Theorem 2.1.1. (The Schönemann-Eisenstein Criterion) *Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ where n is a positive integer. Suppose there exists a prime p such that $p \nmid a_n$, $p | a_j$ for all $j < n$, and $p^2 \nmid a_0$. Then $f(x)$ is irreducible over $\mathbb{Q}$.*

A polynomial $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ is in *Eisenstein form* (with respect to the prime p) if there is a prime p such that $p \nmid a_n$, $p | a_j$ for $j < n$, and $p^2 \nmid a_0$.

An *Eisenstein polynomial* is an $f(x) \in \mathbb{Z}[x]$ for which there is an integer a and a prime p such that $f(x+a)$ is in Eisenstein form with respect to the prime p. In this case, we say $f(x)$ is *Eisenstein with respect to the prime p*.

Example.

$$f(x) = x^3 + 5x^2 + 2x - 1 \quad \text{and} \quad g(x) = 3x^2 + 10x + 2$$

$$R(f,g) = \begin{vmatrix} 1 & 5 & 2 & -1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 3 & 10 & 2 & 0 \\ 0 & 0 & 3 & 10 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 & -1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & -5 & -4 & 3 & 0 \\ 0 & 0 & -5 & -4 & 3 \\ 0 & 0 & 3 & 10 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & 2 & -1 \\ -5 & -4 & 3 & 0 \\ 0 & -5 & -4 & 3 \\ 0 & 3 & 10 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 & -1 \\ 0 & 21 & 13 & -5 \\ 0 & -5 & -4 & 3 \\ 0 & 3 & 10 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 21 & 13 & -5 \\ -5 & -4 & 3 \\ 3 & 10 & 2 \end{vmatrix} = 21(-38) - 13(-19) + (-5)(-38)$$

$$= 19(-42 + 13 + 10) = -19^2$$

Algorithm: Given $f(x) \in \mathbb{Z}[x]$ of degree $n \geq 2$, determine whether $f(x)$ is an Eisenstein polynomial.

Steps:

- **Calculate $R(f, f')$.**
  - $\rightarrow$ **If $R(f, f') = 0$, then $f(x)$ is not Eisenstein with respect to any prime.**
  - $\rightarrow$ **If $R(f, f') \neq 0$, then proceed as follows.**
    - ▶ **Factor $R(f, f')$.**
    - ▶ **For each prime p dividing $R(f, f')$ and each $a \in \{0, 1, ..., p-1\}$, check if $f(x+a)$ is in Eisenstein form with respect p.**
      - $\succ$ **If it is for some such p, then $f(x)$ is an Eisenstein polyomial (with respect to p).**
      - $\succ$ **If it is not for every such p, then $f(x)$ is not an Eisenstein polynomial.**

Example.

$$f(x) = x^3 + 5x^2 + 2x - 1 \quad \text{and} \quad g(x) = 3x^2 + 10x + 2$$

$$R(f,g) = \begin{vmatrix} 1 & 5 & 2 & -1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 3 & 10 & 2 & 0 \\ 0 & 0 & 3 & 10 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 & -1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & -5 & -4 & 3 & 0 \\ 0 & 0 & -5 & -4 & 3 \\ 0 & 0 & 3 & 10 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & 2 & -1 \\ -5 & -4 & 3 & 0 \\ 0 & -5 & -4 & 3 \\ 0 & 3 & 10 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 & -1 \\ 0 & 21 & 13 & -5 \\ 0 & -5 & -4 & 3 \\ 0 & 3 & 10 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 21 & 13 & -5 \\ -5 & -4 & 3 \\ 3 & 10 & 2 \end{vmatrix} = 21(-38) - 13(-19) + (-5)(-38)$$

$$= 19(-42+13+10) = -19^2$$

Example.

$$f(x) = x^3 + 5x^2 + 2x - 1 \quad \text{and} \quad g(x) = 3x^2 + 10x + 2$$

```
> f := x -> x^3 + 5*x^2 + 2*x - 1;
```

$$f := x \to x^3 + 5\,x^2 + 2\,x - 1$$

```
> sort(expand(f(x+11)));
```

$$x^3 + 38\,x^2 + 475\,x + 1957$$

```
> ifactor(475); ifactor(1957);
```

$$(5)^2\,(19)$$

$$(19)\,(103)$$

Note: The prime $p = 19$ is the only p that can "work". From $f(x) \equiv (x - 11)^3 \pmod{19}$ and unique factorization in $\mathbb{F}_{19}[x]$, we get 11 is the only a that can "work".

$$f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{C}[x], \quad g(x) = \sum_{j=0}^{r} b_j x^j \in \mathbb{C}[x]$$

$$n \geq 1, \quad r \geq 1, \quad a_n b_r \neq 0$$

$$R(f,g) = \begin{array}{l} \left| \begin{matrix} a_n & a_{n-1} & a_{n-2} & \dots & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & 0 & a_n & \dots & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_r & b_{r-1} & b_{r-2} & \dots & b_0 & 0 & 0 & \dots & 0 \\ 0 & b_r & b_{r-1} & \dots & b_1 & b_0 & 0 & \dots & 0 \\ 0 & 0 & b_r & \dots & b_2 & b_1 & b_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{matrix} \right| \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} r \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} n \text{ rows} \end{array}$$

Comment: If $\alpha_1, \dots, \alpha_n$ are the roots of $f(x)$, then

$$R(f,g) = a_n^r g(\alpha_1) \cdots g(\alpha_n).$$

Example.

$$f(x) = x^3 + 5x^2 + 2x - 1 \quad \text{and} \quad g(x) = 3x^2 + 10x + 2$$

```
> f := x -> x^3 + 5*x^2 + 2*x - 1;
```

$$f := x \rightarrow x^3 + 5\,x^2 + 2\,x - 1$$

```
> sort(expand(f(x+11)));
```

$$x^3 + 38\,x^2 + 475\,x + 1957$$

```
> ifactor(475); ifactor(1957);
```

$$(5)^2\,(19)$$

$$(19)\,(103)$$

Note: The prime $p = 19$ is the only p that can "work". From $f(x) \equiv (x-11)^3 \pmod{19}$ and unique factorization in $\mathbb{F}_{19}[x]$, we get 11 is the only a that can "work".

(Maple Time)

$0, 1$-Polynomials

$$f_0(x) = 1$$

$$f_1(x) = 1 + x^3$$

$0, 1$-Polynomials

$$\boldsymbol{f_0(x) = 1}$$

$$\boldsymbol{f_1(x) = 1 + x^3}$$

$$\boldsymbol{f_2(x) = 1 + x^3 + x^{15}}$$

$0, 1$-Polynomials

$$f_0(x) = 1$$
$$f_1(x) = 1 + x^3$$
$$f_2(x) = 1 + x^3 + x^{15}$$
$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$

$0, 1$-Polynomials

$$f_0(x) = 1$$

$$f_1(x) = 1 + x^3$$

$$f_2(x) = 1 + x^3 + x^{15}$$

$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$

$$f_4(x) = 1 + x^3 + x^{15} + x^{16} + x^{32}$$

$\mathbf{0, 1}$-Polynomials

$$f_0(x) = 1$$

$$f_1(x) = 1 + x^3$$

$$f_2(x) = 1 + x^3 + x^{15}$$

$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$

$$f_4(x) = 1 + x^3 + x^{15} + x^{16} + x^{32}$$

$$f_5(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33}$$

$0, 1$-Polynomials

$$f_0(x) = 1$$

$$f_1(x) = 1 + x^3$$

$$f_2(x) = 1 + x^3 + x^{15}$$

$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$

$$f_4(x) = 1 + x^3 + x^{15} + x^{16} + x^{32}$$

$$f_5(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33}$$

$$f_6(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34}$$

0, 1-Polynomials

$$f_0(x) = 1$$

$$f_1(x) = 1 + x^3$$

$$f_2(x) = 1 + x^3 + x^{15}$$

$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$

$$f_4(x) = 1 + x^3 + x^{15} + x^{16} + x^{32}$$

$$f_5(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33}$$

$$f_6(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34}$$

$$f_7(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

~~**Problem: Prove that this sequence is infinite.**~~

Definitions and Notations: Let $f(x) \in \mathbb{C}[x]$ with $f(x) \not\equiv 0$. Define $\tilde{f}(x) = x^{\deg f} f(1/x)$. The polynomial $\tilde{f}$ is called the *reciprocal* of $f(x)$. The constant term of $\tilde{f}$ is always non-zero. If the constant term of f is non-zero, then $\deg \tilde{f} = \deg f$ and the reciprocal of $\tilde{f}$ is f. If $\alpha \neq 0$ is a root of f, then $1/\alpha$ is a root of $\tilde{f}$. If $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ in $\mathbb{C}[x]$, then $\tilde{f} = \tilde{g}\tilde{h}$. If $f = \pm\tilde{f}$, then f is called *reciprocal*. If f is not reciprocal, we say that f is *non-reciprocal*. If f is reciprocal and α is a root of f, then $1/\alpha$ is a root of f. The product of reciprocal polynomials is reciprocal so that a non-reciprocal polynomial must have a non-reciprocal irreducible factor. For $f(x) \in \mathbb{Z}[x]$, we refer to the *non-reciprocal part of* $f(x)$ as the polynomial $f(x)$ removed of its irreducible reciprocal factors having a positive leading coefficient. For example, the non-reciprocal part of $3(-x+1)x(x^2+2)$ is $-x(x^2+2)$ (the irreducible reciprocal factors 3 and $x-1$ have been removed from the polynomial $3(-x+1)x(x^2+2)$).

Lemma 9.1.1. *Let $f(x)$ be an arbitrary polynomial in $\mathbb{Z}[x]$. If the non-reciprocal part of $f(x)$ is reducible, then there exist polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ satisfying $u(x)$ and $v(x)$ are both non-reciprocal and $f(x) = u(x)v(x)$.*

Lemma 9.1.1. *Let $f(x)$ be an arbitrary polynomial in $\mathbb{Z}[x]$. If the non-reciprocal part of $f(x)$ is reducible, then there exist polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ satisfying $u(x)$ and $v(x)$ are both non-reciprocal and $f(x) = u(x)v(x)$.*

Lemma 9.1.1. *Let $f(x)$ be an arbitrary polynomial in $\mathbb{Z}[x]$. If the non-reciprocal part of $f(x)$ is reducible, then there exist polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ satisfying $u(x)$ and $v(x)$ are both non-reciprocal and $f(x) = u(x)v(x)$.*

Lemma 9.1.2. *Let $f(x) \in \mathbb{Z}[x]$ with $f(0) \neq 0$, and suppose $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

Lemma 9.1.2. *Let $f(x) \in \mathbb{Z}[x]$ with $f(0) \neq 0$, and suppose $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

Lemma 9.1.3. *Suppose* $f(x)$ *is a* $0,1$*-polynomial with* $f(0) \neq 0$ *and* $f(x) = u(x)v(x)$ *where each of* $u(x)$ *and* $v(x)$ *is non-reciprocal and each of* $u(x)$ *and* $v(x)$ *has a positive leading coefficient. Then the polynomial* $w(x) = u(x)\tilde{v}(x)$ *also has the following properties:*

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$*.*

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$*.*

(iii) $w(1)^2 = f(1)^2$*.*

(iv) $\|w\| = \|f\|$*.*

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

(v) $w(x)$ *is a $0,1$-polynomial with the same number of non-zero terms as* $f(x)$.

(vi) $w(1) = f(1)$.

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

(v) $w(x)$ *is a $0,1$-polynomial with the same number of non-zero terms as* $f(x)$.

(vi) $w(1) = f(1)$.

$$F(x) = x^n + x^{35} + x^{34} + x^{33} + x^{32} + x^{16} + x^{15} + x^3 + 1$$