MATH 532, 736I: MODERN GEOMETRY

Test 2, , Spring 2019   Name _____

Show All Work

Instructions: This test consists of 3 pages (including an information section at the end). Put your name at the top of this page and at the top of the first page of the packet of blank paper. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Fill in the boxes below with your answers. Show <u>ALL</u> of your work. Do <u>NOT</u> use a calculator.

21 pts (1) Let $A = (0,0)$ and $B = (12, 0)$. Let P, Q, and S be 3 other points.

(a) If $R_{\pi, P}(A) = B$, then $P =$ ⬚. (Write a point with specific coordinates.)

(b) If $R_{\pi/2, Q}(B) = A$, then $Q =$ ⬚. (Write a point with specific coordinates.)

(c) If $R_{\pi/3, S}(A) = B$, then $S =$ ⬚. (Write a point with specific coordinates.)

16 pts (2) Give a proof of Theorem 3 stated on the last page of this test. Note the numbering of the theorems on the last page of this test may not be the same as in class. You are proving Theorem 3 as stated on this last page, and you should refer to the numbers of the theorems stated on this last page when doing the proof.

24 pts (3) In the picture, A', C' and Q are points at infinity next to lines that pass through them. Thus, $\overleftrightarrow{PB'}$ passes through A', $\overleftrightarrow{RB'}$ passes through C', and $\overleftrightarrow{CA}$ passes through Q. The line $\overleftrightarrow{XA}$ is parallel to $\overleftrightarrow{PB'}$, so it also passes through A'. The line $\overleftrightarrow{XC}$ is parallel to $\overleftrightarrow{RB'}$, so it also passes through C'. Thus, in the projective plane (with points at infinity) $\overleftrightarrow{AA'} = \overleftrightarrow{XA}$ and $\overleftrightarrow{CC'} = \overleftrightarrow{XC}$. Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ all pass through X, we see that $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point X. The next page contains a proof that $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line. In other words, P, Q and R are collinear. This is the same as saying $\overleftrightarrow{PR}$ is parallel to $\overleftrightarrow{CA}$. Fill in the boxes on the next page to complete the proof.
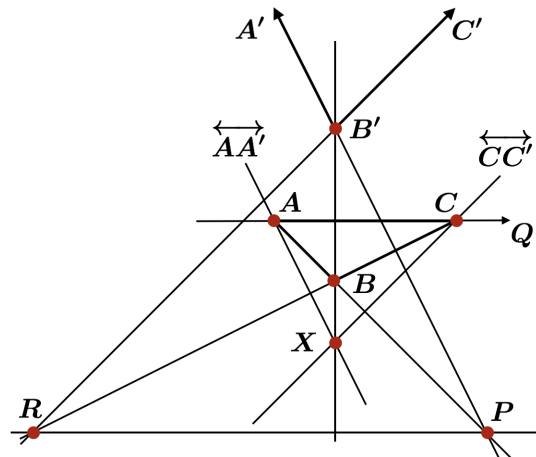
Frank and Ernest
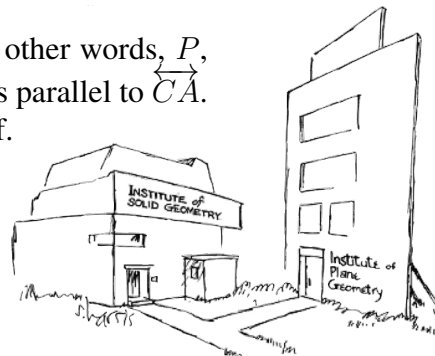
Note: Four theorems are listed at the end of the quiz. Use the numbering given there.

Proof: The point B is on lines $\overleftrightarrow{B'X}$, $\overleftrightarrow{PA}$ and $\overleftrightarrow{RC}$. Hence, by Theorem $\boxed{}$, there are real numbers k_1, k_2, and k_3 with

(∗) $\qquad B = (1 - k_1)B' + k_1 X = (1 - k_2)P + k_2 A = (1 - k_3)R + k_3 C.$

From (∗), we obtain

$$k_1 X - k_2 A = (1 - k_2)P - (1 - k_1)B'.$$

Next, we explain why $k_1 = \boxed{}$. Assume otherwise. Then, dividing by $k_1 - k_2$, we obtain

$$\left(\frac{k_1}{k_1 - k_2}\right)X + \left(\frac{-k_2}{k_1 - k_2}\right)A = \left(\frac{1 - k_2}{k_1 - k_2}\right)P + \left(\frac{k_1 - 1}{k_1 - k_2}\right)B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{XA}$. By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the right above is a point on line $\overleftrightarrow{PB'}$.

This is a contradiction since $\boxed{}$.

Thus, our assumption is wrong, and $k_1 = \boxed{}$. Similarly, we deduce from (∗) that

$$k_1 X - k_3 C = (1 - k_3)R - (1 - k_1)B'.$$

Using this equation and the fact that $\overleftrightarrow{XC}$ and $\overleftrightarrow{RB'}$ are $\boxed{}$, we obtain $k_1 = \boxed{}$.

It follows that $k_2 = \boxed{}$ as well. Since (∗) also implies

$$k_2 A - k_3 C = (1 - k_3)R - (1 - k_2)P$$

we obtain

$$\boxed{} = (1 - k_2)(R - P).$$

We deduce that the vector $\boxed{}$ and the vector $\overrightarrow{PR}$ are parallel. Hence, lines $\boxed{}$ and $\overleftrightarrow{PR}$ are parallel, completing the proof. ∎

23 pts

(4) In the figure, A, B and C are vertices of a triangle. The points D, E and F satisfy that $\triangle DBA$, $\triangle ECB$ and $\triangle EFD$ are equilateral triangles. The point M is the midpoint of $\overline{AC}$. Prove that M is also the midpoint of $\overline{BF}$. Make sure to indicate what happens with each rotation you use to do this problem. (Hint: Let X be the center of the equilateral triangle $\triangle DBA$ as shown and consider
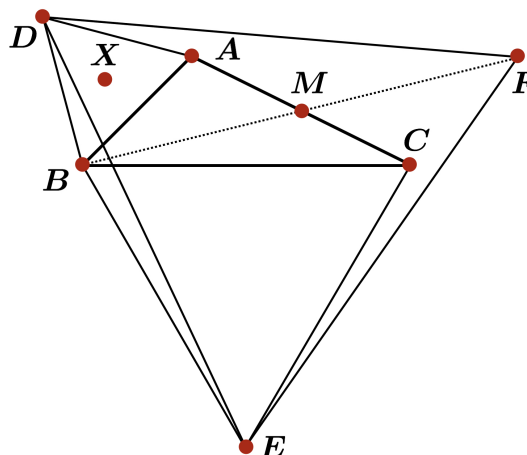
$$f = R_{2\pi/3,X}\, R_{\pi/3,E}.$$

Figure out whether f is a rotation or a translation and be precise. Then determine how that gives you what you want. Don't forget to explain what happens each time you make a rotation for each rotation in the definition of f.)

16 pts

(5) (Note that you don't need to do the previous problem to do this one.) In the problem above, the point M is the midpoint of $\overline{AC}$, and (and as shown there) it is also the midpoint of $\overline{BF}$. You know something about how to tackle problems with midpoints; this problem is one of those. Explain why the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CF}$ (not drawn) are parallel and why the segments $\overline{BA}$ and $\overline{CF}$ (not drawn) have equal length. In other words, explain why the vectors $\overrightarrow{BA}$ and $\overrightarrow{CF}$ are equal.

INFORMATION SECTION

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that $C = (1-t)A + tB$.*

Theorem 2: *Let $\alpha_1, \ldots, \alpha_n$ be real numbers (not necessarily distinct), and let $A_1, \ldots, A_n$ and $B_1, \ldots, B_k$ be points (not necessarily distinct). Let f be a product of the n rotations R_{α_j, A_j} and the k translations T_{B_j} with each of the n rotations and k translations occurring exactly once in the product. If $\alpha_1 + \cdots + \alpha_n$ is not an integer multiple of 2π, then there is point C such that*

$$f = R_{\alpha_1 + \alpha_2 + \cdots + \alpha_n, C}.$$

If $\alpha_1 + \cdots + \alpha_n$ is an integer multiple of 2π, then f is a translation.

Theorem 3: *If A, B and C are collinear, then there are real numbers x, y and z not all 0 such that*
$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0}.$$

Theorem 4: *If A, B and C are points and there are real numbers x, y, and z not all 0 such that*
$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$
then A, B and C are collinear.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1 - \cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1 - \cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$