

MATH 788F

Practice Test Problems

(1) Prove the following:

Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x) = f_1(x)f_2(x)f_3(x)$ where each $f_j(x)$ is irreducible. If

$$f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

where each $\alpha_j \in \mathbb{C}$, then $|\alpha_j| \geq 1$ for at least three values of $j \in \{1, 2, \dots, n\}$.

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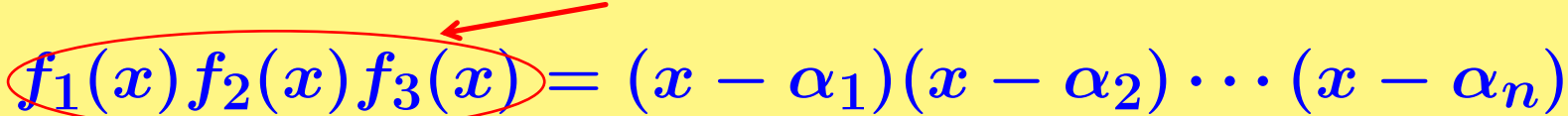
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Show each $f_j(x)$ has a root with absolute value ≥ 1 .

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\implies there exists j such that $|\alpha'_j| \geq 1$

(4) Let

$$f(x) = x^{16} - 8x^{15} - 4x^{14} - 2x^{13} \\ - x^{12} - x^{11} - x^{10} - \dots - x - 1.$$

- (a) Consider $F(x) = (2x - 1)f(x)$. Explain why $F(x)$ has exactly one root α satisfying $|\alpha| \geq 1$.
- (b) Explain why this implies that $f(x)$ is irreducible.

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Lemma 2. Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counting multiplicity) inside the circle \mathcal{C} (i.e., in the interior of the region bounded by \mathcal{C}).

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Idea: Take $m = (2p - 1)(p - 1)$, and show that

$$p\tilde{B}_m(x) = \sum_{j=0}^m pB_j \binom{m}{j} x^j$$

is a rational number times a polynomial in Eisenstein form with respect to p .

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So p divides the constant term and p^2 does not.

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Other Terms where $(p - 1) | j$:

$$\nu_p(m!) = 2p - 2$$

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