Practice Problems for Test 1

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(1) Suppose we want to use induction to prove

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \geq \ln(n+1)$$

for all positive integers n.

(a) What would be the "induction hypothesis" in the proof?

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

(5) Prove that

$$x+y \leq \sqrt{2(x^2+y^2)}$$

for all real numbers x and y.

(6) Let $a_1=\sqrt{6}$, and let

$$a_{n+1}=\sqrt{6+a_n} \quad \text{for each positive integer } n.$$

For example, $a_2=\sqrt{6+a_1}=\sqrt{6+\sqrt{6}}$ and $a_3=\sqrt{6+a_2}=\sqrt{6+\sqrt{6+\sqrt{6}}}$. Prove that

$$a_n \leq 3 \quad \text{for every positive integer } n.$$

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(1) Suppose we want to use induction to prove

$$1^{1999}+2^{1999}+3^{1999}+\cdots+n^{1999} \geq \frac{n^{2000}}{2^{1999}} \qquad \text{for every positive integer } n.$$

(a) What would be the "induction hypothesis" in the proof? Give a complete sentence.

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

(5) Using a proof by contradiction, prove that if n is an integer ≥ 2 and if $P_1, P_2, \ldots, P_n$ are n points ordered clockwise on a circle of radius 1, then the distance between two of these points must be $< 7/n$ (i.e., there exist i and j with $1 \leq i < j \leq n$ such that the distance from P_i to P_j is $< 7/n$).

Hint 1: A circle of radius 1 has circumference 2π and $2\pi < 7$.

Hint 2: The shortest path from a point A to a point B is the path along the line segment joining A to B. Draw a picture and consider the line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ..., $\overline{P_{n-1}P_n}$, and finally $\overline{P_nP_1}$.

(6) Let $a_0 = 4$, $a_1 = 5$, and

$$a_{n+1} = 3a_n - 2a_{n-1} \quad \text{for every integer } n \geq 1.$$

For example, $a_2 = 3a_1 - 2a_0 = 15 - 8 = 7$ and $a_3 = 3a_2 - 2a_1 = 21 - 10 = 11$. Using induction, prove

$$a_n = 2^n + 3$$

for every integer $n \geq 0$.

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(1) Prove that if the product of two positive numbers is < 100, then at least one of the numbers is < 10.

(3) Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$ for every integer $n \geq 1$.

(4) It is not true that the product of a rational number and an irrational number is always irrational. Prove that if α is rational and β is irrational, then $\alpha\beta$ is irrational unless α equals $\boxed{}$. Fill in the box with the correct number (there's just one) and write the proof.

(5) (a) Let $\alpha = e^{1/e}$. Suppose $a_1 = \alpha$, $a_2 = \alpha^\alpha = \alpha^{a_1}$, $a_3 = \alpha^{a_2}$, and so on. Prove that $a_n \leq e$ for all integers $n \geq 1$.

(b) Does the fact that $e = 2.71828\ldots$ have anything to do with your proof? In other words, is it true that if the number e is replaced everywhere in part (a) by any number $t > 0$, then the argument still works?