EXAMPLES ON TRANSLATIONS AND ROTATIONS

(Lecture Notes for Math 532, taught by Michael Filaseta)

1. Let ΔABC be given, and let M_B denote the midpoint of side $\overline{AC}$ and let M_C denote the midpoint of side $\overline{AB}$. Show that $\overleftrightarrow{M_B M_C}$ is parallel to $\overleftrightarrow{BC}$ and that the length of $\overline{M_B M_C}$ is one-half of the length of $\overline{BC}$.

   Solution. A picture would help here. From the theorem, $f = R_{\pi,M_B} R_{\pi,M_C}$ is a translation. One checks that $f(B) = C$. Hence, f is a translation which moves B to C. Also, $f(M_C) = M_C'$ where M_C' is the result of rotating M_C about M_B by π. This means that $\overline{M_C M_B}$ and $\overline{M_B M_C'}$ have the same length and the three points M_C, M_B, and M_C' are collinear. We get that $\overrightarrow{M_C M_C'} = \overrightarrow{BC}$, and the result follows. ■

2. Let A, B, C, and D be the vertices of an arbitrary quadrilateral. Show that the midpoints of the sides of the quadrilateral form a parallelogram.

   Solution. Let M_1 be the midpoint of $\overline{AB}$, M_2 the midpoint of $\overline{BC}$, M_3 the midpoint of $\overline{CD}$, and M_4 the midpoint of $\overline{DA}$. Then from example (1), $\overrightarrow{M_1 M_4}$ and $\overrightarrow{M_2 M_3}$ each have the same direction as $\overrightarrow{BD}$ and half its length. The desired conclusion follows.

   Comment: Instead, one can let $f = R_{\pi,M_4} R_{\pi,M_1}$ and $g = R_{\pi,M_2} R_{\pi,M_3}$. Then f is a translation taking B to D and g is a translation taking D to B. One then essentially repeats the argument in example (1). ■

3. In (2), consider instead midpoints of a $2n-$gon.

   Solution. The problem is a bit vague, but one can conclude the following using the argument in (2). Let $M_1, M_2, \dots, M_{2n}$ denote the midpoints along the edges moving counterclockwise beginning with some edge. Then the segments $\overline{M_1 M_2}, \overline{M_3 M_4}, \dots, \overline{M_{2n-1} M_{2n}}$ can be translated (without rotating them) to form an $n-$gon. Similarly, the segments $\overline{M_{2n} M_1}, \overline{M_2 M_3}, \dots, \overline{M_{2n-2} M_{2n-1}}$ can be translated to form an $n-$gon. ■

4. Let ΔABC be given. Draw an equilateral triangle exterior to ΔABC with one edge $\overline{AB}$, an equilateral triangle exterior to ΔABC with one edge $\overline{BC}$, and an equilateral triangle exterior to ΔABC with one edge $\overline{AC}$. Show that the centers of these 3 equilateral triangles form the vertices of an equilateral triangle.

   Solution. The argument is essentially the same as in the next problem. This is worth going over separately, but we do not do so here. ■

5. Generalize (4) as follows. Let ΔABC and real numbers α, β, and γ be given. Let A', B', and C' be points exterior to ΔABC such that $\angle BA'C = \alpha$, $\angle AB'C = \beta$, and $\angle AC'B = \gamma$. Also, suppose that the lengths of the sides $\overline{BA'}$ and $\overline{A'C}$ are the same, the lengths of the sides $\overline{AB'}$ and $\overline{B'C}$ are the same, and the lengths of the sides $\overline{AC'}$ and $\overline{C'B}$ are the same. Show that if $\alpha + \beta + \gamma = 2\pi$, then the interior angles of $\Delta A'B'C'$ are $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$.

   Solution. Let $f = R_{\alpha,A'}R_{\beta,B'}R_{\gamma,C'}$. Then f is a translation since $\alpha+\beta+\gamma = 2\pi$. Since $f(B) = B$, we get that f is the identity translation. Hence, $f(C') = C'$. Let $C'' = R_{\beta,B'}(C')$. Then $\angle C'B'C'' = \beta$ and the lengths of $\overline{B'C'}$ and $\overline{B'C''}$ are the same. Also,

   $$C' = f(C') = R_{\alpha,A'}R_{\beta,B'}R_{\gamma,C'}(C') = R_{\alpha,A'}R_{\beta,B'}(C') = R_{\alpha,A'}(C'').$$

   This means that $\angle C'A'C'' = \alpha$ and the lengths of $\overline{A'C'}$ and $\overline{A'C''}$ are the same. One easily gets that the triangles $\Delta A'B'C'$ and $\Delta A'B'C''$ are congruent from which it follows that $\angle C'A'B' = \alpha/2$ and $\angle C'B'A' = \beta/2$. It follows that $\angle A'C'B' = \pi - (\alpha/2) - (\beta/2) = \gamma/2$, giving the desired result. ■

   Observe that the following is a consequence of the problem. Suppose that ΔABC is given and D, E, and F are points exterior to ΔABC such that ΔDBC, ΔAEC, and ΔABF are similar so that $\angle D$, $\angle E$, and $\angle F$ are the three angles associated with these similar triangles. Let A', B', and C' be the centers of the circumscribed circles for ΔDBC, ΔAEC, and ΔABF, respectively. Then $\Delta A'B'C'$ is similar to ΔDBC (and, hence, the other two exterior triangles as well).

6. Let n be an odd positive integer, and let $P_1, \ldots, P_n$ be n (not necessarily distinct) points. Let $A = A_0$ be an arbitrary point. For $j \in \{1, \ldots, n\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. For $j \in \{n+1, \ldots, 2n\}$, define A_j as the point you get by rotating A_{j-1} about P_{j-n} by π. Prove that $A_{2n} = A$.

   Solution. Write $n = 2k + 1$ where k is some nonnegative integer. Let f denote the composition of the first n rotations about $P_1, \ldots, P_n$ each by π. Then we want to show that $f(f(A)) = A$. Note that every two rotations by π are equivalent to a translation and the composition of translations is a translation. Hence, we can view f as $T_{(a,b)}R_{\pi,P_1}$ for some (a, b) (where $a = b = 0$ if $k = 0$). By a homework problem, we can rewrite this as

   $$\left(R_{\pi,(a/2,b/2)}\left(R_{\pi,(0,0)}R_{\pi/2,P_1}\right)\right)R_{\pi/2,P_1}.$$

   Taking the product of the matrices as indicated by the parentheses above, we get from the theorem that f is equivalent to a rotation about some point by π. It is clear then that $f(f(A)) = A$, completing the argument. ■

   Comment: The situation when n is even is that A is translated since the compositions of the rotations is a translation. It is possible that the translation is the identity translation in which case $A_{2n} = A$.

7. Let n be an odd positive integer. Suppose that we are given the n midpoints of the sides of an $n-$gon. Show how one can construct an $n-$gon with these given midpoints along its edges.

Solution. Let $M_1, \ldots, M_n$ be the midpoints. Consider the composition f of the rotations about $M_1, \ldots, M_n$ each by π. By the solution to (6), we get that f is equivalent to a rotation about some point, say A, by π. We can construct A as follows. Take any point B and apply f to it. Note that this is done by successively taking the rotations about M_j by π with straightedge and compass for $j = 1, 2, \ldots, n$. We get some point $C = f(B)$. Since B is obtained from C by a rotation about A by π, we deduce that A must be the midpoint of $\overline{BC}$. Since f is equivalent to a rotation about A, we have that $f(A) = A$. Set $A_0 = A$ and rotate it about M_1 to obtain a new point. Call the new point A_1 and rotate it about M_2 to obtain another point A_2. Continue rotating A_j about M_{j+1} to obtain A_{j+1} for $j \in \{1, 2, \ldots, n-1\}$. Since $f(A) = A$, we get that $A_n = A_0$, and the points $A_1, \ldots, A_n$ are the vertices of an $n-$gon as desired. ■

Comments: (i) There are many such $n-$gons since the order of the M_j one chooses to do the above construction will affect the outcome.

(ii) There is another approach to the problem which may be worth discussing. For example, suppose $n = 5$ (though any odd n works here). Call the given midpoints $M_1, \ldots, M_5$. Let $A_0, \ldots, A_4$ be the points we are trying to construct with M_j along edge $\overline{A_{j-1}A_j}$ for $j \in \{1, \ldots, 5\}$ where $A_5 = A_0$. Then the vertices A_0, A_1, A_2, and A_3 are the vertices of a quadrilateral and three of its midpoints M_1, M_2, and M_3 are known. Using the information from example (2), it is not difficult to construct the midpoint M' of $\overline{A_0A_3}$. Now, we know the midpoints of the sides of triangle $\Delta A_0A_3A_4$. Using the information from example (1), we can construct the vertices A_0, A_3, and A_4. One can modify this argument to obtain the other vertices or use the approach in the solution above.

8. Let P_1, P_2, P_3, and P_4 be 4 (not necessarily distinct) points. Let A be an arbitrary point. Beginning with $A_0 = A$, for $j \in \{1, 2, 3, 4\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. Set $Q_1 = P_3$, $Q_2 = P_4$, $Q_3 = P_1$, and $Q_4 = P_2$. Beginning with $B_0 = A$, for $j \in \{1, 2, 3, 4\}$, define B_j as the point you get by rotating B_{j-1} about Q_j by π. Prove that $A_4 = B_4$. (See Figure 1.)

Solution. A rotation about P_1 by π followed by a rotation about P_2 by π is a translation, say T_R. Similarly, a rotation about P_3 by π followed by a rotation about P_4 by π is a translation, say T_S. The problem amounts to establishing that $T_S T_R = T_R T_S$. This is easy to establish (but note that in general the product of two matrices does not commute). ■

9. Let A, B, C, and D be the vertices of a convex quadrilateral labelled counterclockwise. Consider 4 squares exterior to the quadrilateral, one square with an edge $\overline{AB}$,

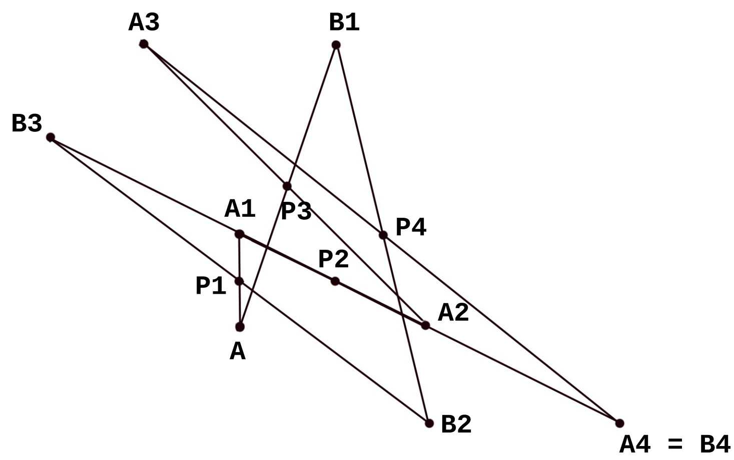

Figure 1

one square with an edge $\overline{BC}$, one square with an edge $\overline{CD}$, and one square with an edge $\overline{DA}$. Let M_1 be the center of the square with edge $\overline{AB}$, let M_2 be the center of the square with edge $\overline{BC}$, let M_3 be the center of the square with edge $\overline{CD}$, and let M_4 be the center of the square with edge $\overline{DA}$. Show that the length of $\overline{M_1M_3}$ is the same as the length of $\overline{M_2M_4}$ and that $\overleftrightarrow{M_1M_3}$ and $\overleftrightarrow{M_2M_4}$ are perpendicular.

Solution. Let

$$g = R_{\pi/2,M_1}R_{\pi/2,M_2}, \quad h = R_{\pi/2,M_3}R_{\pi/2,M_4}, \quad \text{and} \quad f = gh.$$

Then $f(A) = A$, and we get that f is the identity translation. Also, there are points P_1 and P_2 such that g is a rotation about P_1 by π and h is a rotation about P_2 by π. It is easy to see (draw a picture) that since a rotation about P_1 by π followed by a rotation about P_2 by π is the identity, we must have $P_1 = P_2$. Next, we observe that P_1 is the only point such that $g(P_1) = P_1$. It follows that $\Delta M_2P_1M_1$ is a isosceles right triangle labelled counterclockwise (since P_1 so located is mapped to itself by g). Similarly, $\Delta M_4P_2M_3$ is a isosceles right triangle labelled counterclockwise. Since $P_1 = P_2$, we get that $\Delta P_1M_2M_4$ is obtained from $\Delta P_1M_1M_3$ by a rotation about P_1 by $\pi/2$. This implies that the length of $\overline{M_1M_3}$ is the same as the length of $\overline{M_2M_4}$. Let Q_1 be the point of intersection of $\overleftrightarrow{M_1M_3}$ and $\overleftrightarrow{M_2M_4}$ and Q_2 the point of intersection of $\overleftrightarrow{M_2M_4}$ and $\overleftrightarrow{M_1P_1}$ (convince yourself these exist). Then $\angle M_1Q_2Q_1 = \angle P_1Q_2M_2$ and $\angle Q_2M_1Q_1 = \angle P_1M_2Q_2$, and it follows that $\angle M_1Q_1Q_2 = \angle M_2P_1Q_2 = \pi/2$. Thus, the lines $\overleftrightarrow{M_1M_3}$ and $\overleftrightarrow{M_2M_4}$ are perpendicular. ■

Comment: It can be shown that if the original quadrilateral is a parallelogram, then M_1, M_2, M_3, and M_4 form the vertices of a square.

10. Let ΔABC be a given triangle with the angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ all acute. Let $\overline{AP}$ be an altitude drawn from A so that P is on $\overline{BC}$. Similarly, let $\overline{BQ}$ be the altitude drawn from B and $\overline{CR}$ the altitude drawn from C. Show that ΔPQR is a triangle with minimum perimeter that can be inscribed in ΔABC (that is show that if ΔUVW is a triangle with U on $\overline{BC}$, V on $\overline{AC}$, and W on $\overline{AB}$, then its perimeter is at least that of ΔPQR).

Solution. First, we will show that $\angle PRB = \angle QRA$, $\angle RPB = \angle QPC$, and $\angle PQC = \angle RQA$. We establish one of these and the other two can be done in the same way. Let D be the intersection of the altitudes. Since $\angle DRA = \angle DQA = \pi/2$, the points A, Q, D, and R all lie on a circle. Hence,

$$\angle DRQ = \angle DAQ = \frac{\pi}{2} - \angle ACP.$$

Since $\angle DRB = \angle DPB = \pi/2$, the points B, P, D, and R all lie on a circle. Hence,

$$\angle DRP = \angle DBP = \frac{\pi}{2} - \angle BCQ = \frac{\pi}{2} - \angle ACP = \angle DRQ.$$

Since $\angle ARC = \angle BRC$, we get that $\angle PRB = \angle QRA$. As mentioned, in a similar fashion, one obtains $\angle RPB = \angle QPC$ and $\angle PQC = \angle RQA$.
If you haven't already started drawing pictures, get out those crayons. To match some of the discussion here, you should label your points along the triangle as A, B, and C in a counterclockwise direction. We begin with triangle ΔABC and reflect it about side $\overline{BC}$ to get a new tringle ΔA_1BC. The 2 triangles are distinct, congruent, and share the edge $\overline{BC}$. Next, we reflect ΔA_1BC about side $\overline{A_1C}$ to get a new tringle ΔA_1B_1C. Then we reflect ΔA_1B_1C about side $\overline{A_1B_1}$ to get a new tringle $\Delta A_1B_1C_1$. Next, we reflect $\Delta A_1B_1C_1$ about side $\overline{B_1C_1}$ to get a new tringle $\Delta A_2B_1C_1$. Finally, we reflect $\Delta A_2B_1C_1$ about side $\overline{A_2C_1}$ to get a new tringle $\Delta A_2B_2C_1$. If your crayoning technique is mastered, this is what should happen. Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Every point along segment $\overline{AB}$ is first rotated about B by $2\pi - 2\beta$, then about A_1 by $2\pi - 2\alpha$, then about B_1 by 2β, and finally about A_2 by 2α. We can conclude that the segment $\overline{AB}$ has been translated to $\overline{A_2B_2}$. This means that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{A_2B_2}$. If we draw in ΔPQR and its reflections, the information from the first paragraph implies that the points R, P, Q_1 (the first reflection of Q), R_2 (the next reflection of R - there was already an R_1 from the first reflection), P_2 (the next reflection of P), Q_3 (the next reflection of Q), and R_4 (the last reflection of R) are all collinear. Also, the segment $\overline{RR_4}$ has length twice the perimeter of ΔPQR. If one similar reflects ΔUVW, one finds that W is translated to some W_4. Here, $\overline{WW_4}$ and $\overline{RR_4}$ have the same length (since R and W went through the same translation), and the length of $\overline{WW_4}$ is $\leq$ twice the perimeter of ΔUVW. The desired conclusion follows. ■

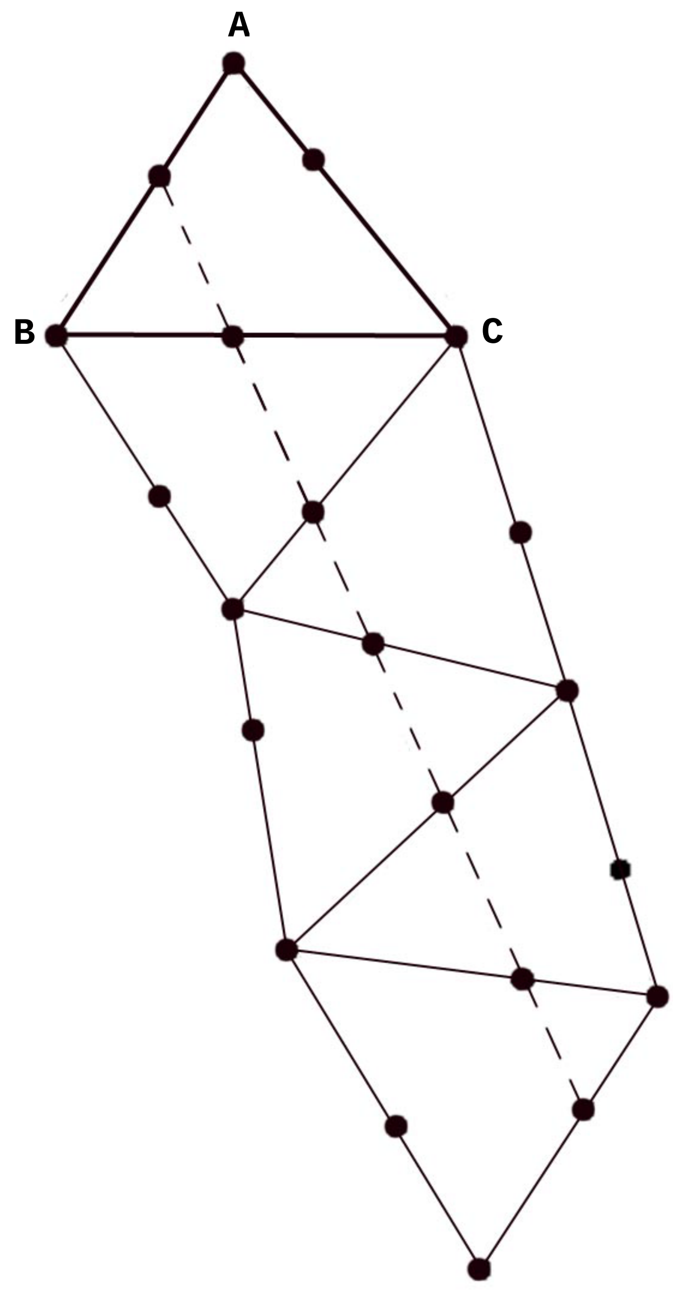

Figure 2