Aaron Dutle

Research Statement

My main research interests lie, broadly speaking, in the field of combinatorics and graph theory. Most of my recent and current work falls under two general categories. Graph modeling, particularly extensions of degree sequence based modeling, and spectral theory for hypergraphs and hypermatrices. In the next few pages, I outline my current and proposed work in these two areas first, followed by a discussion of some other research interests, and finishing with an overview of my interdisciplinary aspirations. While these are my current topics of study, my research interests are ever-expanding, driven both by the exciting problems and people I encounter, and my own desire to expand my mathematical toolset.

1. Graph Modeling

Graphs are natural models for many real-world situations, such as social networks, protein interactions, computer networks, and countless others. I'm interested in creating graphs that attempt to capture some essential features of such networks. One reason for doing so is to discover what makes the original network unique. If we create an ensemble of graphs which match a real network in some collection of parameters, we can analyze the differences between the real and artificial graphs to see if there are other essential features that our model misses. This allows us to see which features are due to the model constraints, and which are unique to the original, perhaps even revealing directions to further refine our model. Secondly, an accurate model allows us to tailor and test algorithms specifically for the type of graph under consideration, and to predict how a real network might evolve.

Joint Degree Matrices. The majority of my work in this area stems from a model called the Joint Degree Matrix (or JDM) [1]. This is a matrix whose i, j entry records the number of edges that connect vertices of degree i to vertices of degree j. It is a more restrictive model than the well studied degree sequence model, and is built to capture whether a graph is assortative (similar degree vertices tend to link), disassortative (high degree vertices link to low degree vertices), or neither.

In [15] my coauthors and I provide a simple proof of exact conditions for when a matrix can be realized as the JDM of a graph. We also show that for any fixed JDM, any realization G can be transformed into any other realization G' using a sequence of simple local moves called restricted swap operations. A restricted swap takes two edges and replaces them with two non-edges, while preserving the JDM. Using these swap operations, we can build a Markov Chain Monte-Carlo algorithm to actually sample a graph with a given JDM. Our result shows that the Markov chain is irreducible. The next important question about this model is the mixing time of this Markov Chain. This is unknown in general, though my coauthors have proven fast mixing in a certain restricted setting [18].

Another open question concerning the JDM is an extremal one. Is there some succinct characterization of the Joint Degree Matrices that have only one realization? The analogous question for degree sequences is answered by *threshold graphs* which have several simple and equivalent characterizations. I've proven that a graph with this property must be highly structured, being either complete or empty between each degree class, but a simple characterization is still elusive.

Generalizations of the JDM. There are a number of ways that one can extend or generalize the JDM model. One method is to ask for graphs that don't exactly match the JDM, but are close in some way. To be more specific, if we divide each entry of a JDM by the total number of edges in the graph, the i, j entry records the probability that a randomly selected edge connects vertices of degree i and j. This is referred to as the Joint Degree Distribution. We've developed two ways to build graphs which approximate this distribution. The first takes a Joint Degree Distribution D and a number m, a desired number of edges, as parameters, and with probability tending to 1 builds a realizable JDM whose distribution approaches D. Unfortunately, this only builds a JDM, and since the maximum degree is fixed, eventually builds sparse realizations. Our second so-called "soft model" adapts the Chung-Lu model [7] to produce a graph which matches the Joint Degree

Distribution in expectation. An obvious next step is to determine what conditions on the JDD allow you to conclude that the graphs produced are concentrated near their expectation.

Another direction to extend the Joint Degree Matrix model is to allow for arbitrary partitions of the vertices, instead of dividing the graph into classes based strictly on degree. We partition a graph into $P_1, P_2, \dots, P_k$ classes, (perhaps) record the degree sequence in each class, and write down a *Partition Adjacency Matrix* (or *PAM*), where the i, j entry records the number of edges connecting P_i to P_j. If the classes are based on degree, we recover the JDM model. If each class is a single vertex, the PAM is the adjacency matrix of the graph, completely describing the graph up to isomorphism. It can also be used to describe many other natural classes of graphs (k-partite, split, etc). My coauthors and I have some results in this area, including conditions for existence of realizations, as well as examples showing that swaps involving only 2 edges are insufficient to transform one PAM realization to another. It should also be noted that a well-studied graph model called the *stochastic block model* can be viewed as a randomized approximation of a PAM.

A final direction the JDM (and PAM) can be taken is to the realm of hypergraphs. A k-uniform hypergraph has "edges" that are size k subsets of the vertices (as such, a graph is a 2-uniform hypergraph). Here, we could specify a Joint Degree Tensor (or hypermatrix), where the $i_1, i_2, \dots i_k$ entry records the number of edges with a vertex in each class P_{i_j}. My collaborators and I have generalized some of the techniques from the graph case to prove characterizations of when such a tensor can be realized. On the other hand, the connectivity of the state space using swaps is still unknown. Essentially this is because the connectivity of the state space of hypergraphs of a fixed degree sequence is still unknown, although there are some results in this direction [2].

2. Hypermatrices and Spectral Hypergraph Theory

Spectral graph theory is the study of how graph properties relate to the eigenvalues of some matrix defined by the graph. In general, one uses either the adjacency matrix A, defined by $(a_{ij}) = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise,} \end{cases}$ or some variation of it.

There have been some previous and ongoing attempts to define eigenvalues for hypergraphs and study their properties, with varying amounts of success. Notable examples include [5, 19, 22, 28, 35]. Most of this work concerns generalizations of the spectrum of a graph by defining some suitable square matrix related to the hypergraph, and studying the eigenvalues. Much of this work has a very different flavor than my research in the area, which instead considers a generalization of a square matrix to a higher-dimensional array.

Hypermatrices and Eigenvalues. A (cubical) *hypermatrix* or *tensor* $\mathcal{A}$ of dimension n and order k is a collection of n^k complex numbers $a_{i_1 i_2 \dots i_k}$, where $1 \le i_j \le n$.

We call a hypermatrix *symmetric* if entries which use the same index sets are the same. That is, $\mathcal{A}$ is symmetric if $a_{i_1 i_2 \dots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(k)}}$ for all permutations σ on $\{1, 2, \dots, k\}$. When $k = 2$, cubical hypermatrices are simply square matrices, and symmetric hypermatrices are just symmetric matrices.

Qi ([34]) and Lim ([27]) offered several generalizations of the eigenvalues of a symmetric matrix to the case of higher order symmetric (or even non-symmetric) hypermatrices. Following [34], call $\lambda \in \mathbb{C}$ an *eigenvalue* of $\mathcal{A}$ if there is a non-zero vector $\mathbf{x} \in \mathbb{C}^n$, which we call an *eigenvector*, satisfying

$$\sum_{i_2, i_3, \dots, i_k = 1}^{n} a_{j i_2 i_3 \dots i_k} x_{i_2} \dots x_{i_k} = \lambda x_j^{k-1} \tag{1}$$

for all $1 \le j \le n$.

For succintness, we consider $\mathbf{x}$ as a vector, write $\mathbf{x}^{[m]}$ for the vector with entries x_i^m and write $\mathcal{A}\mathbf{x}^{k-1}$ for the vector output of the left hand side of 1. Then the eigenvalue equations take on a familiar expression as solutions to $\mathcal{A}\mathbf{x}^{k-1} = \lambda \mathbf{x}^{[k-1]}$.

While this completely defines the set of eigenvalues for a hypermatrix, one can also use a tool from algebraic geometry called the multipolynomial resultant (see [23, 12]) on the eigenvalue defining equations to yield a monic polynomial whose roots are exactly the set of eigenvalues. Because of this analogy, the polynomial is called the characteristic polynomial of $\mathcal{A}$. We denote it by $\phi_{\mathcal{A}}(\lambda)$, and call the multiset of roots the *spectrum* of $\mathcal{A}$.

Even before applications to hypergraphs, there is much work remaining to be done in the field of hypermatrices, and some interesting potential applications. One basic element lacking from hypermatrix theory is a satisfying analogue of matrix multiplication, and in particular, matrix exponentiation. Of course, in the matrix case, this has direct relationship to the characteristic polynomial. Perhaps such a relationship can be developed for hypermatrices as well. Coming from a graph theory perspective, powers of the adjacency matrix of a graph count the number walks between vertices, and such an analogy for hypergraphs would be useful.

Another element lacking from spectral theory of hypermatrices is a method for determining or even bounding the algebraic multiplicity of an eigenvalue based on its eigenvectors. In the matrix case, the number of linearly independent eigenvectors is a lower bound for the algebraic multiplicity, while in the hypermatrix case, no clear relationship is known. Indeed, even a simple formula for calculating the characteristic polynomial is unknown in the hypermatrix case.

Even so, there has been rapid development in area in recent years [4, 21, 27, 30, 34], and some of the results look to be very useful. In particular, let $F_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1,\dots,i_k} \mathcal{A}_{i_1,\dots i_k} x_{i_1} \dots x_{i_k}$ (the obvious generalization of the quadratic form $\mathbf{x}^T A\mathbf{x}$ for a matrix). In the hypermatrix case, there are spectral conditions that guarantee that $F_{\mathcal{A}}$ is copositive [33]. This has many possible applications, one being to the area of Flag Algebras, a tool in graph theory for (among other things) proving density relationships for certain subgraphs of a large host graph. One main step in the process is to find a copositive function, and use as a way to balance some of the densities. In practice, this copositive function is always the quadratic form arising from a positive semidefinite matrix, as these are well-known and relatively easy to find. Applying the copositivity results from hypermatrices to this area may lead to new results in this area.

Spectra of Hypergraphs. In [9], Cooper and I obtain theorems about hypergraphs generalizing many of those from basic spectral graph theory. The hypermatrix we consider is nearly the straightforward generalization of the adjacency matrix.

For a k-uniform hypergraph H on n labeled vertices, the (normalized) *adjacency hypermatrix* $\mathcal{A}_H$ is the order k dimension n hypermatrix with entries

$$a_{i_1,i_2,\dots,i_k} = \frac{1}{(k-1)!} \begin{cases} 1 & \text{if } \{i_1, i_2, \dots i_k\} \in E(H) \\ 0 & \text{otherwise.} \end{cases}$$

Some of the results we obtained are the following.

- If H is the disjoint union of hypergraphs H_1 and H_2, then as sets, $\mathrm{spec}(H) = \mathrm{spec}(H_1) \cup \mathrm{spec}(H_2)$. When considered as multisets, the multiplicities behave in a predictable manner.
- If H is the cartesian product of hypergraphs H_1 and H_2, then $\mathrm{spec}(H_1) + \mathrm{spec}(H_2) \subseteq \mathrm{spec}(H)$.
- The eigenvalue with largest modulus, $\lambda_{\max}$, can be chosen to be a positive real number. If H is connected, then a corresponding eigenvector $\mathbf{x}$ can be chosen to be strictly positive.
- If d is the average degree of H, and Δ is the maximum degree, then $\max\{d, sqrt[k]\Delta\} \leq \lambda_{\max} \leq \Delta$.
- If G is a subhypergraph of H, then $\lambda_{\max}(G) \leq \lambda_{\max}(H)$.
- $\chi \leq \lambda_{\max} + 1$, where χ is the (weak) chromatic number [1].
- If H is k-partite, then the spectrum of H is invariant under multiplication by any k^{th} root of unity.
- The coefficients of the characteristic polynomial count certain subhypergraphs. In partcular, the codegree k coefficient counts edges, and the codegree $k+1$ coefficient counts complete hypergraphs on $k+1$ vertices.

It should be noted that while many of these theorem statements are exactly the same as those in the graph case, the proofs are significantly different than their graph counterparts. Also, a few

[1] the smallest number of colors that can be used to color the vertices of H so that no edge has all of its vertices of the same color.

are not as complete as their graph counterparts, leaving some interesting questions. In particular, the spectrum of a cartesian product of graphs is exactly the set-sum of the factor graphs, while in the hypergraph case, some "sporadic" eigenvalues arise, which we don't yet fully understand. Also, there are hypergraphs which are not k-partite whose spectrum is still invariant under multiplication by k^{th} roots of unity. We understand these somewhat, but lack a complete characterization.

We considered some of the most common classes of hypergraphs and hypermatrices, and characterized the spectrum of the class where possible. We were able to completely determine the spectrum for single-edge hypergraphs, sunflower hypergraphs (hypergraphs with some central set of vertices which are exactly the intersection of every pair of edges), complete k-partite hypergraphs, and complete 3-uniform hypergraphs [9]. We were also able to compute using a recursive technique the characteristic polynomial, and hence the multiset spectrum, for the hypermatrix with every entry being 1, and for 1-seed 3-uniform sunflowers [10]. Of course, this leads to a fount of further problems, as there are many interesting classes of hypergraphs which we have yet to consider.

For example, one could try to characterize the spectrum of cycles (loose or tight), paths (loose or tight), linear hypergraphs, block designs, random hypergraphs, or any other imaginable class of hypergraphs. Indeed, with an appropriate choice of hypergraph class, this is even a problem that could be undertaken as an undergraduate research project.

In [32], the authors use a different, but somewhat related, definition of hypergraph eigenvalues. Some conditions on these eigenvalues are found to be (in a complex way) equivalent to varying strengths of quasirandomness of the hypergraph. The authors have also used them to characterize the existence or absence of certain structures, from an extremal hypergraph theory point of view. I believe that the two eigenvalue concepts can be linked, and that it may bring further tools to bear on these types of problems.

Spectral hypergraph theory is still in its infancy, and there are many unexplored avenues for research in the area. I look forward to being a contributor to its development.

3. Other Interests

While much of my work has concentrated on the topics above, I've the opportunity to work in several other areas of combinatorics as well.

The Lovász Local Lemma. The Lovász Local Lemma [17] is a powerful tool in probabilistic combinatorics. The standard usage allows you to conclude the existence of a favorable outcome to an experiment, if some conditions on the dependency of the events are met.

Given a probability space, and a collection $A_1, \dots, A_k$ of "bad" events, a *dependency graph* has the events as vertices, and the event A_i is independent of any collection of events taken from its non-neighbors. That is, for all i and all $S \subset \{j : (i,j) \notin E(G)\}$ with $P(\bigcap_{j\in S} \overline{A_j}) > 0$,

$$P\left(A_i | \bigcap_{j\in S} \overline{A_j}\right) = P(A_i)$$

In the simplest version, if $p \geq P(A_i)$ for all events A_i, d is an upper bound for the maximum degree in the dependency graph, and $ep(d+1) < 1$, one can conclude that $P(\bigcap_{j=1}^{k} \overline{A_j}) > 0$. That is, there is some outcome avoiding all the bad events.

Following a series of results by others, Moser and Tardos proved [31] that in many situations satisfying the conditions of the Local Lemma, an algorithm with polynomial expected running time can be derived to actually produce the outcome guaranteed to exist by the Lemma. I applied a version of this algorithm to find small dominating sets[2] in some graphs with a power-law distribution. Essentially, we sample all of the vertices, and if vertices are still undominated, choose one and resample its closed neighborhood. The novelty in this application is that the standard Local Lemma tells us nothing, since there is positive probability of choosing *all* vertices, and that we used the solution to the fractional relaxation of the integer linear program that finds a minimum dominating set as our guide to set the probabilities. While this loses the guarantee of fast expected run-time, in

[2] A dominating set is a set of vertices so that every vertex is in the set, or adjacent to a vertex in the set.

practice it produced dominating sets that beat the standard greedy approximation of the minimum dominating set on some classes of graphs.

This technique is ripe for further application. If a problem is provably *close* to meeting the conditions of the Local Lemma with some setting of parameters, the corresponding algorithm may be able to find the desired outcome. This could be used for such things increasing lower bounds for Ramsey numbers, since avoiding cliques and independent sets of particular sizes can be phrased in the language of the Local Lemma[3].

Combinatorics on Words. In [8], my coauthor and I showed that the sequence of players arising from a particular game, which we called a "greedy Galios duel" tends to the Thue-Morse sequence as the shooters' probability of success tends to 0. The occurrence of such a well-know sequence in this setting was delightfully unexpected. Extending this duel to include three or more players may lead to an interesting generalization of the Thue-Morse sequence to larger alphabet size.

A related topic in the area is the following. A word W is called *unavoidable* (on a given alphabet) if every sufficiently long word has W as a pattern inside it. Notably, that the pattern aaa is avoidable on the two letter alphabet is evidenced by the Thue-Morse sequence. Often, a word is proved unavoidable by contradiction rather than construction, which leads to no bounds on what "sufficiently long" means. Finding explicit bounds for the threshold length to see a particular unavoidable word W, or general bounds based on the length and alphabet of W, is a problem I find interesting.

Forbidden Subposets. Given two partially ordered sets P and Q, we say Q is a *subposet* of P if there is an injection $f : Q \to P$ so that if $x \leq_Q y$, then $f(x) \leq_P f(y)$. If Q is not a subposet of P, we say P avoids Q. A typical extremal question then is to fix some small poset Q, and to find the maximum number of elements of a larger poset P that can be taken while still avoiding Q. In many cases, the larger poset is taken as the boolean lattice, i.e., the subsets of an n element set, ordered by inclusion. In this case, $La(n, P)$ is used to denote the size of the largest subposet avoiding P, and $\pi(P) = \lim_{n\to\infty} La(n, P)/\binom{n}{\lfloor n/2\rfloor}$, if the limit exists. It's conjectured that this limit exists, and furthermore, is always an integer.

This conjecture is known true for all posets whose Hasse diagram is a tree, and so the next simplest case is the *diamond* D_2, with four elements $A < B_1, B_2 < C$. The *diamond conjecture* of Griggs and Lu [24] asserts that $\pi(D_2) = 2$. The lower bound is easily achieved by taking two consecutive middle levels. There have been many results that have brought down the upper bound, which currently stands at 2.25 [26]. If the elements allowed to be chosen must be from 3 consecutive levels, this bound can be further reduced.

Our contribution [16] gives multiple constructions on three and four levels that can be made asymptotically close to the conjectured bound. Our constructions select elements from the poset based on weightings of the ground set by elements of abelian groups, and use Markov Chains on these groups to prove that the number of elements is as claimed. While this technique is (provably) not enough to refute the diamond conjecture, it's results give some evidence for why the conjecture has resisted proof.

On the other hand, the technique of weighting by abelian group elements (omitting the Markov chain techniques) still shows promise for finding a counter-example if one exists. The general technique may also be applicable to other forbidden subposet problems.

4. Interdisciplinary interests

As mentioned before, graphs are an excellent model for many real-world situations, in varying disciplines. The following are a few interdisciplinary applications that I'm currently pursuing.

Graphs in Literature. There has been some recent work [29] analyzing classical texts in literature from a graph theory perspective. The researchers track the interactions between characters, and use the data to create a graph. Using some properties of the graphs and assumed properties of social networks, they purport to differentiate works of fiction from possible non-fiction.

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[3]On a practical note, the number of events to avoid is generally to large to be computationally feasible at the current state of technology.

While I found this application interesting, I see another possible application for these graphs in the realm of pedagogy. I believe that someone who understands a book well (such as an instructor) has an intuitive concept of this interaction graph, while someone with little understanding has a much worse approximation. Perhaps there's even a correlation between how well the intuitive approximation matches the true graph, and their overall comprehension of the text in a traditional sense. I'm currently talking with members of the Literature department to develop a way to test this hypothesis, as well as to test whether having the students *actively* create and see the interaction graph throughout the reading of the text helps in understanding.

Graph Clustering. During the recent Summer School in Network Science, I had the opportunity to hear series of talks about graph clustering by Peter Mucha. Essentially, graph clustering attempts to partition your graphs into groups of vertices that are similar, or highly interconnected. While this is an interesting problem mathematically, I was more impressed by the work that is done *after* the clustering. Given additional data about the vertices, we can see which factors agree with the partition. For example, a college social network may split into clusters where the chosen major is dominant characteristic. Using this analysis, we can infer global characteristics (such as what majors are closely related), and local characteristics (the major of a student who didn't volunteer their major, but was placed into a particular cluster).

I see this technique as an excellent chance to work with disciplines that infrequently use graph theory. In disciplines such as sociology, biology, and political science, researchers frequently have data where the recognition of clusters, and causes of clustering, would be helpful in furthering research. I'm interested in working with them to do so.

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