MATH 532, 736I: MODERN GEOMETRY

Test 1, Spring 2016 Name ______________________________

Show All Work

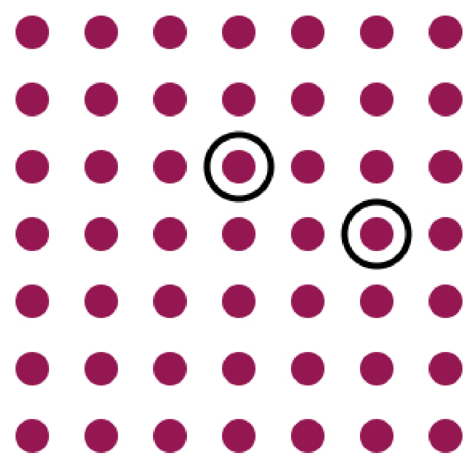
Instructions: This test consists of 4 pages of problems. Put your name at the top of this page and at the top of the first page of the packet of blank paper given to you. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Show ALL of your work. Do NOT use a calculator.

Points: Part I (48 pts), Part II (52 pts)

Part I. The point value for each problem appears to the left of each problem. In Problem 5, I will assume you are using the axioms as you state them in your answer to Problem 1 below.

10 pts (1) Draw a model for a finite AFFINE plane of order 3. Be sure to clearly mark every point and clearly draw every line in your model. If I look at your model, I should be able to tell where each of your points and lines are. In particular, make sure that each line you draw cannot be mistaken for two lines.

10 pts (2) Two points have been circled in the 7×7 array of points to the right. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points. (You do not need to use the packet of blank paper for this problem.)



12 pts (3) Consider the points $(2, 12)$ and $(21, 25)$ in a 37×37 array of points for our model of a finite affine plane of order 37. Find the equation of the line passing through these two points. Put your answer below in the form $y \equiv mx + k \pmod{37}$ where m and k are among the numbers $0, 1, 2, \ldots, 36$. Be sure to show your work in the packet provided with this test. You will not get credit for a correct answer without correct work.

Answer: [ ] (Make sure m and k are in $\{0, 1, \ldots, 36\}$.)

16 pts (4) Recall that the axioms for a finite projective plane are

Axiom P1: There exist at least 4 points no 3 of which are collinear.

Axiom P2: There exists at least 1 line with exactly $n+1$ (distinct) points on it.

Axiom P3: Given 2 distinct points, there is exactly 1 line that they both lie on.

Axiom P4: Given 2 distinct lines, there is at least 1 point on both of them.

Fill in the boxes below to complete a proof of the result stated below. There are two "double" boxes (a box inside of a box) below. These two double boxes should be completed in the same way (that is, whatever you put in one double box should be the same as what you put in the second double box).

Result: *If ℓ is a line with exactly $n+1$ points on it in a finite projective plane of order n and A is a point not on ℓ, then there exist at least $n+1$ distinct lines passing through A.*

This is part of a result done in class and that you were to have learned for this test.

Proof. Let $P_1, P_2, \ldots, P_{n+1}$ be the $n+1$ distinct points on ℓ. Since

[ ]

we have that $A \neq P_j$ for each $j \in \{1, 2, \ldots, n+1\}$. By [ ], there is a line ℓ_j passing through A and P_j for each $j \in \{1, 2, \ldots, n+1\}$. Since [ ] and A is not on ℓ, we see that $\ell_j \neq \ell$ for each $j \in \{1, 2, \ldots, n+1\}$. We justify next that the $n+1$ lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ are different. Assume [ ] for some i and j in $\{1, 2, \ldots, n+1\}$ with $i \neq j$. Then the two points [[ ]] are both on ℓ_i. Since these two points are distinct, [ ] implies that there is [ ] passing through them. Since [[ ]] are two points that are both on ℓ_i and are both on ℓ, we deduce [ ]. This contradicts that [ ]. Thus, our assumption is wrong and the lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ are different. This finishes the proof that there are at least $n+1$ distinct lines passing through A. ■

Part II. The point values appear to the left of each problem. The problems in this section all deal with an axiomatic system consisting of the following axioms.

Axiom 1. There exist at least one point and at least one line with the point on the line.

Axiom 2. Given a point, there are exactly 2 distinct lines that do not pass through the point.

Axiom 3. Given a line, there are exactly 2 distinct points that do not lie on the line.

8 pts (1) Justify that the axiomatic system is consistent. Use the blank paper.

6 pts (2) Justify that the axiomatic system is *not* complete. Include some brief explanation, in complete English sentences, for your answer. Use the blank paper.

8 pts (3) Justify that Axiom 3 is independent of Axiom 1 and Axiom 2. Use the blank paper.

9 pts (4) Write the dual of each axiom below. Use correct English.

Dual of Axiom 1.

Dual of Axiom 2.

Dual of Axiom 3.

5 pts (5) Does the principle of duality hold for this axiomatic system. Explain your answer. Use the blank paper.

(TURN PAGE OVER FOR MORE OF THE TEST.)

16 pts

(6) Fill in the boxes below to finish the proof below that, in the axiomatic system above, if there are exactly 3 points, then there are at least 3 distinct lines parallel to each other. For your convenience, the axioms are repeated here.

Axiom 1. There exist at least one point and at least one line with the point on the line.

Axiom 2. Given a point, there are exactly 2 distinct lines that do not pass through the point.

Axiom 3. Given a line, there are exactly 2 distinct points that do not lie on the line.

Proof. Suppose that there are exactly 3 points. We need to show then that there are 3 distinct lines parallel to each other. By [ ], there is a point P and a line ℓ_1 with P on ℓ_1. By [ ], there are at least 2 distinct points, say Q and R, not on ℓ_1. Since P is on ℓ_1 and Q and R are not on ℓ_1, the points P, Q and R are all distinct. Since there are only 3 points, they are P, Q and R. By [ ], each line passes through exactly one of P, Q and R. By [ ], there are exactly 2 distinct lines, say ℓ_2 and ℓ_3, that do not pass though P. Since P is on ℓ_1, the lines ℓ_1, ℓ_2 and ℓ_3 are distinct. Since each line passes through exactly one of P, Q and R and since P is not on ℓ_2 and not on ℓ_3, each of ℓ_2 and ℓ_3 passes through exactly one of Q and R. Explain in the box below why ℓ_2 and ℓ_3 do not both pass through the same point.

[ ]

Thus, each of ℓ_1, ℓ_2 and ℓ_3 passes through a different point. Since each line passes through exactly one point, we deduce that ℓ_1, ℓ_2 and ℓ_3 are 3 distinct lines which are parallel to each other. ■