MATH 532, 736I: MODERN GEOMETRY

Test 2 Solutions

Test 2 (1992):

Part I:

(1) If A, B, and C are not distinct, then the conclusion that A, B, and C are collinear is clear. Suppose now that A, B, and C are distinct. Recall that not all of x, y, and z are zero. Suppose $x \neq 0$ (a similar argument can be made if $y \neq 0$ or $z \neq 0$). From $xA + yB + zC = \vec{0}$, we obtain $A = (-y/x)B + (-z/x)C$. Let $t = -z/x$. Since $x + y + z = 0$,

$$1 - t = 1 - \left(-\frac{z}{x} \right) = 1 + \frac{z}{x} = \frac{x+z}{x} = -\frac{y}{x}.$$

Hence, $A = (1-t)B + tC$. By Theorem 1, A is on $\overleftrightarrow{BC}$. Thus, A, B, and C are collinear.

(2) The square of the distance from N to M_A is $(N - M_A)^2$ and the square of the distance from N to M_B is $(N - M_B)^2$. It therefore suffices to show that $(N - M_A)^2 = (N - M_B)^2$. Since $\overleftrightarrow{CD}$ and $\overleftrightarrow{BA}$ are perpendicular, $(D - C)(A - B) = 0$. We use that

$$\begin{aligned} D - C &= (A + B + C + D) - (B + C) - (A + C) \\ &= 4N - 2M_A - 2M_B = 2\big((N - M_A) + (N - M_B)\big). \end{aligned}$$

and $$A - B = (A + C) - (B + C) = 2M_B - 2M_A = 2\big((N - M_A) - (N - M_B)\big)$$

Since $(D - C)(A - B) = 0$, we deduce

$$0 = \big((N - M_A) + (N - M_B)\big)\big((N - M_A) - (N - M_B)\big) = (N - M_A)^2 - (N - M_B)^2.$$

Hence, $(N - M_A)^2 = (N - M_B)^2$.

Part II:

(1) Not relevant to the current course.

(2) Let P_A be the intersection of the altitude drawn from A to $\overleftrightarrow{BC}$. Let P_B be the intersection of the altitude drawn from B to $\overleftrightarrow{AC}$. Let P_C be the intersection of the altitude drawn from C to $\overleftrightarrow{AB}$. Let D be the intersection of $\overleftrightarrow{AP_A}$ and $\overleftrightarrow{BP_B}$. Then $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are perpendicular and $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular. Hence, $(D - A)(C - B) = 0$ and $(D - B)(C - A) = 0$. Therefore,

$$0 = (D - A)(C - B) - (D - B)(C - A) = -AC - BD + AD + BC = (D - C)(A - B).$$

Thus, $\overrightarrow{CD}$ and $\overrightarrow{BA}$ are perpendicular. We deduce that the altitude from C also passes through D so that the three altitudes are concurrent.

(3) In this problem, $f = R_{\pi/2,A}T_A R_{\pi,A}$. From the last page of this test (and as shown on homework from class) $T_A = R_{\pi,A/2}R_{\pi,(0,0)}$. Thus, we can view f as a product of four rotations with the sum of the angles in the rotations being $7\pi/2$. Since a rotation by $7\pi/2$ is the same as a rotation by $3\pi/2$, we deduce that $f = R_{3\pi/2,B}$. Using $A = (1,3)$, one writes $R_{\pi/2,A}T_A R_{\pi,A}$ as a product of 3 matrices and obtains

$$R_{\pi/2,A}T_A R_{\pi,A} = \begin{pmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Note that the matrix above can be used to justify $f = R_{3\pi/2,B}$ as well.) If $B = (x,y)$, then

$$R_{3\pi/2,B} = \begin{pmatrix} 0 & 1 & x-y \\ -1 & 0 & x+y \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $x - y = -5$ and $x + y = 5$. Solving, we deduce $x = 0$ and $y = 5$. Thus, $B = (0,5)$.

Part III:

(1) The vector $B - A$ (going in the direction of ℓ) is parallel to the vector $D - C$ (going in the direction of m). Thus, $B - A = k(D - C)$ for some constant k. Since E is the midpoint of $\overline{AC}$ and F is the midpoint of $\overline{BD}$, we obtain

$$E = \frac{1}{2}(A + C) \qquad \text{and} \qquad F = \frac{1}{2}(B + D).$$

Thus,

$$F - E = \frac{1}{2}(B + D) - \frac{1}{2}(A + C) = \frac{1}{2}(B - A) + \frac{1}{2}(D - C) = \frac{k+1}{2}(D - C).$$

Thus, $\overrightarrow{EF}$ is a constant times $\overrightarrow{CD}$. This constant is non-zero since $E \neq F$. It follows that $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{CD}$. Hence, $\overleftrightarrow{EF}$ is parallel to both ℓ and m.

(2) You do not need to know this for Test 2.

(3) **Proof.** By Theorem 1 (from the Information Page at the end of this test), there are real numbers k_1, k_2, and k_3 such that

$$X = (1 - k_1)A + k_1 A' = (1 - k_2)B + k_2 B' = (1 - k_3)C + k_3 C'.$$

Next, we show that $k_1 \neq k_2$. Assume $k_1 = k_2$. Observe that $k_1 \neq 0$ since otherwise we would have $X = A = B$, contradicting that A and B are distinct points. Also, $k_1 \neq 1$ since otherwise we would have $X = A' = B'$, contradicting that A' and B' are distinct points. We get that

$$(1 - k_1)A - (1 - k_2)B = k_2 B' - k_1 A'$$

and that the vectors $\overrightarrow{BA}$ and $\overrightarrow{A'B'}$ either have the same direction or the exact opposite direction. This contradicts that the point P exists. Hence, $k_1 \neq k_2$. Thus,

$$\frac{1 - k_1}{k_2 - k_1}A + \frac{k_2 - 1}{k_2 - k_1}B = \frac{k_2}{k_2 - k_1}B' + \frac{-k_1}{k_2 - k_1}A'.$$

By Theorem 1 with $t = (k_2 - 1)/(k_2 - k_1)$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$. By Theorem 1 with $t = -k_1/(k_2 - k_1)$, we see that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

$$P = \frac{1 - k_1}{k_2 - k_1}A + \frac{k_2 - 1}{k_2 - k_1}B.$$

Hence,

$$(k_2 - k_1)P = (1 - k_1)A + (k_2 - 1)B. \tag{1}$$

Using that

$$(1 - k_2)B - (1 - k_3)C = k_3C' - k_2B',$$

we similarly obtain that $k_2 \neq k_3$, that

$$\frac{1 - k_2}{k_3 - k_2}B + \frac{k_3 - 1}{k_3 - k_2}C = \frac{k_3}{k_3 - k_2}C' + \frac{-k_2}{k_3 - k_2}B',$$

and that

$$(k_3 - k_2)Q = (1 - k_2)B + (k_3 - 1)C. \tag{2}$$

From

$$(1 - k_3)C - (1 - k_1)A = k_1A' - k_3C',$$

we similarly obtain that either

$$k_3 = k_1 \tag{3}$$

or

$$\frac{1 - k_3}{k_1 - k_3}C + \frac{k_1 - 1}{k_1 - k_3}A = \frac{k_1}{k_1 - k_3}A' + \frac{-k_3}{k_1 - k_3}C'. \tag{4}$$

If (4) holds, then we could deduce that there is a point on both line $\overleftrightarrow{AC}$ and line $\overleftrightarrow{A'C'}$, giving a contradiction. Thus, (3) must hold. We get from (1) and (2) that

$$(k_2 - k_1)P + (k_3 - k_2)Q = (1 - k_1)A + (k_3 - 1)C$$

so that

$$(k_2 - k_1)(P - Q) = (1 - k_1)(A - C).$$

Observe that $P \neq Q$ since otherwise we would have that the points A, B, and C are collinear, which isn't the case. Since $k_1 \neq k_2$, we obtain that the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel, completing the proof. ■

Test 2 (1993):

Part I:

(1) If $A = B$, then take $x = 1$, $y = -1$, and $z = 0$. Suppose now that $A \neq B$. By Theorem 1, there is a real number t such that $C = (1-t)A + tB$. Let $x = 1-t$, $y = t$, and $z = -1$. Then x, y, and z are not all 0, $x + y + z = 0$, and $xA + yB + zC = \vec{0}$.

(2) See Problem (2) of the 1992 test.

(3) One answer is: $\Delta AA'Z$ and $\Delta BB'X$ are perspective from point Y.

Part II:

(1) It suffices to show that $G = \frac{1}{2}(C + F)$. The given information implies

$$D = \frac{B+C}{2}, \quad E = \frac{A+C}{2}, \quad F = \frac{A+B}{2}, \quad \text{and} \quad G = \frac{D+E}{2}.$$

Hence,

$$G = \frac{1}{2}(D+E) = \frac{1}{2}\left(\frac{B+C}{2} + \frac{A+C}{2}\right) = \frac{1}{2}\left(C + \frac{A+B}{2}\right) = \frac{1}{2}(C+F),$$

as desired.

(2) From the given information, $f = R_{\pi/2,(1,1)}R_{\pi,(1,0)}R_{\pi/2,(0,0)}$. Since the sum of the angles ($\pi/2$, π, and $\pi/2$) is 2π (an integer times 2π), we deduce that f is a translation. In other words, $f = T_B$. To determine B, one can compute f by multiplying matrices. However, it is probably easier to take a point and see what f does to it (where it is mapped under f). Consider $(x, y) = (0, 0)$. Rotating this point about $(0, 0)$ by $\pi/2$ does not change it. Now, rotating the point about $(1, 0)$ by π moves it to $(2, 0)$. Finally, rotating this point about $(1, 1)$ by $\pi/2$ moves it to $(2, 2)$. Thus, f takes $(0, 0)$ to $(2, 2)$. It follows that $B = (2, 2)$.

(3) You do not need to know this for Test 2.

(4) Recall in class that we discussed how one could obtain the point C where $R_{\alpha+\beta,C} = R_{\beta,B}R_{\alpha,A}$. The points A, B, and C form a triangle with $\angle BAC = \alpha/2$ and $\angle ABC = \beta/2$. Since $\alpha + \beta = \pi$ in this problem, the sum of the measures of $\angle BAC$ and $\angle ABC$ is $\pi/2$. Thus, ΔABC is a right triangle with $\angle ACB = \pi/2$. Recall that we showed in class that in this situation, C is on the circle having diameter $\overline{AB}$. Since the center of this circle is the midpoint of $\overline{AB}$, C is on the circle centered at $\frac{1}{2}(A + B)$ passing through A and B.

Test 2 (1994):

(1) See Problem 1 on the 1992 test. Note that the theorems are numbered differently.

(2) Observe that
$$2N - M_A - Q_A = \frac{A+B+C+D}{2} - \frac{B+C}{2} - \frac{A+D}{2} = \vec{0}.$$
Thus,
$$\begin{aligned}0 &= (2N - M_A - Q_A)(M_A - Q_A) = 2NM_A - M_A^2 - 2NQ_A + Q_A^2 \\ &= -N^2 + 2NM_A - M_A^2 + N^2 - 2NQ_A + Q_A^2 = -(N - M_A)^2 + (N - Q_A)^2.\end{aligned}$$
Thus, $(N - M_A)^2 = (N - Q_A)^2$, and we deduce that the distance from N to M_A is the same as the distance from N to Q_A.

(3) (a) $(3, -2)$

(b) 4

(c) $(4, 4)$ (you should be able to do this with and without matrices)

(d) $(1, 4)$ (you should be able to do this with and without matrices)

(4) One answer is: $\Delta AZA'$ and $\Delta BXB'$ are perspective from point Y. There are other answers. Another one that was suggested is: $\Delta A'YA$ and $\Delta C'XC$ are perspective from point Z.

(5) Observe that $a^2 = (B - A)^2$, $b^2 = (C - B)^2$, and $c^2 = (C - A)^2$. Since $a^2 + b^2 = c^2$, we obtain
$$(B - A)^2 + (C - B)^2 = (C - A)^2$$
so that
$$B^2 - 2AB + A^2 + C^2 - 2BC + B^2 = C^2 - 2AC + A^2.$$
Rearranging, we obtain $2B^2 - 2AB - 2BC + 2AC = 0$. Dividing by 2, we obtain $0 = B^2 - AB - BC + AC = (A - B)(C - B)$. Therefore, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular and, hence, $\angle ABC$ is a right angle.

(6) You do not need to know this for Test 2.

(7) Since f is three successive rotations with the angles of these rotations summing to $8\pi/3$, $f = R_{8\pi/3,(x,y)} = R_{\alpha,D}$ where $\alpha = 2\pi/3$ and $D = (x, y)$ for some numbers x and y. We need to determine x and y. Since $f(1, 1) = (5, 7)$, we deduce that
$$\begin{pmatrix}5\\7\\1\end{pmatrix} = R_{2\pi/3,(x,y)}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1/2 & -\sqrt{3}/2 & (3x+\sqrt{3}y)/2\\ \sqrt{3}/2 & -1/2 & (-\sqrt{3}x+3y)/2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$
Thus,
$$10 = -1 - \sqrt{3} + 3x + \sqrt{3}y \qquad \text{and} \qquad 14 = \sqrt{3} - 1 - \sqrt{3}x + 3y.$$
Rewriting these, we obtain
$$3x + \sqrt{3}y = 11 + \sqrt{3} \qquad \text{and} \qquad \sqrt{3}x - 3y = -15 + \sqrt{3}.$$
Multiplying the first of these by $\sqrt{3}$ and adding the result to the second, we deduce $4\sqrt{3}\,x = 12\sqrt{3} - 12$ so that $x = 3 - \sqrt{3}$. Substituting, we obtain (after a little work) $y = 4 + (2\sqrt{3}/3)$.

(8) **Proof:** Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq 1$. Assume $k_1 = 1$. Then $A - B = A' - B'$ so that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel. This contradicts that P exists. Therefore, $k_1 \neq 1$. Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = -k_1/(1-k_1)$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $$(k_1 - k_2)Q = k_1 B - k_2 C.$$

Also, from $k_2 C - A = k_2 C' - A'$, we deduce that

(3) $$(k_2 - 1)R = k_2 C - A.$$

Therefore, from (1), (2), and (3),

$$(1 - k_1)P + (k_1 - k_2)Q + (k_2 - 1)R = \overrightarrow{0}.$$

The result follows from Theorem 3 (on the last page of this test).

Test 2 (1995):

(1) See Problem 1 on the 1993 test. Note that the theorems are numbered differently.

(2) The square of the distance from N to M_A is $(N - M_A)^2$ and the square of the distance from N to M_C is $(N - M_C)^2$. It therefore suffices to show that $(N - M_A)^2 = (N - M_C)^2$. Since $\overleftrightarrow{BD}$ and $\overleftrightarrow{CA}$ are perpendicular, $(D - B)(A - C) = 0$. We use that

$$\begin{aligned} D - B &= (A + B + C + D) - (B + C) - (A + B) \\ &= 4N - 2M_A - 2M_C = 2\big((N - M_A) + (N - M_C)\big). \end{aligned}$$

and $$A - C = (A + B) - (B + C) = 2M_C - 2M_A = 2\big((N - M_A) - (N - M_C)\big)$$

Since $(D - B)(A - C) = 0$, we deduce

$$0 = \big((N - M_A) + (N - M_C)\big)\big((N - M_A) - (N - M_C)\big) = (N - M_A)^2 - (N - M_C)^2.$$

Hence, $(N - M_A)^2 = (N - M_C)^2$.

(3) (a) $(-2, 0)$

(b) 1

(c) $(12, 6)$

(4) Observe that $a^2 = (B - A)^2$, $b^2 = (C - B)^2$, and $c^2 = (C - A)^2$. Since $a^2 + b^2 = c^2$, we obtain

$$(B - A)^2 + (C - B)^2 = (C - A)^2$$

so that

$$B^2 - 2AB + A^2 + C^2 - 2BC + B^2 = C^2 - 2AC + A^2.$$

Rearranging, we obtain $2B^2 - 2AB - 2BC + 2AC = 0$. Dividing by 2, we obtain $0 = B^2 - AB - BC + AC = (A - B)(C - B)$. Therefore, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular and, hence, $\angle ABC$ is a right angle.

(5) One answer is: $\Delta TB'B$ and $\Delta SA'A$ are perspective from point R.

(6) Let D be the midpoint of $\overline{BC}$. Then $D = (B + C)/2$. Since the distance from A to B is equal to the distance from A to C, we obtain $(A - B)^2 = (A - C)^2$. Thus,

$$\begin{aligned} 0 &= (A - B)^2 - (A - C)^2 = (2A - B - C)(C - A) \\ &= 2\left(A - \frac{B + C}{2}\right)(C - A) = 2(A - D)(C - A) \end{aligned}$$

(where the second equality follows by considering the factorization of the difference of two squares). It follows that $\overrightarrow{DA}$ and $\overrightarrow{AC}$ are prependicular. In other words, the line passing through A and the midpoint of $\overline{BC}$ is perpendicular to line $\overleftrightarrow{BC}$.

(7) **Proof:** Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq 1$. Assume $k_1 = 1$. Then $A - B = A' - B'$ so that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel. This contradicts that P exists. Therefore, $k_1 \neq 1$. Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = -k_1/(1-k_1)$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $$(k_1 - k_2)Q = k_1 B - k_2 C.$$

Using that

$$A - k_2 C = A' - k_2 C'$$

and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel (so R is a point at "infinity"), we obtain

(3) $$k_2 = 1.$$

From (1) and (2), we obtain

(4) $$(1 - k_1)P + (k_1 - k_2)Q = A - k_2 C.$$

Using (3), we can rewrite (4) in the form

$$(1 - k_1) \times (P - Q) = A - C.$$

Recall that $k_1 \neq 1$. Therefore, the line $\overleftrightarrow{AC}$ is parallel to the line $\overleftrightarrow{PQ}$.