MATH 532, 736I: MODERN GEOMETRY

Test 1, Spring 2011 Name ______________________

Show All Work

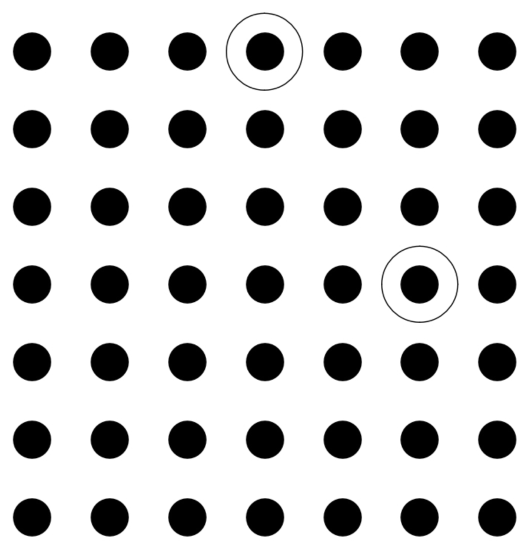
Instructions: Put your name at the top of this page and at the top of the first page of the packet of blank paper given to you. Work each problem on the paper provided, using a separate page for each problem. Show ALL of your work. Put your answers in the boxes below where appropriate. Do NOT use a calculator.
Points: Part I (52 pts), Part II (48 pts)

Part I. The point value for each problem appears to the left of each problem. In Problem 5, I will assume you are using the axioms as you state them in your answer to Problem 1 below.

12 pts (1) In the packet of white paper provided to you, state the axioms for a finite AFFINE plane of order n. (Number or name the axioms so you can refer to them in Problem 5.)

8 pts (2) In the packet of white paper provided to you, give a model for a finite PROJECTIVE plane of order 3. Be sure to clearly mark every point and clearly draw every line in your model.

8 pts (3) Two points have been circled in the 7×7 array of points to the right. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points.



8 pts (4) Consider the points $(4, 2)$ and $(10, 9)$ in an 11×11 array of points for our model of a finite affine plane of order 11. Find the equation of the line passing through these two points. Put your answer below in the form $y \equiv mx + k \pmod{11}$ where m and k are among the numbers $0, 1, 2, \ldots, 10$. Be sure to show your work in the packet provided with this test.

Answer: [ ]

16 pts (5) Using only the theorem below and the axioms you stated in Problem 1, fill in the boxes below to complete a proof that in an affine plane of order n, for each line ℓ, there are at least $n-1$ lines parallel to ℓ. This is part of a proof you were to have memorized for class. (More precisely, the problem you were to have memorized for class involved showing that there are "exactly" $n-1$ such lines. I have not used the word "exactly" in my statement of the problem above.)

Theorem. *In an affine plane of order n, each line contains exactly n points.*

Note: The theorem is to be used in the proof below. The proof is establishing that there are at least $n-1$ lines parallel to ℓ as stated above.

Proof. Let ℓ be an arbitrary line. By [ ], there is a point P_1 not on ℓ. By [ ], line ℓ has at least one point, say P_2, on it. From [ ], there is a line ℓ' passing through P_1 and P_2. Since

[ ],

we have that $\ell' \neq \ell$. [ ] implies that P_2 is the only point on both ℓ' and ℓ. By [ ], ℓ' has exactly [ ] points on it, two of which are P_1 and P_2. Let $P_3, \ldots, P_n$ denote the remaining points on ℓ'. For each $j \neq 2$, P_j is not on ℓ so that [ ] implies that there is a line ℓ_j passing through P_j and parallel to ℓ. Each such ℓ_j is different from ℓ' since

[ ].

It follows that the lines ℓ_j (with $j \neq 2$) are distinct by [ ] (since ℓ' is the unique line passing through any two of the P_j's). Thus, there are at least $n-1$ distinct lines parallel to ℓ (namely, the lines ℓ_j with $j \neq 2$). ■

Part II. The point values appear to the left of each problem. The problems in this section all deal with an axiomatic system consisting of the following axioms.

Axiom 1. There exist 3 collinear points (that is, 3 points and a line with the 3 points on the line).

Axiom 2. There exist exactly 3 distinct lines.

Axiom 3. Given two distinct lines, there is at least one point on both lines.

Axiom 4. Given two distinct points, there is at most one line passing through them.

Note: Axiom 1 is saying that there exist 3 collinear points. This does NOT mean "exactly". There may be more points in the axiomatic system, and there may even be more points on the same line as these 3 points.

8 pts (1) Justify that the axiomatic system is consistent.

8 pts (2) Justify that the axiomatic system is *not* complete. Include some brief explanation for your answer.

12 pts (3) Justify that the axiomatic system is independent.

6 pts (4) What is the dual of Axiom 3?

6 pts (5) Justify that the principle of duality does not hold for this axiomatic system.

8 pts (6) Prove the following:

Given two distinct lines, there is *exactly* one point on both lines.