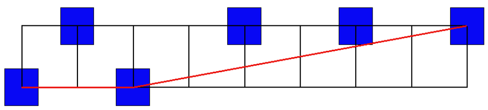
MATH 788F TEST

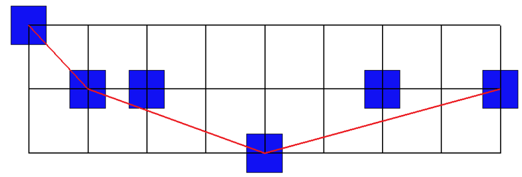
(1) Let

$$f(x) = 45x^8 - 30x^7 - 15x^6 - 2x^4 + 30x^2 + 150.$$

Using Newton polygons, explain why $f(x)$ is irreducible. Below are two Newton polygons to get you started. You may want to use others.



Newton polygon of $f(x)$ with respect to 2



Newton polygon of $f(x)$ with respect to 3

(2) Let $f(x) = x^3 + k$ where k is an arbitrary integer. Suppose that $f(x)$ is Eisenstein with respect to a prime p. Prove that either $p = 3$ or p is a divisor of k.

(3) Let $m = p^2 + p - 2$. Recall that $p\widetilde{B}_m(x) = \sum_{j=0}^{m} pB_j \binom{m}{j}$. Explain why each of the coefficients $pB_j\binom{m}{j}$ for $1 \le j \le m-1$ is a rational number which, when reduced, has its numerator divisible by p. (I am *not* asking you to prove $p\widetilde{B}_m(x)$ is a rational number times an Eisenstein polynomial. I am, however, asking you to give part of a proof that $p\widetilde{B}_m(x)$ is a rational number times an Eisenstein polynomial.)

(4) Let $d_n d_{n-1} \dots d_0$ be the decimal representation of a product of three primes. Let $f(x) = \sum_{j=0}^{n} d_j x^j$. Prove that $f(x)$ is the product of at most three irreducible polynomials. In other words, show that if $f(x) = f_1(x)f_2(x)f_3(x)f_4(x)$ where each $f_j(x) \in \mathbb{Z}[x]$, then $f_j(x) \equiv \pm 1$ for at least one $j \in \{1, 2, 3, 4\}$. Prove any lemmas from class you use except you may use Lemma 5, without proof, given in the handout.

(5) Let $f(x) = \sum_{j=0}^{n} a_j x^j$ where $a_n = 1$, $a_{n-1} = 0$, and

Note that the coefficients should be integers and the constant term non-zero.

$$a_{n-2} > 1 + |a_{n-3}| + |a_{n-4}| + \cdots + |a_1| + |a_0|.$$

Thus, a_{n-2} is positive and greater than the sum of the absolute values of the other coefficients.

(a) Show that $f(x)$ has exactly two roots (counting multiplicity) with absolute values ≥ 1.

(b) Show that $f(x)$ has no real roots with absolute value ≥ 1. (Hint: If $z \in \mathbb{R}$, then the terms $a_n z^n$ and $a_{n-2}z^{n-2}$ have the same sign. Also, the inequality in this problem is mighty strong.)

(c) Explain why $f(x)$ is irreducible.