# Solutions to Practice Problems for Test 1

(1) Let $a$ and $b$ be the two positive numbers so that $ab < 100$. Assume that both $a$ and $b$ are $\geq 10$. Then $ab \geq 10 \times 10 = 100$. This contradicts that $ab < 100$. Hence, our assumption is wrong and at least one of $a$ or $b$ is $< 10$.

(2) Since $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \geq 1$, we want to prove

$$(*) \qquad 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for every positive integer $n$. We use induction on $n$. Since $1^3 = 1 = \dfrac{1^2 \times (1+1)^2}{4}$, $(*)$ holds when $n = 1$. Now, suppose that $(*)$ holds for some $n$. We want to show that

$$(**) \qquad 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}.$$

From $(*)$, we obtain

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right) \\ &= (n+1)^2 \frac{n^2 + 4n + 4}{4} = \frac{(n+1)^2(n+2)^2}{4}. \end{aligned}$$

Hence, $(**)$ holds. By induction, we deduce that $(*)$ holds for every positive integer $n$.

(3) We prove

$$(*) \qquad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$$

for every integer $n \geq 1$ by induction on $n$. Since the sum on the left of $(*)$ is simply $1/\sqrt{1} = 1$ when $n = 1$ and since the right of $(*)$ is $2\sqrt{1} = 2$ when $n = 1$, we see that $(*)$ holds when $n = 1$. Now, suppose that $(*)$ is true for some integer $n$. We want to prove

$$(**) \qquad \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \leq 2\sqrt{n+1}.$$

From $(*)$, we obtain

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{2\sqrt{n(n+1)} + 1}{\sqrt{n+1}}.$$

Since $(n + \frac{1}{2})^2 = n^2 + n + \frac{1}{4}$, we see that

$$\sqrt{n(n+1)} = \sqrt{n^2 + n} < \sqrt{\left(n + \frac{1}{2}\right)^2} = n + \frac{1}{2}.$$

Therefore,

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \le \frac{2\sqrt{n(n+1)}+1}{\sqrt{n+1}} < \frac{2(n+\frac{1}{2})+1}{\sqrt{n+1}} = \frac{2n+2}{\sqrt{n+1}} = \frac{2(n+1)}{\sqrt{n+1}} = 2\sqrt{n+1}.$$

Thus, $(**)$ holds, and we have by induction that $(*)$ holds for every positive integer $n$.

(4) Let $\alpha$ be rational and $\beta$ be irrational. We prove that $\alpha\beta$ is irrational unless $\alpha$ equals $\boxed{\ 0\ }$. Assume $\alpha\beta$ is rational where we consider only the case that $\alpha \neq 0$. Then $\alpha = a/b$ (since $\alpha$ is rational) and $\alpha\beta = c/d$ (since $\alpha\beta$ is rational) for some integers $a$, $b$, $c$, and $d$ with $c \neq 0$ and $d \neq 0$. Since $\alpha \neq 0$, we deduce $a \neq 0$. Thus,

$$\beta = \frac{\alpha\beta}{\alpha} = \frac{c/d}{a/b} = \frac{bc}{ad}$$

where $bc$ and $ad$ are integers and $ad \neq 0$. Hence, $\beta$ is rational, contradicting that $\beta$ is given to be irrational. Therefore, $\alpha\beta$ is irrational.

(5) (a) We prove $a_n \le e$ for every integer $n \ge 1$ by induction on $n$. Since $e > 1$, we deduce that $a_1 = e^{1/e} \le e$. Suppose now that $a_n \le e$ for some integer $n \ge 1$. Since $e > 1$ and $1/e > 0$, $e^{1/e} > 1$. Hence,

$$a_{n+1} = \left(e^{1/e}\right)^{a_n} \le \left(e^{1/e}\right)^{e} = e.$$

Therefore, by induction, $a_n \le e$ for every integer $n \ge 1$.

(b) The argument above made use of the fact that $e > 1$ and that $e^{1/e} > 1$ (though you might not have noticed where the latter was used). We did not need to use that $e > 1$ since $t^{1/t} \le t$ is true for all $t > 0$. However, $e^{1/e} > 1$ was needed. The same argument works fine if $t > 1$ (then $t^{1/t} > 1$). The argument does not work if $0 < t < 1$. In fact, in this case, $a_2 > t$.

(6) (a) The proof can be completed as follows:

**Proof:** We want to show that there is an integer $q$ such that $a^2 = 4q + r$ with $r = 0$ or $r = 1$. The remainder when $a$ is divided by 4 is one of 0, 1, 2, or 3. If the remainder is 0, then $a = 4k$ for some integer $k$ so that $a^2 = 16k^2 = 4(4k^2) + 0$. Thus, in this case, one can take $q = 4k^2$ and $r = 0$. If the remainder is 1, then $a = 4k + 1$ for some integer $k$ so that $a^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$. In this case, one can take $q = 4k^2 + 2k$ and $r = 1$. If the remainder is 2, then

$$a = \boxed{4k+2}$$

for some integer $k$ so that

$$a^2 = \boxed{16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) + 0}.$$

In this case, one can take

$$q = \boxed{4k^2 + 4k + 1} \qquad \text{and} \qquad r = \boxed{0}.$$

If the remainder is 3, then

$$a = \boxed{4k + 3}$$

for some integer $k$ so that

$$a^2 = \boxed{16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1}.$$

In this case, one can take

$$q = \boxed{4k^2 + 6k + 2} \qquad \text{and} \qquad r = \boxed{1}.$$

Thus, no matter what the remainder is when $a$ is divided by 4, we deduce that the remainder when $a^2$ is divided by 4 is either 0 or 1. This completes the proof. ■

(b) Assume that $N = 342039283547533429990284934820226101890873292014 3$ is the sum of two squares. Then $N = a^2 + b^2$ for some integers $a$ and $b$. Observe that

$$\begin{aligned} N &= 3420392835475334299902849348202261018908732920143 \\ &= 34203928354753342999028493482022610189087329201 \times 100 + 43. \end{aligned}$$

In other words, there is an integer $m$ such that $N = 100m + 43$. Since $100m + 43 = 4(25m + 10) + 3$, there is an integer $k$ (namely, $k = 25m + 10$) such that $N = 4k + 3$. By part (a), we know that the remainder when we divide $a^2$ or $b^2$ by 4 is in each case either 0 or 1. Hence, $a^2 = 4q_1 + r_1$ and $b^2 = 4q_2 + r_2$ for some integers $q_1$, $q_2$, $r_1$, and $r_2$ with each of $r_1$ and $r_2$ either 0 or 1. Since $N = a^2 + b^2$, we deduce that

$$0 = N - \left(a^2 + b^2\right) = (4k + 3) - \left(4q_1 + r_1 + 4q_2 + r_2\right) = 4(k - q_1 - q_2) + (3 - r_1 - r_2).$$

Thus,

$$3 - r_1 - r_2 = 4(-k + q_1 + q_2).$$

In other words, $3 - r_1 - r_2$ is a multiple of 4. Each of $r_1$ and $r_2$ is either 0 or 1 so that the only possible values for $3 - r_1 - r_2$ are $3 - 0 - 0 = 3$, $3 - 0 - 1 = 2$, $3 - 1 - 0 = 2$, and $3 - 1 - 1 = 1$. Since none of these values (3, 2, and 1) is a multiple of 4, we have a contradiction. Therefore, $N$ is not the sum of two squares.

**Comment:** The same argument works for any integer $N$ which has a remainder of 3 when divided by 4. There are other numbers, like 21, which do not have this property and are not the sum of two squares.