MATH 532/736I, LECTURE NOTES 8

Theorem 1. *Let A and B be distinct points. Then C is on $\overleftrightarrow{AB}$ if and only if there is a real number t such that $C = (1-t)A + tB$.*

Theorem 3. *If A, B and C are points and there exist real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$

then A, B and C are collinear.

Definition: Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are *perspective from a point X* if the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ all pass through X. (See Figure 1.)

Definition: Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are *perspective from a line* if P, Q and R are collinear where P is the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$, Q is the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ and R is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$. (See Figure 2.)

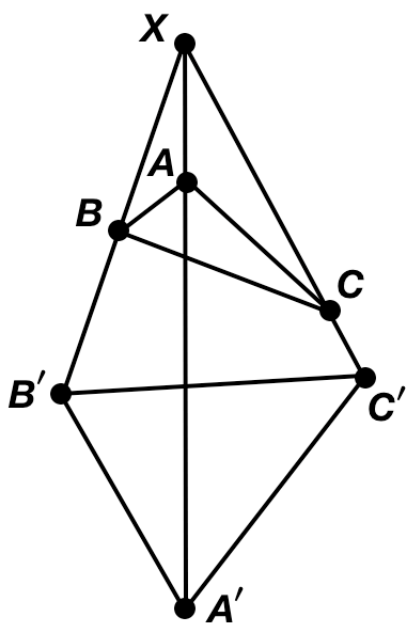

Figure 1

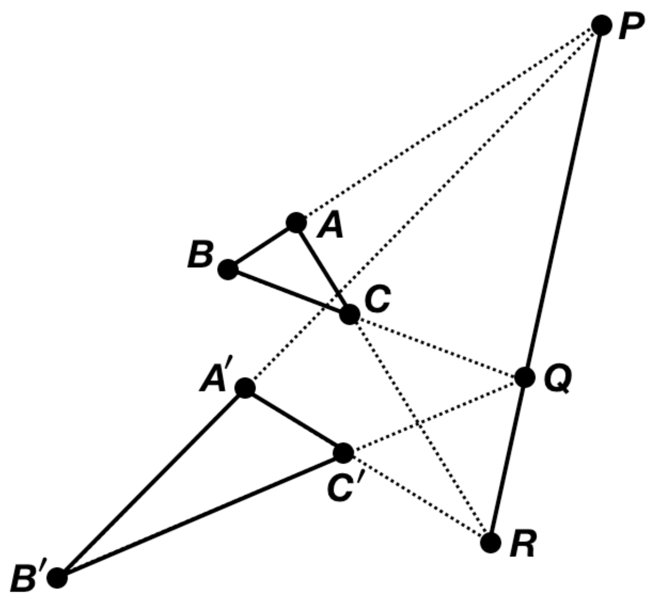

Figure 2

Desargues' Theorem: *If two triangles are perspective from a point, then they are perspective from a line.*

Comment: Below is the proof for the case that X, P, Q and R are points in the "affine" plane and the points A, B, C, A', B' and C' are distinct (and also in the affine plane).

Proof. By Theorem 1, there are real numbers k_1, k_2, and k_3 such that

$$X = (1-k_1)A + k_1 A' = (1-k_2)B + k_2 B' = (1-k_3)C + k_3 C'. \tag{$*$}$$

We show that $k_1 \neq k_2$. Assume $k_1 = k_2$. Observe that $k_1 \neq 0$ since otherwise we would have $X = A = B$, contradicting that A and B are distinct points. Also, $k_1 \neq 1$ since otherwise we would have $X = A' = B'$, contradicting that A' and B' are distinct points. We get that

$$(1 - k_1)A - (1 - k_2)B = k_2 B' - k_1 A'.$$

Hence, $(1 - k_1)\overrightarrow{BA} = k_1 \overrightarrow{A'B'}$ and the vectors $\overrightarrow{BA}$ and $\overrightarrow{A'B'}$ either have the same direction or the exact opposite direction. This contradicts that the point P exists. Hence, $k_1 \neq k_2$.

From $(*)$, we obtain that

$$\frac{1 - k_1}{k_2 - k_1} A + \frac{k_2 - 1}{k_2 - k_1} B = \frac{k_2}{k_2 - k_1} B' + \frac{-k_1}{k_2 - k_1} A'.$$

By Theorem 1 with $t = (k_2 - 1)/(k_2 - k_1)$, we see that the expression on the left is a point on line $\overleftrightarrow{AB}$. By Theorem 1 with $t = -k_1/(k_2 - k_1)$, we see that the expression on the right is a point on line $\overleftrightarrow{A'B'}$. Therefore,

$$P = \frac{1 - k_1}{k_2 - k_1} A + \frac{k_2 - 1}{k_2 - k_1} B.$$

Hence,

$$(k_2 - k_1)P = (1 - k_1)A + (k_2 - 1)B. \tag{1}$$

From $(*)$, we use now that

$$(1 - k_2)B - (1 - k_3)C = k_3 C' - k_2 B'.$$

Similarly to the above, we obtain that $k_2 \neq k_3$, that

$$\frac{1 - k_2}{k_3 - k_2} B + \frac{k_3 - 1}{k_3 - k_2} C = \frac{k_3}{k_3 - k_2} C' + \frac{-k_2}{k_3 - k_2} B',$$

and that

$$(k_3 - k_2)Q = (1 - k_2)B + (k_3 - 1)C. \tag{2}$$

Using $(*)$ once again, we obtain

$$(1 - k_3)C - (1 - k_1)A = k_1 A' - k_3 C'.$$

Similarly to the above, we deduce that $k_1 \neq k_3$, that

$$\frac{1 - k_3}{k_1 - k_3} C + \frac{k_1 - 1}{k_1 - k_3} A = \frac{k_1}{k_1 - k_3} A' + \frac{-k_3}{k_1 - k_3} C',$$

and that

$$(k_1 - k_3)R = (1 - k_3)C + (k_1 - 1)A. \tag{3}$$

Take

$$A = P, \quad B = Q, \quad C = R,$$

$$x = k_2 - k_1, \;\; y = k_3 - k_2, \;\; \text{and} \;\; z = k_1 - k_3$$

in Theorem 3. Note that $x \neq 0$. We deduce that P, Q and R are collinear, finishing the proof. ■