MATH 574: DISCRETE MATH Name ____________________

Quiz #2

Use English Sentences Throughout Your Proof!!

Points: The quiz is one problem worth 10 points.

(1) Using induction, prove that $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n+1}$ for every integer $n \geq 3$. Be sure to do a proof by induction. Note also that this quiz problem differs from the related homework problem in two ways: this quiz problem is for $n \geq 2$ rather than $n \geq 1$ and you are being asked to prove the sum is $\geq \sqrt{n+1}$ rather than $\geq \sqrt{n}$.

Solution: We prove

$$(*) \qquad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n+1}$$

for every integer $n \geq 3$ by induction on n. Since

$$\sum_{k=1}^{3} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \geq 1 + \frac{1}{2} + \frac{1}{2} = 2 = \sqrt{3+1},$$

we see that $(*)$ holds when $n = 3$. Suppose $(*)$ holds for some n. We want to prove that

$$(**) \qquad \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \geq \sqrt{n+2}.$$

By the induction hypothesis,

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}\,\sqrt{n+1}+1}{\sqrt{n+1}} = \frac{n+2}{\sqrt{n+1}}.$$

Since $\sqrt{n+2} > \sqrt{n+1}$, we obtain $\frac{n+2}{\sqrt{n+1}} > \frac{n+2}{\sqrt{n+2}}$. Hence,

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \geq \frac{n+2}{\sqrt{n+1}} > \frac{n+2}{\sqrt{n+2}} = \sqrt{n+2}.$$

Therefore, $(**)$ holds. By induction, we deduce that $(*)$ holds for every positive integer n.