MATH 532, 736I: MODERN GEOMETRY
Test 1 Solutions

Test 1 (1992):

Part I:

(1) **Axiom P1:** There exist at least 4 distinct points no 3 of which are collinear.
Axiom P2: There exists at least 1 line with exactly $n+1$ (distinct) points on it.
Axiom P3: Given 2 distinct points, there is exactly 1 line that they both lie on.
Axiom P4: Given 2 distinct lines, there is at least 1 point on both of them.

(2) **Axiom A1:** There exist at least 4 distinct points no 3 of which are collinear.
Axiom A2: There exists at least 1 line with exactly n points on it.
Axiom A3: Given any 2 distinct points, there exists exactly one line passing through the 2 points.
Axiom A4: Given any line ℓ and any point P not on ℓ, there is exactly 1 line through P that does not intersect ℓ.

(3) By Axiom P1, there is a point P. By Axiom P3, for each point $Q \neq P$, there is exactly one line passing through P and Q. Thus, each point $Q \neq P$ is on a line through P and each point $Q \neq P$ is on only one such line. By Theorem 1, there are exactly $n+1$ lines through P. By Theorem 2, each of these lines has exactly n points other than P on it. Thus, there are exactly $(n+1)n+1 = n^2+n+1$ points. By the principle of duality, there are exactly n^2+n+1 lines.

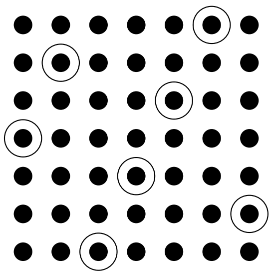
(4) Let the points on ℓ be denoted $Q_1, Q_2, \dots, Q_n$. By Axiom A3, for each $j \in \{1, 2, \dots, n\}$, there is a line ℓ_j passing through A and Q_j. Since A is on each ℓ_j and A is not on ℓ, each ℓ_j is different from ℓ. Recall $n \geq 2$ (since no affine plane of order 1 exists). By Axiom A3, ℓ is the unique line passing through any two of the Q_j's. It follows that the ℓ_j's are distinct. By Axiom A4, there is a line ℓ' parallel to ℓ and passing through A. Since ℓ' does not intersect ℓ and each ℓ_j does, the line ℓ' is different from each ℓ_j. Thus, there are at least $n+1$ lines passing through A. If ℓ'' is a line different from ℓ' passing through A, then Axiom A4 implies that ℓ'' must intersect ℓ. Hence, ℓ'' passes through A and some Q_j. By Axiom A3, we deduce $\ell'' = \ell_j$. Hence, there are at most $n+1$ lines passing through A. Thus, there are exactly $n+1$ lines passing through A.

Part II:

(1) See the first page of the Appendix.

(2) See the first page of the Appendix.

(3)



(4) The points $(1, 5)$ and $(6, 3)$ are on the line. The equation of the line in the $xy-$plane passing through these points is $y = mx + b$ where

$$m = \frac{3-5}{6-1} = \frac{-2}{5} \quad \text{and} \quad b = 5 - m = 5 + \frac{2}{5} = \frac{27}{5}.$$

Since $5 \times 3 \equiv 1 \pmod 7$, we view $1/5$ as being 3 modulo 7. This gives that $m \equiv -2 \times 3 \equiv -6 \equiv 1 \pmod 7$ and $b \equiv 27 \times 3 \equiv 4 \pmod 7$. Thus, the line is $y \equiv x + 4 \pmod 7$. Note that we can check are answering by verifying directly that $(1, 5)$ and $(6, 3)$ are on this line.

Part III (These were Problems (8)-(12) on Homework 1):

(1) Yes. As a model, you can use the 4 vertices of a square for the points and the 4 sides and 2 diagonals for the lines. Another model is Euclidean geometry. And now that we're further in the course, note that any finite affine plane satisfies the axioms.

(2) Yes. Consider a line with two points on it or the empty set to see that Axiom 1 is independent of the other axioms. Consider three points and no lines to see that Axiom 2 is independent of the other axioms. Consider 5 points, no three collinear, with a line going through each pair of points to see that Axiom 3 is independent of the other axioms. Consider 3 vertices of a triangle for the points with the 3 sides of the triangle being the lines to see that Axiom 4 is independent of the other axioms.

(3) No. Use two (non-isomorphic) models from (1).

(4) No. The first model described in (1) has two lines that do not intersect (for example, the opposite sides of the square). The dual of Axiom 2 would assert that any two lines intersect in exactly one point. So the principle of duality does not hold.

(5) By Axiom 1, there are 3 points, A, B, and C, which are not collinear. By Axiom 2, there is a line ℓ_1 through B and C and a line ℓ_2 through A and B. Since the points A, B, and C are not collinear, A is not on ℓ_1 and C is not on ℓ_2. By Axiom 4, there is a line ℓ_3 through A which does not intersect ℓ_1 and there is a line ℓ_4 through C which does not intersect ℓ_2. Since ℓ_3 does not intersect ℓ_1 and ℓ_4 does (at C), we get from Axiom 3 that ℓ_3 intersects ℓ_4, say at D. Since D is on ℓ_3 and ℓ_3 does not intersect ℓ_1, D cannot be B or C (points on ℓ_1). Since D is on ℓ_4 and ℓ_4 does not intersect ℓ_2, D cannot be A (a point on ℓ_2). Thus, A, B, C, and D are distinct points. We know that A, B, and C are not collinear. By Axiom 2, ℓ_2 is the only line passing through A and B. Since D is on ℓ_4 and since ℓ_4 and ℓ_2 do not intersect, D is not on the line passing through A and B. Therefore, A, B, and D are not collinear. By Axiom 2, ℓ_1 is the only line passing through B and C. Since D is on ℓ_3 and since ℓ_3 and ℓ_1 do not intersect, D is not on the line passing through B and C. Therefore, B, C, and D are not collinear. By Axiom 2, ℓ_3 is the only line passing through A and D and ℓ_4 is the only line passing through C and D. Since ℓ_4 intersects ℓ_1 and ℓ_3 does not, $\ell_4 \neq \ell_3$. Thus, the line passing through A and D is not the same as the line passing through C and D. Therefore, A, C, and D are not collinear. Hence, no 3 of A, B, C, and D are collinear.

Test 1 (1993):

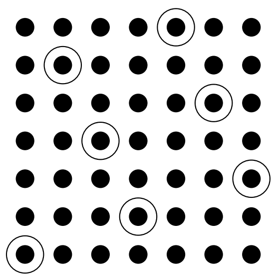
Part I:

(1) See 1992 Test, Problem (1).

(2) See 1992 Test, Problem (2).

(3) See the first page of the Appendix.

(4)



(5) The slope of the line through $(3, 8)$ and $(9, 10)$ is $1/3$, so we want m to satisfy $3m \equiv 1 \pmod{17}$. One checks that $m = 6$. Thus, $y \equiv 6x + k \pmod{17}$. By plugging in the point $(3, 8)$, we obtain $8 \equiv 6 \cdot 3 + k \pmod{17}$ so that $k \equiv 8 - 18 \equiv -10 \equiv 7 \pmod{17}$. The equation of the line is therefore $y \equiv 6x + 7 \pmod{17}$.

(6) By Axiom A1, there is a point P_1 not on ℓ. By the given theorem, line ℓ has at least one point, say P_2, on it. By Axiom A3, there is a line ℓ' passing through P_1 and P_2. Since P_1 is on ℓ' but not on ℓ, $\ell' \neq \ell$. Axiom A3 implies that P_2 is the only point on both ℓ' and ℓ. By the theorem, ℓ' has exactly n points on it, two of which are P_1 and P_2. Let $P_3, \dots, P_n$ denote the remaining points on ℓ'. For each $j \neq 2$, P_j is not on ℓ so that Axiom A4 implies that there is a line ℓ_j passing through P_j and parallel to ℓ. Each such ℓ_j is different from ℓ' since ℓ' intersects ℓ. It follows that the lines ℓ_j (with $j \neq 2$) are distinct by Axiom A3 (since ℓ' is the unique line passing through any two of the P_j's). Thus, there are at least $n - 1$ distinct lines parallel to ℓ (namely, the lines ℓ_j with $j \neq 2$).

Part II:

(1) Yes. See the second page of the Appendix. Also, note that a line with 4 points on it is a model for this system.

(2) Yes. See the second page of the Appendix.

(3) No. See the second page of the Appendix.

(4) (a) The model consisting of a single line with 4 points on it satisfies the axioms. The principle of duality does not hold for the model since the number or points in the model is different from the number of lines. (One can use an affine plane for a model in this part as well.)

(b) Finite projective planes satisfy the axioms. Since finite projective planes satisfy the principle of duality, there are models for the axiomatic system for which the principle of duality holds.

(5) By Axiom 4, there exist 3 noncollinear points. Call them A, B, and C. a line ℓ_1 through A and C, and a line ℓ_2 through B and C. By Axiom 2, there is a point P different from A and C on line ℓ_1. By Axiom 2, there is a point Q different from B and C on line ℓ_2. We show that A, B, P, and Q are 4 distinct points, no 3 of which are collinear. Note that $A \neq B$. By the defintion of P, we have $P \neq A$. By the defintion of Q, we have $Q \neq B$. Since A, B, and C

are noncollinear, $\ell_1 \neq \ell_2$ so that B is not on ℓ_1 and A is not on ℓ_2. Hence, $P \neq B$ and $Q \neq A$. By Axiom 3, ℓ_1 is the only one line through P and C and ℓ_2 is the only one line through Q and C. Since $\ell_1 \neq \ell_2$, P is not on ℓ_2 and Q is not on ℓ_1. In particular, $P \neq Q$. Thus, the points A, B, P, and Q are distinct. Since ℓ_1 is the only line through A and P and each of B and Q is not on ℓ_1, the points A, P, and B are noncollinear and the points A, P, and Q are noncollinear. Since ℓ_2 is the only line through B and Q and each of A and P is not on ℓ_2, the points B, A, and Q are noncollinear and the points B, P, and Q are noncollinear. Therefore, A, B, P, and Q are 4 distinct points, no 3 of which are collinear.

Test 1 (1994):

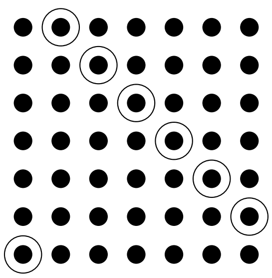
Part I:

(1) See 1992 Test, Problem (1).

(2) See 1992 Test, Problem (2).

(3) See the first page of the Appendix.

(4)



(5) The slope of the line through $(2, 8)$ and $(15, 17)$ is $9/13$, so we want m to satisfy $13m \equiv 9 \pmod{19}$. One checks that $m = 8$. Thus, $y \equiv 8x + k \pmod{19}$. By plugging in the point $(2, 8)$, we obtain $8 \equiv 8 \cdot 2 + k \pmod{19}$ so that $k \equiv 8 - 16 \equiv -8 \equiv 11 \pmod{19}$. The equation of the line is therefore $y \equiv 8x + 11 \pmod{19}$.

(6) Let P be an arbitrary point. By Axiom A2, there is a line ℓ with exactly n points on it. If P is not on ℓ, then applying Result 1 with line ℓ and the point P gives that P has exactly $n + 1$ lines passing through it. Now, consider the case that P is on ℓ. By Axiom A1, there are at least two points A and B not on ℓ and at least two other points C and D such that no 3 of these 4 points are collinear. By Axiom A3, there is a line ℓ_1 through A and C and a line ℓ_2 through A and D. Since no 3 of A, B, C, and D are collinear, $\ell_1 \neq \ell_2$, B is not on ℓ_1, and B is not on ℓ_2. Note that $P \neq A$ since A is not on ℓ and P is. Axiom A3 implies that P can be on at most one of ℓ_1 and ℓ_2 (since there is only one line through both A and P). Let ℓ' be an ℓ_j ($j = 1$ or 2) such that P is not on ℓ'. Thus, both P and B are not on ℓ' and B is not on ℓ. We apply Result 1 with line ℓ and point B to deduce that there are exactly $n+1$ lines passing through B. We then apply Result 2 with line ℓ' and point B to deduce that there are exactly n points on ℓ'. Finally, we apply Result 1 with line ℓ' and point P to deduce that there are exactly $n + 1$ lines passing through P. This completes the proof.

Part II:

(1) Yes. See the third page of the Appendix.

(2) Yes. See the third page of the Appendix.

(3) No. See the third page of the Appendix.

(4) **Dual of Axiom 1:** There exist at least 4 distinct lines, no 3 of which are concurrent.

Does this dual hold? Yes.

Explanation: By Axiom 1, there exist 4 distinct points no 3 of which are collinear. Call them A, B, C, and D. By Axiom 4, there exist lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 passing through A and B, B and C, C and D, and A and D, respectively. Since no 3 of A, B, C, and D are collinear, each of ℓ_1, ℓ_2, ℓ_3, and ℓ_4 passes through exactly 2 of the points A, B, C, and D and the lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are distinct. We will complete the proof by showing that no 3 of these 4 lines are concurrent.

Assume 3 (or more) of the lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 intersect at a common point P. Then the above implies that $P \neq A$, $P \neq B$, $P \neq C$, and $P \neq D$. Observe that given any 3 of the 4 lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4, from among those 3 lines, there must be at least 2 which have one of A, B, C, or D in common. Thus, by considering the 3 (or more) lines among ℓ_1, ℓ_2, ℓ_3, and ℓ_4 which intersect at P, we can find at least 2 lines which intersect at P and at some other point (A, B, C, or D). This contradicts Axiom 4, so our assumption that ℓ_1, ℓ_2, ℓ_3, and ℓ_4 intersect at a common point P must be incorrect. Therefore, no 3 of the 4 lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are concurrent.

Dual of Axiom 2: There exists at least 1 point with exactly 2 lines passing through it.

Does this dual hold? No.

Explanation: In the model given for (1), no point has exactly 2 lines passing through it. This model then shows that the dual of Axiom 2 does not hold for every model in the axiomatic system.

Dual of Axiom 3: There exists at least 1 point with at least 3 distinct lines passing through it.

Does this dual hold? Yes.

Explanation: By Axiom 1, there are 4 points, no 3 of which are collinear. Call them A, B, C, and D. By Axiom 4, there is a line ℓ_1 through A and B, a line ℓ_2 through A and C, and a line ℓ_4 through A and D. Since no 3 of A, B, C, and D are collinear, these lines must be distinct. Hence, A is a point with at least 3 lines passing through it.

Dual of Axiom 4: Given any 2 distinct lines, there is exactly one point where the lines intersect.

Does this dual hold? No.

Explanation: In the model given for (1), the lines ℓ_1 and ℓ_2 do not intersect. This model then shows that the dual of Axiom 4 does not hold for every model in the axiomatic system.

(5) No (since the dual of Axioms 2 and 4 do not hold).

Test 1 (1995):

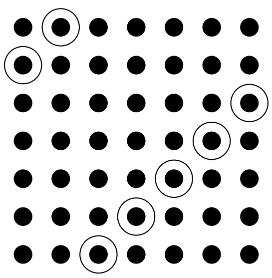
Part I:

(1) See 1992 Test, Problem (1).

(2) See 1992 Test, Problem (2).

(3) See 1993 Test, Problem (3).

(4)



(5) The slope of the line through $(6,1)$ and $(12,12)$ is $11/6$, so we want m to satisfy $6m \equiv 11 \pmod{17}$. One checks that $m = 16$ (note that $m \equiv -1 \pmod{17}$). Thus, $y \equiv 16x + k \pmod{17}$. By plugging in the point $(6,1)$, we obtain $1 \equiv 16 \cdot 6 + k \pmod{17}$ so that $k \equiv 1 - 96 \equiv -95 \equiv 7 \pmod{17}$. The equation of the line is therefore $y \equiv 16x + 7 \pmod{17}$.

(6) Let the points on ℓ be denoted $Q_1, \ldots, Q_n$. By Axiom A3, there is a line ℓ_j passing through A and Q_j for each $j \in \{1, 2, \ldots, n\}$. Since A in on each ℓ_j and not on ℓ, we deduce that $\ell \neq \ell_j$ for each j. Axiom A3 implies that ℓ is the only line passing through two distinct Q_j's. Thus, the ℓ_j's are distinct. By Axiom A4, there is a line ℓ' passing through A and parallel to ℓ. Since each ℓ_j intersects ℓ, $\ell' \neq \ell_j$ for each j. Thus, there are at least $n+1$ lines passing through A (the line ℓ' and the ℓ_j's). If $\ell'' \neq \ell'$ and ℓ'' passes through A, then Axiom 4 implies ℓ'' must intersect ℓ. Since the points on ℓ are $Q_1, \ldots, Q_n$, we deduce that ℓ'' passes through some Q_j as well as A. From Axiom A3, we deduce $\ell'' = \ell_j$. Hence, there are no more than $n+1$ lines passing through A. Therefore, there are exactly $n+1$ lines passing through A.

Part II:

(1) Yes. See the fourth page of the Appendix.

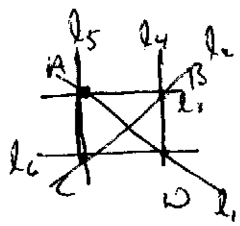
(2) Yes. See the fourth page of the Appendix.

(3) No. See the fourth page of the Appendix.

(4) No. Since the dual of Axiom 2 is that "every point has at least 3 distinct lines passing through it" and since this was not true for our model in (1), the principle of duality does not hold.

(5) By Axiom 1, there are at least two lines ℓ_1 and ℓ_2. By Axiom 2, each of these has at least 3 distinct points on it. By Axiom 3, these two lines have at most 1 point in common. Thus, there are two distinct points, say A and B, on ℓ_1 neither of which is on ℓ_2 and there are two distinct points, say C and D, on ℓ_2 neither of which is on ℓ_1. Note that these four points are distinct. By Axiom 3, ℓ_1 is the only line passing through both A and B and ℓ_2 is the only line passing through both C and D. This implies that A, B, and C are noncollinear and that A, B, and D are noncollinear (since C and D are not on ℓ_1). Also, C, D, and A are noncollinear and that C, D, and B are noncollinear (since A and B are not on ℓ_2). Therefore, A, B, C, and D are 4 points no 3 of which are collinear.

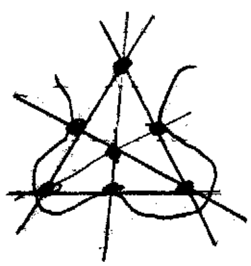
APPENDIX



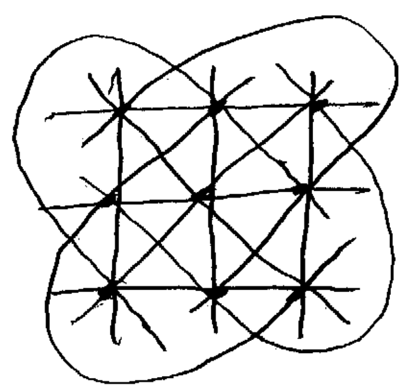
Part II (1), 1992.

	l_1	l_2	l_3	l_4	l_5	l_6
A	1	0	1	0	1	0
B	0	1	1	1	0	0
C	0	1	0	0	1	1
D	1	0	0	1	0	1

Part II (2), 1992



Part I (3), 1993.



Part I (3), 1994

Pt 2:

(1)

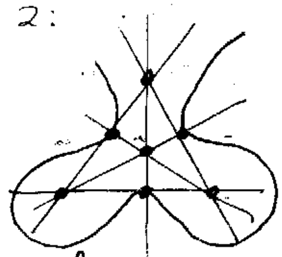


fig 1

It is consistent by fig. 1

(2) is A1 indepent:

l has 3 pts but there is 1 line thru any two pts

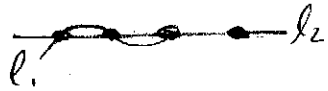
Therefore A1 is indepent

A2 indp.?

A1 is satisfied
A3 is satisfied
A2 is not

$\Rightarrow$ A2 is independent

A3 indp?



A1 satisfied ✓
A2 satisfied ✓
A3 not ✗

$\Rightarrow$ A3 is Independent

A1, A2, A3 indp $\Rightarrow$ system independent

(3)

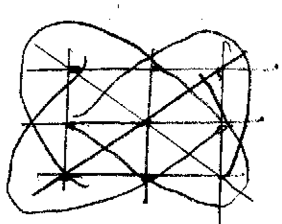
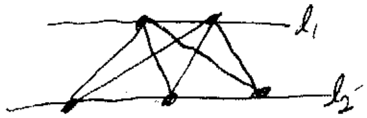


fig 2

fig 2 and fig 1 are consistent with A1–A3
$\Rightarrow$ <u>not</u> <u>complete</u>

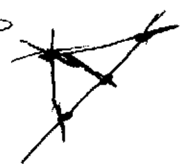
(1) Is the axiomatic system consistent? Justify your answer.



Yes, the axiomatic system is consistent as a model can be made that satisfies the system of axioms.

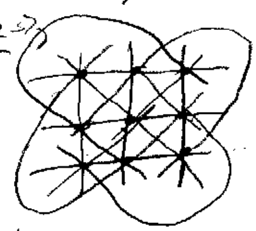
(2) Is the axiomatic system independent? Justify your answer.

Axiom 1 is independent by ⇒



Axioms 2–4 hold + Axiom 1 does not as 3 of the 4 pts are collinear.

Axiom 2 is independent ⇒

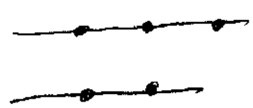


Axioms 1, 3+4 hold + Axiom 2 does not as all lines connect 3 pts

Axiom 3 is independent ⇒

Axioms 1, 2 + 4 hold while Axiom 3 does not as there is no line containing 3 pts.

Axiom 4 is independent ⇒



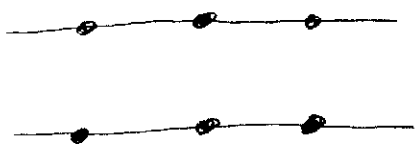
Axioms 1–3 hold while Axiom 4 does not as there is not a line through any 2 pts.

Since each axiom is independent, the axiomatic system is then said to be independent. (Yes).

(3) Is the axiomatic system complete? Justify your answer.

; since these models are different than the model for (1) we have more than one model, each different (non isomorphic) for the system. Therefore, the axiomatic system is not complete. (No).

(1) Is the axiomatic system consistent? Justify your answer.

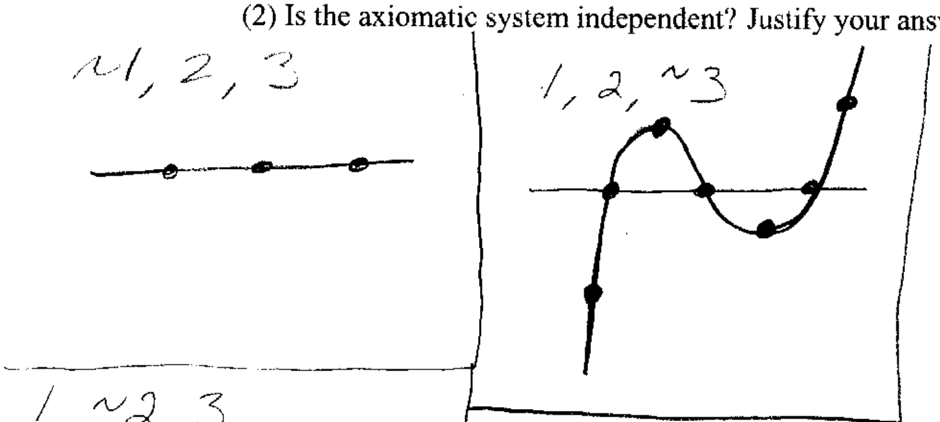


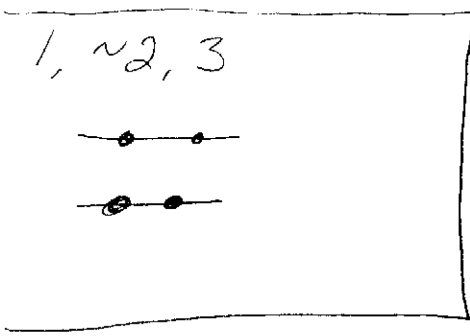
A1 ✓
A2 ✓
A3 ✓

yes

all the axioms hold

(2) Is the axiomatic system independent? Justify your answer.

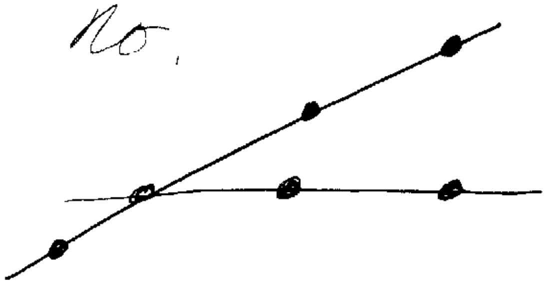




each axiom taken as false ⇒ the axiomatic system is independent so yes

(3) Is the axiomatic system complete? Justify your answer.

No.



A1 ✓
A2 ✓
A3 ✓

This example the lines cross the original did not hence not complete.