad F(x_1, \ldots, x_r) = a_0 + \sum_{j=1}^{r} a_j x_j$$

$$(1) \qquad \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = (m_{ij})_{r \times t} \begin{pmatrix} v_1 \\ \vdots \\ v_t \end{pmatrix}$$

$$d_i = m_{i1} v_1 + \cdots + m_{it} v_t, \ 1 \leq i \leq r$$

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(m_{ij}) **will come from a finite set depending only on F**

$v_j \in \mathbb{Z}$ **show exist for some (m_{ij})**

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}, \quad F(x_1, \ldots, x_r) = a_0 + \sum_{j=1}^{r} a_j x_j$$

$$(1) \quad d_i = m_{i1}v_1 + \cdots + m_{it}v_t, \; 1 \le i \le r$$

$$F(y_1^{m_{11}} \cdots y_t^{m_{1t}}, \ldots, y_1^{m_{r1}} \cdots y_t^{m_{rt}})$$

$$y_j = x^{v_j}, \quad 1 \le j \le t$$

$$F\left(x^{d_1}, x^{d_2}, \ldots, x^{d_r}\right) = f(x)$$

Thought: A factorization in $\mathbb{Z}[y_1, \ldots, y_t]$ implies a factorization of $f(x)$ in $\mathbb{Z}[x]$.

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}, \quad F(x_1, \ldots, x_r) = a_0 + \sum_{j=1}^{r} a_j x_j$$

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Counter-Thought: We want m_{ij} and v_j in $\mathbb{Z}$, but not necessarily positive.

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}, \qquad F(x_1, \ldots, x_r) = a_0 + \sum_{j=1}^{r} a_j x_j$$

$$(1) \qquad d_i = m_{i1}v_1 + \cdots + m_{it}v_t, \; 1 \le i \le r$$

$$J\, F(y_1^{m_{11}} \cdots y_t^{m_{1t}}, \ldots, y_1^{m_{r1}} \cdots y_t^{m_{rt}})$$

Do what you have to do to make this in $\mathbb{Z}[y_1, y_2, \ldots, y_t]$.

$$f(x)=\sum_{j=0}^{r} a_j x^{d_j}, \qquad F(x_1,\ldots,x_r)=a_0+\sum_{j=1}^{r} a_j x_j$$

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$$y_1^{u_1}\cdots y_t^{u_t} F(y_1^{m_{11}}\cdots y_t^{m_{1t}},\ldots,y_1^{m_{r1}}\cdots y_t^{m_{rt}})$$

Recall: Factor and substitute $y_j = x^{v_j}$.

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j}, \qquad F(x_1, \ldots, x_r) = a_0 + \sum_{j=1}^{r} a_j x_j$$

(1) $$d_i = m_{i1}v_1 + \cdots + m_{it}v_t, \; 1 \le i \le r$$

(2) $$y_1^{u_1} \cdots y_t^{u_t} F(y_1^{m_{11}} \cdots y_t^{m_{1t}}, \ldots, y_1^{m_{r1}} \cdots y_t^{m_{rt}}) = F_1(y_1, \ldots, y_t) \cdots F_s(y_1, \ldots, y_t)$$

(3) $$f(x) = \prod_{i=1}^{s} x^{w_i} F_i(x^{v_1}, \ldots, x^{v_t})$$

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Conclusion: (1) and (2) imply (3)

$$f(x)=\sum_{j=0}^{r}a_jx^{d_j},\qquad F(x_1,\ldots,x_r)=a_0+\sum_{j=1}^{r}a_jx_j$$

(1) $$d_i=m_{i1}v_1+\cdots+m_{it}v_t,\ 1\le i\le r$$

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Question: Are $s-1$ of the factors 1?

Theorem (A. Schinzel, 1969): *Fix*

$$F = a_r x_r + \cdots + a_1 x_1 + a_0,$$

with a_j nonzero integers. There exists a finite computable set of matrices S with integer entries, depending only on F, with the following property:

Suppose the vector

$$\overrightarrow{d} = \langle d_1, d_2, \ldots, d_r \rangle \in \mathbb{Z}^r,$$

with $d_r > \cdots > d_1 > 0$, is such that

$$f(x) = F(x^{d_1}, x^{d_2}, \ldots, x^{d_r})$$

has no non-constant reciprocal factor.

Then $\exists\ r \times t$ *matrix* $M = (m_{ij}) \in S$ *of rank* $t \leq r$ *and a vector*

$$\overrightarrow{v} = \langle v_1, v_2, \ldots, v_t \rangle \in \mathbb{Z}^t$$

such that

$$\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = M \begin{pmatrix} v_1 \\ \vdots \\ v_t \end{pmatrix}$$

holds and the factorization given by

$$y_1^{u_1} \cdots y_t^{u_t} F(y_1^{m_{11}} \cdots y_t^{m_{1t}}, \ldots, y_1^{m_{r1}} \cdots y_t^{m_{rt}})$$

$$= F_1(y_1, \ldots, y_t) \cdots F_s(y_1, \ldots, y_t)$$

in $\mathbb{Z}[y_1, \ldots, y_t]$ *into irreducibles implies*

$$f(x) = \prod_{i=1}^{s} x^{w_i} F_i(x^{v_1}, \ldots, x^{v_t})$$

as a product of polynomials in $\mathbb{Z}[x]$ *each of which is either irreducible over* $\mathbb{Q}$ *or a constant.*

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- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**

We've checked:

f does not have a cyclotomic factor.

We want to know:

Does f have a reciprocal factor?

- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**

Does f have a reciprocal factor?

Suppose $w(x)$ is a reciprocal factor.

$$w(\alpha) = 0 \implies \alpha \neq 0 \ \text{ and } \ w(1/\alpha) = 0$$
$$\implies f(\alpha) = 0 \ \text{ and } \ g(\alpha) = 0,$$

where $g(x) = x^{\deg f} f(1/x) \neq f(x)$

We want to compute $\gcd(f, g)$.

- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**

In general, if f and g are sparse polynomials around degree n in $\mathbb{Z}[x]$, how does one compute $\gcd(f, g)$?

Some items to keep in mind:

$\longrightarrow$ **The Euclidean algorithm will run in time that is polynomial in n, not $\log n$.**

- If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.

In general, if f and g are sparse polynomials around degree n in $\mathbb{Z}[x]$, how does one compute $\gcd(f, g)$?

Some items to keep in mind:

→ Plaisted (1977) has shown that this problem is at least as hard as any problem in NP.

- **If f has no cyclotomic factor but has a reciprocal factor, then the algorithm will give an explicit reciprocal factor.**

→ Plaisted (1977) has shown that this problem is at least as hard as any problem in NP.

Plaisted's takes f and g to be divisors of $x^N - 1$ where N is a product of small primes.

We are interested in the case that both f and g do not have a cyclotomic factor.

Problem: *Find an algorithm which takes given sparse polynomials*

$$f(x) = \sum_{j=1}^{k} a_j x^{d_j}, \quad g(x) = \sum_{j=1}^{k} b_j x^{d_j},$$

in $\mathbb{Z}[x]$ *having no cyclotomic factors, with*

$$d_1 = 0 < d_2 < \cdots < d_k,$$

and computes $\gcd(f, g)$ *in time that is polynomial in* $\log d_k$.

Theorem (Granville, Schinzel, F.): *There is an algorithm which takes as input two polynomials* $f(x)$ *and* $g(x)$ *in* $\mathbb{Z}[x]$*, each of degree* $\le n$ *and height* $\le H$ *and having* $\le r+1$ *nonzero terms, with at least one of* $f(x)$ *and* $g(x)$ *free of any cyclotomic factors, and outputs the value of* $\gcd_{\mathbb{Z}}(f(x), g(x))$ *and runs in time* $O_{r,H}\big(\log n\big)$*.*

Corollary: *If* $f(x), g(x) \in \mathbb{Z}[x]$ *with* $f(x)$ *or* $g(x)$ *not divisible by a cyclotomic polynomial, then* $\gcd_{\mathbb{Z}}(f(x), g(x))$ *has* $O_{r,H}(1)$ *terms.*

Corollary: *If $f(x), g(x) \in \mathbb{Z}[x]$ with $f(x)$ or $g(x)$ not divisible by a cyclotomic polynomial, then $\gcd_{\mathbb{Z}}(f(x), g(x))$ has $O_{r,H}(1)$ terms.*

Note that if a and b are relatively prime positive integers, then

$$\gcd\left(x^{ab} - 1, (x^a - 1)(x^b - 1)\right)$$

$$= \frac{(x^a - 1)(x^b - 1)}{x - 1},$$

which can have arbitrarily many terms.

Theorem (Granville, Schinzel, F.): *There is an algorithm which takes as input two polynomials $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$, each of degree $\le n$ and height $\le H$ and having $\le r+1$ nonzero terms, with at least one of $f(x)$ and $g(x)$ free of any cyclotomic factors, and outputs the value of $\gcd_{\mathbb{Z}}(f(x), g(x))$ and runs in time $O_{r,H}(\log n)$.*

$$f(x) = \sum_{j=1}^{k} a_j x^{d_j} \longrightarrow F_1(\vec{x}) = \sum_{j=1}^{k} a_j x_j$$

$$g(x) = \sum_{j=1}^{k} b_j x^{d_j} \longrightarrow F_2(\vec{x}) = \sum_{j=1}^{k} b_j x_j$$

Lemma (Bombieri and Zannier): *Let*

$$F_1, F_2 \in \mathbb{Q}[x_1, \ldots, x_k]$$

be coprime polynomials. There exists a number $c_1(F_1, F_2)$ *with the following property. If* $\overrightarrow{u} = \langle u_1, \ldots, u_k \rangle \in \mathbb{Z}^k$, $\xi \neq 0$ *is algebraic and*

$$F_1(\xi^{u_1}, \ldots, \xi^{u_k}) = F_2(\xi^{u_1}, \ldots, \xi^{u_k}) = 0,$$

then either ξ *is a root of unity or there exists a non-zero vector* $\overrightarrow{v} \in \mathbb{Z}^k$ *having length at most* c_1 *and orthogonal to* $\overrightarrow{u}$.

$$f(x)=\sum_{j=1}^{k} a_j x^{d_j} \rightarrow F_1(\vec{x})=\sum_{j=1}^{k} a_j x_j$$

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There exists a number $c_1(\vec{a},\vec{b},k)$ *with the following property. If* $\vec{u} = \langle d_1,\ldots,d_k\rangle \in \mathbb{Z}^k$, $\xi \neq 0$ *is algebraic and*

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There exists a number $c_1(\overrightarrow{a}, \overrightarrow{b}, k)$ *with the following property. If* $f(\xi) = g(\xi) = 0$*, then there exists a non-zero vector* $\overrightarrow{v} \in \mathbb{Z}^k$ *having length at most* c_1 *and orthogonal to* $\overrightarrow{u}$*.*

$$\overrightarrow{u} = \langle d_1, \ldots, d_k \rangle$$

Note: It is important that c_1 is computable.

Idea: The lattice of vectors orthogonal to $\overrightarrow{v}$ is $(k-1)$-dimensional so that there exists a vector $\langle e_1, \ldots, e_{k-1} \rangle$ and a matrix $\mathcal{M}$ in $\mathbb{Z}^{k-1}$ satisfying

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So

$$d_i = \sum_{j=1}^{k-1} m_{ij} e_j,$$

with the $m_{ij} \in \mathbb{Z}$ bounded.

$$d_i = \sum_{j=1}^{k-1} m_{ij} e_j$$

$$x^{d_i} = \prod_{j=1}^{k-1} \left(x^{e_j}\right)^{m_{ij}}$$

$$d_i = \sum_{j=1}^{k-1} m_{ij} e_j \qquad x^{d_i} = \prod_{j=1}^{k-1} \left(x^{e_j}\right)^{m_{ij}}$$

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$$F_1^{(2)}(y_1, \ldots, y_{k-1}) = \sum_{i=1}^{k} a_i \prod_{j=1}^{k-1} y_j^{m_{ij}}$$

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$$F_1^{(2)}\left(x^{e_1}, \ldots, x^{e_{k-1}}\right) = f(x)$$

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