Renormings and symmetry properties of one-greedy bases

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Abstract

We continue the study of 1-greedy bases initiated by F. Albiac and P. Wojtaszczyk [1]. We answer several open problems they raised concerning symmetry properties of 1-greedy bases and the improving of the greedy constant by renorming. We show that 1-greedy bases need not be symmetric nor subsymmetric. We also prove that one cannot in general make a greedy basis 1-greedy as demonstrated for the Haar basis of dyadic Hardy space $H_1(\mathbb{R})$ and for the unit vector basis of Tsirelson space. On the other hand, we give a renorming of L_p $(1 < p < \infty)$ that makes the Haar basis 1-unconditional and 1-democratic. Other results in this paper clarify the relationship between various basis constant that arise in the context of greedy bases.

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Introduction

In approximation theory one is often concerned with approximating a signal (a vector in some Banach space) with a finite linear combination from some dictionary (a basis of the space). The greedy algorithm is perhaps the simplest theoretical scheme for m-term approximation, which can be described as follows. Let X be a Banach space and (e_i) be a *(Schauder) basis* of X. Recall that this

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means that for any $x \in X$ there is a unique sequence (x_i) of scalars with $x = \sum_{i=1}^{\infty} x_i e_i$. For such an x we fix a permutation ρ of $\mathbb{N}$ (not necessarily unique) such that $|x_{\rho(1)}| \geq |x_{\rho(2)}| \geq \dots$. We then define the m^{th} greedy approximant to x by

$$\mathcal{G}_m(x) = \sum_{i=1}^{m} x_{\rho(i)} e_{\rho(i)} \ .$$

For this to make sense we need $\inf \|e_i\| > 0$, otherwise (x_i) may be unbounded. In fact, we will always assume that (e_i) is *seminormalized*, *i.e.*, $0 < \inf \|e_i\| \leq \sup \|e_i\| < \infty$, and often that (e_i) is *normalized*, *i.e.*, $\|e_i\| = 1$ for all $i \in \mathbb{N}$.

We measure the efficiency of the greedy algorithm by comparing it to the best m-term approximation: for $m \in \mathbb{N}$ we let

$$\sigma_m(x) = \inf \left\{ \Big\| x - \sum_{i \in A} a_i e_i \Big\| : \ A \subset \mathbb{N},\ |A| \leq m,\ (a_i)_{i \in A} \subset \mathbb{R} \right\} .$$

We say that (e_i) is a *greedy basis* for X if there exists $C > 0$ (C*-greedy*) such that

$$\|x - \mathcal{G}_m(x)\| \leq C \sigma_m(x) \qquad \text{for all } x \in X \text{ and for all } m \in \mathbb{N} \ .$$

The smallest C is the *greedy constant* of the basis. An important example is the Haar basis of $L_p[0,1]$ ($1 < p < \infty$) which was shown to be greedy by V. N. Temlyakov [10]. This result was later established by P. Wojtaszczyk [11] using a different method which extended to the Haar system in one-dimensional dyadic Hardy space $H_p(\mathbb{R})$, $0 < p \leq 1$. Other examples include the unit vector basis of ℓ_p ($1 \leq p < \infty$) and c_0, or an orthonormal basis of a separable Hilbert space.

As well as in approximation theory, Schauder bases play a very important rôle in abstract Banach space theory. The idea behind introducing a co-ordinate system is that it makes an a priori abstract space into a concrete space of scalar sequences. There are many deep and beautiful results in the geometry of Banach spaces that were solved using basis techniques. Often knowing that a space has a basis, however, is not sufficient and one needs to consider special bases. A particularly useful notion is that of an *unconditional basis*: a basis (e_i) of a Banach space X is said to be *unconditional* if there is a constant K (K*-unconditional*) such that

$$\Big\| \sum a_i e_i \Big\| \leq K \cdot \Big\| \sum b_i e_i \Big\| \qquad \text{whenever } |a_i| \leq |b_i| \text{ for all } i \in \mathbb{N} \ .$$

The best constant K is the *unconditional constant* of the basis which we denote by K_U. The property of being unconditional is easily seen to be equivalent to that of being *suppression unconditional* which means that for some constant K (*suppression K-unconditional*) the natural projection onto any subsequence of the basis has norm at most K:

$$\Big\| \sum_{i \in A} a_i e_i \Big\| \leq K \cdot \Big\| \sum_{i=1}^{\infty} a_i e_i \Big\| \qquad \text{for all } (a_i) \subset \mathbb{R},\ A \subset \mathbb{N} \ .$$

The smallest K is the *suppression unconditional constant* of the basis and is denoted by K_S. It is easy to verify that $K_S \le K_U \le 2K_S$.

It is perhaps somewhat surprising that various properties of bases that have arisen independently in abstract Banach space theory on the one hand and in approximation theory on the other hand turned out to be closely related. In [7], S. V. Konyagin and V. N. Temlyakov introduced the notion of greedy and democratic bases and proved the following characterization.

Theorem 0.1 ([7, Theorem 1])**.** *A basis of a Banach space is greedy if and only if it is unconditional and democratic.*

A basis (e_i) is said to be *democratic* if there is a constant $\Delta > 0$ (Δ-*democratic*) such that

$$\Big\| \sum_{i\in A} e_i \Big\| \le \Delta \Big\| \sum_{i\in B} e_i \Big\| \qquad \text{whenever } |A| \le |B| \ .$$

(In the original definition, A and B have the same cardinality, however when (e_i) is weakly null, which will always be the case for us, the two definitions are equivalent.) The proof of this theorem provides the following estimates for the various basis constants involved (*c.f.* [12, Theorem 1]):

$$K_S \le C \ , \quad \Delta \le C \qquad \text{and} \qquad C \le K_S + K_S^3 \cdot \Delta \ . \tag{1}$$

That is, a C-greedy basis is suppression C-unconditional and C-democratic, and conversely, a suppression unconditional and Δ-democratic basis is C-greedy with $C \le K_S + K_S^3 \cdot \Delta$.

In this paper we continue the study of the "isometric case", *i.e.,* the case when $C = 1$, initiated by F. Albiac and P. Wojtaszczyk. In [1] they give a characterization of 1-greedy bases in terms of symmetry properties of the basis. They raise several open problems about symmetry properties of 1-greedy bases and about the possibility of improving various basis constants by renorming. In this paper we shall provide solutions to these problems. In order to explain their characterization, we need some definitions. Let (e_i) be a basis of a real Banach space X. The *support* (with respect to the basis (e_i)) of a vector $x = \sum x_i e_i$ is the set $\text{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}$. The subspace of vectors with finite support (*i.e.,* the linear span of (e_i)) can be indentified in the obvious way with the space c_{00} of real sequences that are eventually zero. The basis (e_i) then corresponds to the unit vector basis of c_{00}. Given vectors $x = \sum x_i e_i$ and $y = \sum y_i e_i$ in c_{00}, we say y *is a greedy rearrangement of* x (with respect to the basis (e_i)) if there exists a bijection $\pi\colon \text{supp}(x) \to \text{supp}(y)$ such that $|y_{\pi(i)}| = |x_i|$ for all $i \in \text{supp}(x)$, and $\pi(i) = i$ and $y_i = x_i$ whenever $|x_i| < \|x\|_{\ell_\infty}$. To put it informally, y is obtained from x by moving (and possibly changing the sign of) some of the coefficients of x of maximum modulus to co-ordinates where x is zero. Note that this definition differs slightly from that given in [1] in that we allow changing the signs of coefficients of maximum modulus that are not "moved" to other co-ordinates. This subtle difference is an important one but only for finite bases (*c.f.* Definition 3.1, Proposition 3.2 and Example 5.1 in [1]). In this paper we will only consider infinite bases, and for those the two definitions of Property (A) below resulting from the two definitions of greedy rearrangement are equivalent.

Theorem 0.2 ([1, Theorem 3.4]). *A basis (e_i) of a Banach space is* 1*-greedy if and only if it satisfies the following two properties:*

(i) (e_i) *is suppression* 1*-unconditional, and*

(ii) (e_i) *satisfies Property (A):* $\|x\| = \|y\|$ *whenever* y *is a greedy rearrangement of* x*.*

Property (A) is a sort of weak symmetry property for largest coefficients. Recall that the basis (e_i) of X is *symmetric* if there is a constant $K > 0$ (K-*symmetric*) such that for all $x = \sum_{i=1}^{\infty} x_i e_i$ in X, for all permutations π of $\mathbb{N}$ and for all sequences (ε_i) of signs we have

$$\Big\| \sum_{i=1}^{\infty} \varepsilon_i x_{\pi(i)} e_i \Big\| \leq K \cdot \Big\| \sum_{i=1}^{\infty} x_i e_i \Big\| \ .$$

Thus, by the above characterization, the property of being 1-greedy is formally weaker than being 1-symmetric. Albiac and Wojtaszczyk gave an example of a 1-greedy but not 1-symmetric basis [1, Example 5.6]. Their example, being the unit vector basis of c_0 with an equivalent norm, is still symmetric, and they raise the question whether there is a 1-greedy basis that is not symmetric [1, Problem 6.1]. In this paper we give a positive answer to this question:

Theorem A. *There is a Banach space with a normalized basis that is* 1*-greedy and not symmetric.*

We can strengthen the above result with considerably more work to obtain the following.

Theorem B. *There is a norm on* $\mathrm{c}_{00} \oplus \mathrm{c}_{00}$ *equivalent to the norm* $\|\cdot\|_{\ell_2} + \|\cdot\|_{\ell_{2,1}}$ *with respect to which the unit vector basis is* 1*-greedy and not subsymmetric.*

The definition of Lorentz space $\ell_{2,1}$ will be recalled in Section 4. A basis (e_i) of a Banach space X is *subsymmetric* if it is equivalent to all its subsequences which implies that there is a constant $K > 0$ (K-*subsymmetric*) such that for all $x = \sum_{i=1}^{\infty} x_i e_i$ in X, for all sequences (ε_i) of signs and for all increasing sequences $n_1 < n_2 < \dots$ of positive integers we have

$$\Big\| \sum_{i=1}^{\infty} \varepsilon_i x_i e_{n_i} \Big\| \leq K \Big\| \sum_{i=1}^{\infty} x_i e_i \Big\| \ .$$

Note that a 1-subsymmetric spaces is 1-unconditional and 1-democratic.

What justifies studying the isometric case in general is the fact that various approximation algorithms converge trivially when some appropriate constant is 1. By contrast, when this constant is strictly greater than 1, the problem of convergence can be very difficult to resolve. We mention as an example (see [3] and [9] for details and recent developments) the so called X-greedy algorithm whose convergence for the Haar basis of $L_p[0,1]$ (whose unconditional constant is strictly greater than 1) is still an open problem. The question of improving the greedy or democracy constant of a basis by renorming is therefore of interest. This question was raised explicitly by Albiac and Wojtaszczyk [1, Problems 6.2 and 6.3]. In this paper we present a solution by giving as counterexamples two important Banach spaces with greedy bases.

Theorem C. *There is no renorming of the dyadic Hardy space H_1 that makes its natural basis 1-greedy. There is no renorming of Tsirelson space T for which there is any 1-greedy, normalized basis of T.*

It remains an open problem whether the Haar basis of $L_p[0,1]$ ($1 < p < \infty$) can become 1-greedy under an equivalent norm [1, Problem 6.2]. Towards a positive answer Albiac and Wojtaszczyk showed [1, Proposition 4.5] that under an equivalent norm, for any $\varepsilon > 0$, there is a subsequence of the Haar basis which is 1-unconditional and $(1+\varepsilon)$-democratic, and whose closed linear span is isomorphic to $L_p[0,1]$. In this paper we strengthen this result considerably:

Theorem D. *For any $1 < p < \infty$ there is a renorming of $L_p[0,1]$ such that the Haar basis is 1-unconditional and 1-bidemocratic.*

A glance at the estimates (1) reveals that this renorming makes the Haar basis 2-greedy. We are going to show that in general one cannot deduce anything better.

Theorem E. *There exists a 1-unconditional, 1-democratic basis which is not C-greedy for any $C < 2$.*

In fact, the example we construct is 1-subsymmetric. In [1, Example 5.4] it is observed that the first example of a subsymmetric but not symmetric basis due to D. J. H. Garling [5] is also an example for a 1-unconditional, 1-democratic but not 1-greedy basis. The greedy constant of Garling's example turns out to be 2 (this is not computed in [1] but it is not hard to see), and thus Garling's example also proves Theorem E. Our example has the additional property of being 2-symmetric, which is best possible since, by [1, Theorem 2.5], a C-symmetric basis is C-greedy.

In the other direction, a 1-greedy basis is suppression 1-unconditional by Theorem 0.2, and hence 2-unconditional. For finite bases, Examples 5.1 and 5.2 of [1] show that 1-greedy need not imply 1-unconditional, but this possibility was left open for infinite bases [1, Problem 6.4]. We solve this problem by proving

Theorem F. *There is a renorming of Lorentz space $\ell_{2,1}$ such that the unit vector basis is 1-greedy, 2-unconditional but not $(2-\varepsilon)$-unconditional for any $\varepsilon > 0$.*

The paper is organized as follows. The first two sections are concerned with renorming results: we prove Theorems C and D in Sections 1 and 2, respectively. In Sections 3 and 4 we give examples that clarify the relationship between the greedy, democratic and unconditional constants of a basis (Theorems E and F). In Section 5 we construct a basis that is 1-greedy but not symmetric, and finally, in Section 6, we produce a 1-greedy basis that is not even subsymmetric.

We follow standard Banach space terminology and work with real scalars throughout. We use $|\cdot|$ for absolute value (of a real number) and size (of a finite set). The indicator function of a set A is denoted by $\mathbf{1}_A$. For $A \subset \mathbb{N}$ we think of $\mathbf{1}_A$ as the vector $\sum_{i\in A} e_i$ in c_{00} or the functional $\sum_{i\in A} e_i^*$, where (e_i^*) is the sequence of biorthogonal functionals to the unit vector basis (e_i) of c_{00}. For a vector x and functional f, the standard pairing $f(x)$ will sometimes be written as $\langle x, f\rangle$.

1. Renormings of greedy bases

The following questions were raised by Albiac and Wojtaszczyk (see Problems 6.2 and 6.3 in [1]).

(i) Can a greedy basis (e_i) be renormed so that it is 1-greedy?

(ii) Can a democratic basis (e_i) be renormed so that it is 1-democratic?

By the characterization of 1-greedy bases, Theorem 0.2, a 1-greedy basis is 1-democratic, and so the following proposition solves both problems in the negative for spaces like the Tsirelson space T (*i.e.,* the dual of the original Tsirelson space as described in [4]) with any basis, or for the dyadic Hardy space H_1 with the Haar basis.

Proposition 1.1. *Assume that X is a Banach space with a normalized suppression 1-unconditional basis (e_i) and that there is a sequence $(\rho_n) \subset (0,1]$ with $\rho = \inf_{n\in\mathbb{N}} \rho_n > 0$ so that*

$$\Big\|\sum_{i\in E} e_i\Big\| = \rho_n n \qquad \textit{whenever } n\in\mathbb{N} \textit{ and } E\subset\mathbb{N} \textit{ with } |E| = n\ .$$

Then (e_i) is $\frac{2}{\rho}$-equivalent to the unit vector basis of ℓ_1.

Proof. First note that the sequence (ρ_n) is non-increasing. Indeed,

$$\begin{aligned} n(n-1)\rho_n &= \Big\|(n-1)\sum_{i=1}^{n} e_i\Big\| \\ &= \Big\|\sum_{i=1}^{n}\ \sum_{j\in\{1,\dots,n\}\setminus\{i\}} e_j\Big\| \\ &\le \sum_{i=1}^{n}\Big\|\sum_{j\in\{1,\dots,n\}\setminus\{i\}} e_j\Big\| = n(n-1)\rho_{n-1}\ , \end{aligned}$$

which implies the claim.

Denote the biorthogonal functionals of (e_i) by (e_i^*). For each finite subset E of $\mathbb{N}$ we choose $(a_i^{(E)})_{i\in E}\subset\mathbb{R}$, so that

$$\sum_{i\in E} a_i^{(E)} e_i^* \in S_{X^*} \qquad \text{and} \tag{2}$$

$$\Big(\sum_{i\in E} a_i^{(E)} e_i^*\Big)\Big(\sum_{i\in E} e_i\Big) = \sum_{i\in E} a_i^{(E)} = \Big\|\sum_{i\in E} e_i\Big\| = n\rho_n. \tag{3}$$

Since (e_i) is suppression 1-unconditional, it follows that $a_i^{(E)}\ge 0$ for all $i\in E$.

Since for any finite $E\subset\mathbb{N}$ with $|E|=n$, and for any $j\in E$ we have

$$\begin{aligned}\sum_{i\in E\setminus\{j\}} a_j^{(E)} &\le \Big\|\sum_{i\in E\setminus\{j\}} e_i\Big\| = (n-1)\rho_{n-1} = \sum_{i\in E\setminus\{j\}} a_i^{(E\setminus\{j\})}\\ &\le \Big\|\sum_{i\in E} e_i\Big\| = n\rho_n = \sum_{i\in E} a_i^{(E)}\ ,\end{aligned}$$

it follows, after choosing $j_0\in E$ so that $\min_{j\in E} a_j^{(E)} = a_{j_0}^{(E)}$, that

$$\begin{aligned}\min_{j\in E} a_j^{(E)} &= a_{j_0}^{(E)} \qquad\qquad (4)\\ &= \sum_{i\in E} a_i^{(E)} - \sum_{i\in E\setminus\{j_0\}} a_i^{(E)}\\ &\ge n\rho_n-(n-1)\rho_{n-1} = \rho_n-(\rho_{n-1}-\rho_n)(n-1)\ .\end{aligned}$$

Let $b_n=\min_{1\le j\le n} a_j^{(\{1,2,\dots,n\})}$. Since $1-\rho=\sum_{n\ge 2}(\rho_{n-1}-\rho_n)<\infty$ we can find an infinite set $N\subset\mathbb{N}$ for which

$$\lim_{n\to\infty,\ n\in N}(n-1)(\rho_{n-1}-\rho_n)=0\ .$$

Fix $\varepsilon\in(0,\rho)$. By (4) we can pass to an infinite $N_\varepsilon\subset N$ such that $b_n\ge\rho_n-\varepsilon\ge\rho-\varepsilon$ whenever $n\in N_\varepsilon$. Finally, using the suppression 1-unconditionality of (e_i^*), it follows that $\sum_{i\in A}(\rho-\varepsilon)e_i^*\in B_{X^*}$ for all $A\subset\{1,\dots n\}$ and for all $n\in N_\varepsilon$, and thus for all $n\in\mathbb{N}$.

Since $\varepsilon>0$ can be chosen arbitrarily small, it follows that (e_i^*) is $2/\rho$-equivalent to the unit vector basis of c_0, and thus (e_i) is $2/\rho$-equivalent to the unit vector basis of ℓ_1. □

In the next two results we use Proposition 1.1 to deduce Theorem C stated in the Introduction, and answer questions (i) and (ii) above.

Corollary 1.2. *There is no renoming of Tsirelson space T for which T has an unconditional and 1-democratic basis.*

Proof. Assume a renorming $|||\cdot|||$ of T existed admitting an unconditional and 1-democratic basis (x_i). We can clearly assume that (x_i) is normalized. We can also assume that (x_i) is suppression 1-unconditional. Indeed, if it is not, then we can simply pass to the equivalent norm given by

$$\sup_{E\subset\mathbb{N}} \Big|\Big|\Big| \sum_{i\in E} a_i x_i \Big|\Big|\Big|$$

for any $\sum a_i x_i \in T$, with respect to which (x_i) is suppression 1-unconditional and still normalized 1-democratic.

(x_i) is weakly null (since T is reflexive), and therefore it has a subsequence (x_i') with ℓ_1 as spreading model. It follows that there is a $\rho > 0$ and a sequence $(\rho_n) \subset (0, 1]$ such that for any $n \in \mathbb{N}$ and any $A \subset \mathbb{N}$ with $|A| = n$ we have

$$\Big\| \tfrac{1}{n} \sum_{i\in A} x_i \Big\| = \lim_{k_1\to\infty,\ k_2\to\infty\ldots k_n\to\infty} \Big\| \tfrac{1}{n} \sum_{i=1}^{n} x'_{k_i} \Big\| = \rho_n \geq \rho\ .$$

This implies, by Proposition 1.1, that (x_i) is equivalent to the unit vector basis of ℓ_1, which is a contradiction. □

Remark. The above argument applies more generally to show that if Y is any reflexive Banach space not containing ℓ_1, and if ℓ_1 is the only spreading model of Y, then Y contains no unconditional 1-democratic basic sequence.

Corollary 1.3. *The dyadic Hardy space H_1 admits no equivalent norm with respect to which the Haar system is 1-greedy.*

Proof. Let us denote by (h_n) the Haar system in H_1. It was shown in [11, Lemma 9] that

$$\Big|\Big|\Big| \sum_{n\in A} h_n \Big|\Big|\Big| \geq \frac{1}{2}|A| \tag{5}$$

for any finite set $A \subset \mathbb{N}$. Here $|||\cdot|||$ denotes the natural square-function norm of H_1 (see [11] for details). Now assume that $\|\cdot\|$ is an equivalent norm on H_1 with respect to which (h_n) is 1-democratic. As in the proof of Corollary 1.2 we may assume that (h_n) is normalized and suppression 1-unconditional with respect to $\|\cdot\|$. Then by (5) the conditions of Proposition 1.1 are satisfied with $e_n = h_n$ for all n. This implies that (h_n) is equivalent to the unit vector basis of ℓ_1, which is a contradiction. □

Remark. In [11] Wojtaszczyk studied the efficiency of the greedy algorithm for multi-dimensional Haar systems. In particular he showed that in dimension one the Haar basis is a greedy basis for $H_p,\ 0 < p < \infty$. For $p > 1$ this reproves the result of Temlyakov [10] that the Haar basis of L_p is greedy.

2. Renormings of bidemocratic greedy bases

In this section we show that $L_p[0, 1]$ $(1 < p < \infty)$ may be renormed so that the Haar basis is 1-unconditional and 1-democratic (Theorem D). This answers a question raised implicitly in [1, page 78]. We first prove some more general results on renorming bidemocratic greedy bases, of which Theorem D will be an easy consequence.

Suppose that (e_i) is a normalized 1-unconditional greedy basis of a Banach space X with biorthogonal sequence (e_i^*). Recall that the *fundamental function* φ is defined by

$$\varphi(n) = \sup_{|A| \le n} \Big\| \sum_{i \in A} e_i \Big\| \ .$$

The *dual fundamental function* φ^* is given by

$$\varphi^*(n) = \sup_{|A| \le n} \Big\| \sum_{i \in A} e_i^* \Big\| \ .$$

We recall that $(\varphi(n)/n)$ is a decreasing function of n, since for any $A \subset \mathbb{N}$ with $|A| = n$ we have

$$\sum_{i \in A} e_i = \frac{1}{n-1} \sum_{i \in A} \sum_{j \in A \setminus \{i\}} e_j$$

(*c.f.* the first claim in the proof of Proposition 1.1). Clearly, $\varphi(n)\varphi^*(n) \ge n$. We say that (e_i) is *bidemocratic* if there is a constant $\Delta > 0$ (Δ-*bidemocratic*) such that

$$\phi(n)\phi^*(n) \le \Delta n \qquad \text{for all } n \in \mathbb{N} \ .$$

It is known [2, Proposition 4.2] that if (e_i) is bidemocratic with constant Δ, then both (e_i) and (e_i^*) are democratic with constant Δ.

Theorem 2.1. *Suppose that (e_i) is a 1-unconditional and Δ-bidemocratic basis for a Banach space X. Then*

$$|||x||| = \max\Big\{ \|x\|, \sup_{|A| < \infty} \frac{\phi(|A|)}{|A|} \sum_{i \in A} |e_i^*(x)| \Big\} \tag{6}$$

is an equivalent norm on X. Moreover, (e_i) is 1-unconditional and 1-bidemocratic with respect to $|||\cdot|||$. In particular, (e_i) and (e_i^*) are 1-democratic and 2-greedy.

Proof. For $x \in X$ and $|A| < \infty$, note that

$$\frac{\phi(|A|)}{|A|} \sum_{i \in A} |e_i^*(x)| \le \frac{\Delta}{\phi^*(|A|)} \sum_{i \in A} |e_i^*(x)| \le \Delta \|x\| \ .$$

Hence

$$\|x\| \le |||x||| \le \Delta \|x\| \ .$$

Since $\big\| \sum_{i \in A} e_i \big\| \le \phi(|A|)$, we have

$$\begin{aligned} \Big|\Big|\Big| \sum_{i \in A} e_i \Big|\Big|\Big| &= \sup_{|B| < \infty} \frac{|A \cap B| \phi(|B|)}{|B|} \\ &= \sup_{n \ge |A|} \frac{|A| \phi(n)}{n} \\ &= \phi(|A|) \ , \end{aligned}$$

using the fact that $(\phi(n)/n)$ is a decreasing function of n. Thus, (e_i) is 1-democratic with respect to $|||\cdot|||$.

From (6) we have $|||\sum_{i\in A} e_i^*||| \le \dfrac{|A|}{\phi(|A|)}$. On the other hand,

$$\Big|\Big|\Big|\sum_{i\in A} e_i^*\Big|\Big|\Big| \ge \frac{|A|}{|||\sum_{i\in A} e_i|||} = \frac{|A|}{\phi(|A|)} \ .$$

Hence (e_i) and (e_i^*) have fundamental functions with respect to $|||\cdot|||$ of $(\phi(n))$ and $(n/\phi(n))$, respectively, which implies that (e_i) is 1-bidemocratic. □

Corollary 2.2. *Suppose that X has nontrivial type and that (e_i) is a greedy basis for X. Then X admits an equivalent norm $|||\cdot|||$ such that (e_i) is 1-unconditional and 1-bidemocratic with respect to $|||\cdot|||$.*

Proof. First, we use the fact that there is an equivalent norm on X for which (e_i) is 1-unconditional (and greedy). By [2, Prop. 4.1] every greedy basis for a space with nontrivial type is bidemocratic. So by Theorem 2.1 there is an equivalent norm for which (e_i) is 1-unconditional and 1-bidemocratic. □

$L_p[0,1]$ has nontrivial type for $1 < p < \infty$, so we obtain Theorem D in the Introduction which improves [1, Proposition 4.5].

Corollary 2.3. *There is a renorming of $L_p[0,1]$ $(1 < p < \infty)$ for which the Haar basis is 1-unconditional and 1-bidemocratic.*

3. 1-unconditional and 1-democratic does not imply 1-greedy

In this section we give an example of a 1-unconditional and 1-democratic, and hence 2-greedy, basis (e_i) that is not C-greedy for any $C < 2$. This establishes Theorem E. Our example is in fact 2-symmetric which is best possible since a C-symmetric basis is C-greedy by [1, Theorem 2.5]. It is also 1-subsymmetric: for all $(a_i) \in c_{00}$, for all sequences (ε_i) of signs and for all increasing sequences $n_1 < n_2 < \dots$ of positive integers we have

$$\Big\|\sum_{i=1}^{\infty} \varepsilon_i a_i e_{n_i}\Big\| = \Big\|\sum_{i=1}^{\infty} a_i e_i\Big\| \ .$$

Thus, subject to being 1-unconditional and 1-democratic but not C-greedy for any $C < 2$, our example has as much symmetry as one can hope for. The construction is motivated by the proof by Kutzarova and Lin that Schlumprecht space S contains ℓ_∞^n's uniformly [8]. We first need to introduce a piece of notation. For subsets E, F of $\mathbb{N}$ we write $E < F$ if $m < n$ for all $m \in E,\ n \in F$. If $m \in \mathbb{N}$ and $F \subset \mathbb{N}$, then we write $m < F$ instead of $\{m\} < F$.

Fix $(\varepsilon_i) \subset \mathbb{R}^+$ such that $\sum_{i=1}^{\infty}\sum_{j\ge i} \varepsilon_j < 1$. Then choose sequences $n_i \nearrow \infty$ of positive integers and $r_i \searrow 0$ of positive reals such that $r_1 = n_1 = 1$ and

$$\max\left\{\frac{r_i n_i}{r_j n_j}, \frac{r_j}{r_i}\right\} < \varepsilon_j \qquad \text{whenever } i < j \ .$$

This can clearly be done, and moreover we can also arrange that n_i divides n_j whenever $i < j$. Let

$$\mathcal{F} = \big\{ s\mathbf{1}_E + t\mathbf{1}_F : s, t \in \mathbb{R},\ E, F \subset \mathbb{N},\ E < F, \\ \exists\, i < j,\ s = r_i,\ |E| = n_i,\ t = r_j,\ |F| = n_j \big\}$$

and define

$$\|x\| = \sup_{f \in \mathcal{F}} \langle |x|, f \rangle \qquad (x \in \mathrm{c}_{00}),$$

where $|x|$ denotes the sequence $\big(|x_i|\big)$ if $x = (x_i)$. It is easy to verify that $\|\cdot\|$ is a norm on c_{00} and that the unit vector basis (e_i) is a normalized (*n.b.*, $r_1 = n_1 = 1$) 1-subsymmetric basis of the completion X of $(\mathrm{c}_{00}, \|\cdot\|)$. It is also not hard to verify that (e_i) is 2-symmetric. Indeed,

$$|||x||| = \sup \big\{ r_i \langle x, \mathbf{1}_E \rangle : i \in \mathbb{N},\ E \subset \mathbb{N},\ |E| = n_i \big\} \qquad (x \in \mathrm{c}_{00})$$

is a 2-equivalent norm on X with respect to which (e_i) is 1-symmetric. It remains to show that the greedy constant of (e_i) is 2.

Theorem 3.1. *Let X be the space defined above. The unit vector basis (e_i) of X is not C-greedy for any $C < 2$.*

For the proof we need norm-estimates for two types of vectors.

Lemma 3.2. *Fix $i < j$ in $\mathbb{N}$. Let $x = \frac{1}{r_i n_i} \mathbf{1}_G + \frac{1}{r_j n_j} \mathbf{1}_H$, where*

$$G = \{g_1, \dots, g_{n_i}\}\ , \quad H = \bigcup_{m=1}^{n_i} H_m\ , \quad H_1 < g_1 < H_2 < g_2 < \dots < H_{n_i} < g_{n_i}$$

and $|H_m| = \frac{n_j}{n_i}$ for each $m = 1, \dots, n_i$. Then $\|x\| \leq 1 + \frac{1}{n_i} + 2 \sum_{p \geq i} \varepsilon_p$.

Proof. Let $k \in \mathbb{N}$ and let $E \subset \mathbb{N}$ with $|E| = n_k$. We first estimate $\langle x, r_k \mathbf{1}_E \rangle$. We define

$$\lambda(E) = \min\{m \geq 1 : \min E \leq g_m\}$$
$$\rho(E) = \min\{m \geq 1 : \max E < g_m\}$$

and, for this to be always well-defined, we set $g_{n_i+1} = \infty$. We have

$$\langle x, r_k \mathbf{1}_E \rangle = \frac{r_k}{r_i n_i} |G \cap E| + \frac{r_k}{r_j n_j} |H \cap E|$$

$$\leq \frac{r_k}{r_i n_i} \min \Big\{ \rho(E) - \lambda(E), n_i, n_k \Big\} + \frac{r_k}{r_j n_j} \min \Big\{ \big(\rho(E) - \lambda(E) + 1\big) \frac{n_j}{n_i}, n_j, n_k \Big\}\ .$$

Observe that

$$\frac{r_k}{r_i n_i} \min \Big\{ \rho(E) - \lambda(E), n_i, n_k \Big\} \leq \begin{cases} \frac{\rho(E) - \lambda(E)}{n_i} & \text{if } k = i \\ \min \Big\{ \frac{r_k}{r_i}, \frac{r_k n_k}{r_i n_i} \Big\} & \text{if } k \neq i\ . \end{cases}$$

Similarly, we have

$$\frac{r_k}{r_j n_j}\min\left\{\big(\rho(E)-\lambda(E)+1\big)\frac{n_j}{n_i}, n_j, n_k\right\} \le \begin{cases} \frac{\rho(E)-\lambda(E)+1}{n_i} & \text{if } k=j \\ \min\left\{\frac{r_k}{r_j}, \frac{r_k n_k}{r_j n_j}\right\} & \text{if } k\neq j \ . \end{cases}$$

Hence, by the choice of (n_l) and (r_l), we have

$$\langle x, r_k \mathbf{1}_E\rangle \le \begin{cases} \frac{\rho(E)-\lambda(E)}{n_i} + \varepsilon_j & \text{if } k=i \\ \varepsilon_j + \frac{\rho(E)-\lambda(E)+1}{n_i} & \text{if } k=j \\ \varepsilon_{\max\{i,k\}} + \varepsilon_{\max\{j,k\}} & \text{otherwise.} \end{cases}$$

We can finally estimate $\|x\|$. Let $f \in \mathcal{F}$. Then $f = s\mathbf{1}_E + t\mathbf{1}_F$, where $s = r_k$, $|E| = n_k$, $t = r_l$ and $|F| = n_l$ for some $k < l$; moreover $E < F$, which implies that $\rho(E) \le \lambda(F)$, and hence $(\rho(E)-\lambda(E)) + (\rho(F)-\lambda(F)) \le n_i$. It follows immediately that

$$\langle x, f\rangle = \langle x, r_k\mathbf{1}_E\rangle + \langle x, r_l \mathbf{1}_F\rangle \le 1 + \frac{1}{n_i} + 2\sum_{p\ge i}\varepsilon_p$$

and this completes the proof of the lemma. □

Lemma 3.3. *Fix $i < j$ in $\mathbb{N}$. Let $y = \frac{1}{r_i n_i}\mathbf{1}_G + \frac{1}{r_j n_j}\mathbf{1}_H$, where $G < H$, $|G| = n_i$ and $|H| = n_j$. Then $\|y\| \ge 2$.*

Proof. Simply apply to y the element $f = r_i\mathbf{1}_G + r_j\mathbf{1}_H$ of $\mathcal{F}$. □

Remark. The vectors x and y in the previous two lemmata are rearrangements of each other, so we already know that the unit vector basis is not C-symmetric for any $C < 2$.

Proof of Theorem 3.1. Fix $\mu > 1$ and fix positive integers $i < j$. Let

$$z = s'\mathbf{1}_{G'} + s\mathbf{1}_G + t\mathbf{1}_H$$

where G and H are defined as in Lemma 3.2, $G' < H$ and $|G'| = |G| = n_i$, and $s' = \frac{1}{r_i n_i}$, $s = \frac{\mu}{r_i n_i}$, $t = \frac{1}{r_j n_j}$. Thus z combines two vectors of type x and y from the two lemmata above.

Consider now N-term approximations to z where $N = n_i$. By Lemma 3.2 we have

$$\sigma_N(z) \le \|z - s'\mathbf{1}_{G'}\| < 1 + \frac{1}{n_i} + 3\sum_{p\ge i}\varepsilon_p$$

provided $\mu - 1$ is sufficiently small. On the other hand, the N^{th} greedy approximant to z is $\mathcal{G}_N(z) = s\mathbf{1}_G$, and

$$\|z - \mathcal{G}_N(z)\| = \|z - s\mathbf{1}_G\| \ge 2$$

by Lemma 3.3. Letting i go to infinity, we have shown that the basis is not C-greedy if $C < 2$. □

4. 1-greedy does not imply 1-unconditional

In this section we answer Problem 6.4 of [1] by constructing a 1-greedy basis that is not 1-unconditional. This will be achieved by renorming the Lorentz space $\ell_{2,1}$. We begin with recalling the definition of $\ell_{2,1}$.

Consider the weight sequence $w_i = \sqrt{i} - \sqrt{i-1},\ i = 1, 2, \dots$. Let $\mathcal{E}$ be the collection of all formal sums f of the form

$$f = \sum_{i=1}^{\infty} \pm w_i e^*_{m_i}$$

for any choice of signs, where $m_1, m_2, \dots$ is a permutation of $\mathbb{N}$. Then

$$\|x\|_{\ell_{2,1}} = \sup_{f \in \mathcal{E}} \langle x, f \rangle \qquad (x \in \mathrm{c}_{00})$$

defines a norm on c_{00}, and we denote by $\ell_{2,1}$ the completion of c_{00} with respect to this norm. The unit vector basis (e_i) is a normalized 1-symmetric, and hence 1-greedy [1, Theorem 2.5], basis of $\ell_{2,1}$. Note that for $x = \sum_{i=1}^{\infty} x_i e_i$ we have

$$\|x\|_{\ell_{2,1}} = \sum_{i=1}^{\infty} w_i |x_{\rho(i)}| \ ,$$

where ρ is the decreasing rearrangement of x: $|x_{\rho(1)}| \geq |x_{\rho(2)}| \geq \dots$. The space $\ell_{2,1}$ will also be featured in Sections 5 and 6 where we shall recall further properties.

Let $\mathcal{F}$ now be the collection of all functionals f of the form

$$f = \frac{1}{\sqrt{n}} \sum_{i \in E} \pm e^*_i + \sum_{i=n+1}^{\infty} w_i e^*_{m_i} \ , \tag{7}$$

for any choice of signs, where $n \geq 1$, $|E| = n$, (m_i) is a sequence of distinct positive integers, and $\{m_i \colon i > n\} \cap E = \emptyset$. Then

$$|||x||| = \sup_{f \in \mathcal{F}} |\langle x, f \rangle| \qquad (x \in \ell_{2,1})$$

is a renorming of $\ell_{2,1}$ satisfying

$$|||x||| \leq \|x\|_{\ell_{2,1}} \leq 2|||x||| \qquad \text{for all } x \in \ell_{2,1} \ .$$

The second inequality is straightforward. To see the first one, it is sufficient to prove the following: for $n \in \mathbb{N}$ and positive reals $a_1 \geq a_2 \geq \dots \geq a_n$, the inequality $\frac{1}{\sqrt{n}} \sum a_i \leq \sum w_i a_i$ holds. One way to see this is as follows.

$$\sum w_i a_i = \sup_{\rho} \sum w_i a_{\rho(i)} \geq \mathrm{Ave}_\rho \sum w_i a_{\rho(i)} = \frac{1}{n}\Big(\sum w_i\Big)\Big(\sum a_i\Big) = \frac{1}{\sqrt{n}} \sum a_i \ ,$$

where the first equality, in which the sup is taken over all permutations ρ of $\{1, \dots, n\}$, follows by a standard inequality for decreasing rearrangements (see [6, p. 261]), whereas the inequality, in which we replace sup by the average over all ρ, is clear as are the next two equalities.

Theorem 4.1. *With respect to $|||\cdot|||$, (e_i) is 1-greedy and 2-unconditional but not $(2-\varepsilon)$-unconditional for any $\varepsilon > 0$.*

Proof. Note that (e_i) is a normalized, suppression 1-unconditional, and hence 2-unconditional, basis for $(\ell_{2,1}, |||\cdot|||)$. To show that (e_i) is 1-greedy it suffices, by Theorem 0.2, to verify Property (A). To that end, consider $x = y + \sum_{i\in B} \pm e_i$, for some choice of signs, where $\|y\|_{\ell_\infty} < 1$, $B \cap \operatorname{supp}(y) = \emptyset$ and $|\operatorname{supp}(y)| < \infty$. Suppose that $|\langle x, f\rangle| = |||x|||$, where f is given by (7) for an appropriate choice of signs. We may assume without loss of generality that $\langle x, f\rangle > 0$, which implies that $\langle x, e^*_{m_i}\rangle \geq 0$ for all $i > n$. We shall show that f may be chosen so that $B \subset E$. This will immediately imply Property (A), since if $\bar{x}$ is a greedy rearrangement of x then the same greedy rearrangement applied to f yields $\bar{f} \in \mathcal{F}$ which norms $\bar{x}$. Indeed, fixing $N \in \mathbb{N}$ with $N > \operatorname{supp}(x)$ and $N > \operatorname{supp}(\bar{x})$, we can assume, after changing f if necessary, that

$$\{m_i : i > n\} \cap \{1, \ldots, N\} = \{m_i : i > n\} \cap \operatorname{supp}(x) ,$$

so the change from f to $\bar{f}$ is permissible.

Assume now that $B \not\subset E$, and fix $r \in B \setminus E$. If $r = m_i$ for some $i > n$, then interchanging m_i and m_{n+1} does not decrease $\langle x, f\rangle$. So we may assume that either $r \notin \{m_i : i > n\}$ or $r = m_{n+1}$. In either case we set $E_1 = E \cup \{r\}$ and consider the linear functional $f_1 \in \mathcal{F}$ defined (for an appropriate choice of signs) by

$$f_1 = \frac{1}{\sqrt{n+1}} \sum_{i\in E_1} \pm e^*_i + \sum_{i=n+2}^{\infty} w_i e^*_{m_i} .$$

(In fact, the signs do not change for $i \in E$, and we use a plus sign for $i = r$.) Putting $\alpha = \sum_{i\in E} |\langle x, e^*_i\rangle|$, we note that $\alpha \leq n$ and obtain

$$\begin{aligned}\langle x, f_1\rangle - \langle x, f\rangle &\geq \frac{\alpha+1}{\sqrt{n+1}} - \frac{\alpha}{\sqrt{n}} - w_{n+1} \\ &= \frac{1}{\sqrt{n+1}} - \Big(\frac{\alpha}{\sqrt{n}\sqrt{n+1}} + 1\Big) \cdot w_{n+1} \geq 0 .\end{aligned}$$

So $\langle x, f_1\rangle = |||x|||$ and $|B \setminus E_1| = |B \setminus E| - 1$. Iterating this argument a total of $|B \setminus E|$ times shows that without loss of generality we may assume that $B \subseteq E$. Thus, as explained above, (e_i) has Property (A) and hence is 1-greedy.

We now show that (e_i) is not $(2-\varepsilon)$-unconditional. To that end, let $(n_i)_{i=0}^{\infty}$ be a rapidly increasing sequence of integers with $n_0 = 0$. Consider the sequence $(x_i)_{i=1}^{\infty}$ defined by

$$x_i = \frac{1}{\sqrt{n_i - n_{i-1}}} \sum_{j=n_{i-1}+1}^{n_i} e_j .$$

Then $|||x_i||| = \|x_i\|_{\ell_{2,1}} = 1$, and, provided (n_i) increases sufficiently rapidly, we obtain

$$\Big|\Big|\Big| \sum_{i=1}^{N} x_i \Big|\Big|\Big| > N - 1 \qquad (N \geq 1)$$

by applying the functional $f = \sum_{i=1}^{\infty} w_i e_i^*$. On the other hand, we also have

$$\Big|\Big|\Big| \sum_{i=1}^{N} (-1)^i x_i \Big|\Big|\Big| \leq \frac{N+4}{2} \qquad (N \geq 1) \ .$$

To see this, let $f \in \mathcal{F}$ be given by (7) and write $f = g + h$, where $g = \frac{1}{\sqrt{n}} \sum_{i \in E} \pm e_i^*$ and $h = \sum_{i=n+1}^{\infty} w_i e_{m_i}^*$. For each $i = 1, \ldots, N$ we have

$$\begin{aligned} |\langle x_i, g \rangle| &\leq \frac{\min\{n, n_i\}}{\sqrt{n}\sqrt{n_i - n_{i-1}}} \qquad \text{and} \\ 0 \leq \quad \langle x_i, h \rangle &\leq \|x_i\|_{\ell_{2,1}} = 1 \ , \end{aligned}$$

and hence

$$\Big| \Big\langle \sum_{i=1}^{N} (-1)^i x_i, g \Big\rangle \Big| \leq \frac{3}{2} \qquad \text{and}$$

$$\Big| \Big\langle \sum_{i=1}^{N} (-1)^i x_i, h \Big\rangle \Big| \leq \frac{N+1}{2} \ ,$$

where for the first inequality we need (n_i) to increase sufficiently rapidly.

Thus, we have obtained

$$\lim_{N \to \infty} \frac{\big|\big|\big| \sum_{i=1}^{N} x_i \big|\big|\big|}{\big|\big|\big| \sum_{i=1}^{N} (-1)^i x_i \big|\big|\big|} = 2 \ ,$$

which implies that (e_i) is not $(2-\varepsilon)$-unconditional for any $\varepsilon > 0$. □

The following more general result is proved in similar fashion.

Theorem 4.2. *Let $1 \leq q < p < \infty$. Then there is a renorming of Lorentz space $\ell_{p,q}$ for which the unit vector basis is 1-greedy but not 1-unconditional.*

5. A 1-greedy basis need not be symmetric

In this section we answer the most important problem raised in [1], Problem 6.1, which asks whether there exists a 1-greedy basis that is not symmetric in an infinite-dimensional Banach space. We give a positive answer by constructing an example based on the space $\ell_{2,1}$ used in the previous section. This will prove Theorem A.

As before, we consider the weight sequence $w_i = \sqrt{i} - \sqrt{i-1},\ i = 1, 2, \ldots$. This time $\mathcal{F}$ will denote the collection of all formal sums f of the form

$$f = \frac{1}{\sqrt{n}} \sum_{i \in E} \pm e_i^* + \frac{1}{2} \sum_{i=n+1}^{\infty} \pm w_i e_{m_i}^* \ , \tag{8}$$

for any choice of signs, where $n \geq 1$, $|E| = n$, $m_1 < m_2 < \dots$ are positive integers, and $\{m_i \colon i > n\} \cap E = \emptyset$. We then define a norm $\|\cdot\|$ on c_{00} by setting

$$\|x\| = \sup_{f \in \mathcal{F}} \langle x, f \rangle \qquad \text{for } x \in c_{00} \ .$$

It is straightforward that (e_i) is a normalized 1-unconditional basis for the completion of $(c_{00}, \|\cdot\|)$.

Theorem 5.1. *With respect to $\|\cdot\|$, the unit vector basis (e_i) is 1-greedy and not symmetric.*

Proof. An argument similar to the first part of the proof of Theorem 4.1 shows that (e_i) satisfies Property (A), and hence it is 1-greedy. Indeed, let $x = y + \sum_{i \in B} \pm e_i$ for some choice of signs, where $\|y\|_{\ell_\infty} < 1$, $B \cap \operatorname{supp}(y) = \emptyset$ and $|\operatorname{supp}(y)| < \infty$. Suppose that $\langle x, f \rangle = \|x\|$, where f is given by (8) for an appropriate choice of signs. As in the proof of Theorem 4.1, it suffices to show that f may be chosen so that $B \subset E$. Property (A) will then follow immediately.

Assume that in fact $B \not\subset E$, and fix $r \in B \backslash E$. If $r = m_j$ for some $j > n$, then set $m_i^1 = m_i$ for $i < j$ and $m_i^1 = m_{i+1}$ for $i \geq j$. Otherwise, if $r \notin \{m_i : i > n\}$, then we set $m_i^1 = m_i$ for all $i \in \mathbb{N}$. In either case we set $E_1 = E \cup \{r\}$, $\alpha = \sum_{i \in E} |\langle x, e_i^* \rangle|$, and consider the linear functional $f_1 \in \mathcal{F}$ defined (for an appropriate choice of signs) by

$$f_1 = \frac{1}{\sqrt{n+1}} \sum_{i \in E_1} \pm e_i^* + \frac{1}{2} \sum_{i=n+2}^{\infty} \pm w_i e_{m_i}^* \ .$$

Noting that $\alpha \leq n$, we obtain

$$\langle x, f_1 \rangle - \langle x, f \rangle \geq \frac{\alpha + 1}{\sqrt{n+1}} - \frac{\alpha}{\sqrt{n}} - \frac{1}{2} w_{n+1} - \frac{1}{2} w_{n+2} \geq 0 \ .$$

As before, this completes the proof that (e_i) has Property (A) and is 1-greedy. It remains to show that (e_i) is not symmetric.

Fix $k \in \mathbb{N}$ and choose a rapidly increasing sequence $n_1 < n_2 < \dots < n_k$ of positive integers. Let $E_1 < E_2 < \dots < E_k$ and $F_1 < F_2 < \dots < F_k$ be finite subsets of $\mathbb{N}$ with $|E_i| = |F_{k+1-i}| = n_i$ such that

$$\bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} F_i = \left\{1, 2, \dots, \textstyle\sum_{i=1}^{k} n_i\right\} \ .$$

Set

$$x = \sum_{i=1}^{k} \tfrac{1}{\sqrt{n_i}} \mathbf{1}_{E_i} \qquad \text{and} \qquad y = \sum_{i=1}^{k} \tfrac{1}{\sqrt{n_{k+1-i}}} \mathbf{1}_{F_i} \ .$$

We show that $\|x\| \geq \frac{k}{3}$ and $\|y\| \leq 3$. Since y is a rearrangement of x, and since $k \in \mathbb{N}$ is arbitrary, it will follow that (e_i) is not symmetric.

Since $e_1^* + \frac{1}{2}\sum_{i=2}^\infty w_i e_i^* \in \mathcal{F}$, we have

$$\|x\| \geq \frac{1}{2}\sum_{i=1}^{k} \frac{\sqrt{n_1+\cdots+n_i} - \sqrt{n_1+\cdots+n_{i-1}}}{\sqrt{n_i}} \geq \frac{k}{3}$$

provided $n_1, \dots, n_k$ increases sufficiently fast.

Next, assume that $\|y\| = \langle y, f\rangle$ where f is given by (8). Write $f = g + \frac{1}{2}h$ where $g = \frac{1}{\sqrt{n}}\sum_{i\in E} e_i^*$ and $h = \sum_{i=n+1}^\infty w_i e_{m_i}^*$. Since

$$\left\langle \tfrac{1}{\sqrt{n_i}}\mathbf{1}_{F_{k+1-i}}, g\right\rangle \leq \frac{\min\{n, n_i\}}{\sqrt{n_i \cdot n}} = \min\left\{\sqrt{\tfrac{n}{n_i}}, \sqrt{\tfrac{n_i}{n}}\right\} ,$$

it follows that $|\langle y, g\rangle| \leq 3/2$, provided the n_i increase sufficiently fast.

For each $j = 1, \dots, k$ let $p_j = |F_j \cap \{m_i : i > n\}|$. Observe that

$$\left\langle \tfrac{1}{\sqrt{n_{k+1-j}}}\mathbf{1}_{F_j}, h\right\rangle \leq \frac{\sqrt{p_j}}{\sqrt{n_{k+1-j}}} .$$

It follows that if $\frac{p_j}{n_{k+1-j}} < 4^{-j}$ for all j, then $|\langle y, h\rangle| \leq 1$. Otherwise there is a least value of j, which we denote by j_0, such that $\frac{p_j}{n_{k+1-j}} \geq 4^{-j}$. Set $i_0 = \lfloor 4^{-j_0} n_{k+1-j_0}\rfloor + 1$. Then

$$\left\langle \tfrac{1}{\sqrt{n_{k+1-j}}}\mathbf{1}_{F_j}, h\right\rangle \leq \begin{cases} 2^{-j} & \text{if } j < j_0 \\ 1 & \text{if } j = j_0 \\ w_{i_0}\sqrt{n_{k+1-j}} & \text{if } j > j_0 . \end{cases}$$

Thus, assuming the n_i were suitably chosen, we obtain $|\langle y, h\rangle| \leq 3$.

The above estimates yield $\|y\| = \langle y, g\rangle + \frac{1}{2}\langle y, h\rangle \leq 3$, as claimed. □

6. A 1-greedy basis need not be subsymmetric

In this final section we construct a normalized 1-greedy basis that is not subsymmetric. This example is more involved than the previous ones, and so we divided the section into four subsections. We first fix our notation repeating some of the earlier definitions. Next we describe a general procedure for constructing 1-greedy bases starting with a given norm on c_{00}. We then apply this procedure to norms on $c_{00} \oplus c_{00}$ that are 1-symmetric in each co-ordinate. In the final subsection we specialize to the norm $\|\cdot\|_{\ell_2} + \|\cdot\|_{\ell_{2,1}}$ and prove that the resulting 1-greedy basis is not subsymmetric.

6.1. Notation.

As usual, (e_i) denotes the unit vector basis of c_{00} with biorthogonal functionals (e_i^*). Let $x, y \in c_{00}$. The following is a list of notation that will be used in this section.

- $M(x) = \big\{k \in \mathbb{N} : |\langle x, e_k^* \rangle| = \|x\|_{\ell_\infty}\big\}$.
- For $A \subset \mathbb{N}$ write $Ax = \sum\limits_{k \in A} \langle x, e_k^* \rangle e_k$.
- Define $|x|$ by $\langle |x|, e_k^* \rangle = |\langle x, e_k^* \rangle|,\ k \in \mathbb{N}$.
- Define $x \cdot y$ by $\langle x \cdot y, e_k^* \rangle = \langle x, e_k^* \rangle \langle y, e_k^* \rangle,\ k \in \mathbb{N}$.
- Write $x \leq y$ if $\langle x, e_k^* \rangle \leq \langle y, e_k^* \rangle$ for all $k \in \mathbb{N}$.
- Given $t \in \mathbb{R}$, write $x \leq t$ to mean $\langle x, e_k^* \rangle \leq t$ for all $k \in \mathbb{N}$.
- Write $x \overset{\text{dist}}{\equiv} y$ if there exists a permutation $\pi \colon \mathbb{N} \to \mathbb{N}$ such that $\langle y, e_k^* \rangle = \langle x, e_{\pi(k)}^* \rangle$ for all $k \in \mathbb{N}$.
- Write $x \sim y$ if y is a greedy rearrangement of x, *i.e.*, if we can write $x = z + \lambda\varepsilon \cdot \mathbf{1}_A$ and $y = z + \lambda\eta \cdot \mathbf{1}_B$, where $\varepsilon \colon A \to \{\pm 1\},\ \eta \colon B \to \{\pm 1\}$ are functions on finite sets A, B, and $A \cap \operatorname{supp}(z) = B \cap \operatorname{supp}(z) = \emptyset,\ |A| = |B|$ and $\|z\|_{\ell_\infty} \leq \lambda$.

Thus $M(x)$ denotes the set of co-ordinates of x of maximum modulus. Next, a subset A of $\mathbb{N}$ is identified with the projection onto it. The following four pieces of notation define the standard lattice structure of c_{00}. Note that for $A \subset \mathbb{N}$ and $x \in \mathrm{c}_{00}$ we have $Ax = \mathbf{1}_A \cdot x$, where $\mathbf{1}_A = \sum_{k \in A} e_k$ denotes, as usual, the indicator function of A. In the last line we repeated the definition of greedy rearrangement in a slightly different form.

6.2. Greedification

Given a seminorm $\|\cdot\|$ on c_{00}, we define $f \colon \mathrm{c}_{00} \to \mathbb{R}$ by

$$f(x) = \inf\big\{\|y\| : y \sim x\big\}\ ,$$

and we then define $|||\cdot|||$ on c_{00} by

$$\begin{aligned} |||x||| &= \inf\Big\{\sum_{i=1}^m \|x_i'\| : m \geq 0,\ x = \sum_{i=1}^m x_i,\ x_i \sim x_i'\Big\} \\ &= \inf\Big\{\sum_{i=1}^m f(x_i) : m \geq 0,\ x = \sum_{i=1}^m x_i\Big\}\ . \end{aligned}$$

Note that $|||\cdot|||$ is a seminorm on c_{00} dominated by $\|\cdot\|$. We shall write $\mathcal{G}\big(\|\cdot\|\big)$ for $|||\cdot|||$.

Starting with a seminorm $\|\cdot\|$ on c_{00}, we define a sequence of seminorms $\|\cdot\|_{(n)}$ recursively as follows. We set $\|\cdot\|_{(0)} = \|\cdot\|$, and for $n \in \mathbb{N}$ we define $\|\cdot\|_{(n)} = \mathcal{G}\big(\|\cdot\|_{(n-1)}\big)$. We then let $\|\cdot\|_{(\infty)}$ be the pointwise limit of the sequence

$\big(\|\cdot\|_{(n)}\big)$. Observe that $\|\cdot\|_{(\infty)}$ is a seminorm with respect to which the unit vector basis satisfies Property (A): if $x\sim y$ then $\|x\|_{(\infty)} = \|y\|_{(\infty)}$.

The notation above will be fixed for the rest of this subsection. We write $K_S = K_S^{\|\cdot\|}$, $K_U = K_U^{\|\cdot\|}$ and $\Delta = \Delta^{\|\cdot\|}$ for the the suppression-unconditional, unconditional and democratic constants, respectively, of the unit vector basis (e_i) with respect to $\|\cdot\|$, which take values in $\mathbb{R}^+ \cup \{\infty\}$ in general. These constants as well as Property (A) were defined in the Introduction with respect to a norm, but they clearly make sense for seminorms, too.

Routine verification gives the following result.

Proposition 6.1. *(i) Assume that $\|\cdot\|$ dominates another seminorm $\|\cdot\|'$ on c_{00}. Then $\mathcal{G}(\|\cdot\|)$ dominates $\mathcal{G}(\|\cdot\|')$.*

(ii) If (e_i) has Property (A) with respect to $\|\cdot\|$, then $|||\cdot||| = \|\cdot\|$. It follows that $\|\cdot\|_{(\infty)} = \|\cdot\|$.

(iii) For $x \in c_{00}$ and $A \subset \mathbb{N}$ we have $f(Ax) \leq K_S f(x)$. It follows that $K_S^{|||\cdot|||} \leq K_S^{\|\cdot\|}$.

(iv) For $x \in c_{00}$ and $\varepsilon\colon \mathbb{N} \to \{\pm 1\}$ we have $f(\varepsilon \cdot x) \leq K_U f(x)$. It follows that $K_U^{|||\cdot|||} \leq K_U^{\|\cdot\|}$.

(v) For $x \in c_{00}$ we have $\|x\| \geq f(x) \geq \frac{1}{K_S(1+4\Delta)}\|x\|$. It follows that $\|x\| \geq |||x||| \geq \frac{1}{K_S(1+4\Delta)}\|x\|$.

(vi) If $K_S^{\|\cdot\|} = 1$, then

$$|||x||| = \inf\Big\{\sum_{i=1}^m f(x_i) : m \geq 0,\ x = \sum_{i=1}^m x_i \text{ and } \operatorname{supp}(x_i) \subset \operatorname{supp}(x)\ \forall\, i\Big\}\ .$$

The point about the "greedification" procedure is to produce 1-greedy bases. This is an easy consequence of properties (i), (ii) and (iii) above, the characterization of 1-greedy bases (Theorem 0.2) and the observation above that (e_i) always has Property (A) with respect to $\|\cdot\|_{(\infty)}$.

Corollary 6.2. *Assume that $K_S^{\|\cdot\|} = 1$ and $\|e_i\| = 1$ for all $i \in \mathbb{N}$. Then $\|\cdot\|_{(\infty)}$ is a norm on c_{00} that dominates the c_0-norm, and moreover (e_i) is a normalized 1-greedy basis with respect to $\|\cdot\|_{(\infty)}$.*

Remark. Recall from the Introduction the (still open) problem raised by Albiac and Wojtaszczyk [1, Problem 6.2] which asks if there is an equivalent renorming ϕ, say, of $L_p[0,1]$ $(1 < p < \infty)$ with respect to which the Haar basis (h_i) is (normalized and) 1-greedy. Assume that such an equivalent norm ϕ exists. Denote by $\|\cdot\|_p$ the L_p-norm, and consider the equivalent norm

$$\Big\|\sum a_i h_i\Big\| = \sup_{A\subset\mathbb{N}} \Big\|\sum_{i\in A} a_i h_i\Big\|_p\ ,$$

with respect to which (h_i) is normalized and suppression 1-unconditional. The above corollary and Proposition 6.1(i) and (ii) then tell us that (h_i) is normalized and 1-greedy with respect to the equivalent norm $\|\cdot\|_{(\infty)}$. In other words, if there is a positive answer to [1, Problem 6.2], then the greedification procedure produces that positive answer.

We now continue with the general discussion of greedification. We say that (e_i) is *strictly unconditional* with respect to $\|\cdot\|$ if the following holds:

$$\forall\ x, y \in \mathrm{c}_{00} \quad \big(|x| \leq |y| \text{ and } |x| \neq |y|\big) \Longrightarrow \|x\| < \|y\| \ .$$

Note that this implies $K_U^{\|\cdot\|} = 1$. The next result shows that greedification preserves strict unconditionality under certain conditions.

Proposition 6.3. *Assume that in the definition of f and $|||\cdot|||$ the infimum is always attained. If, in addition, (e_i) is strictly unconditional with respect to $\|\cdot\|$, then the same holds with respect to $|||\cdot|||$.*

Proof. Assume that $|x| = A|y|$ for some $A \not\supset \operatorname{supp}(y)$. We show that $|||x||| < |||y|||$, and the result then follows by convexity.

Write $x = \varepsilon \cdot Ay$ for suitable $\varepsilon \colon \mathbb{N} \to \{\pm 1\}$. By assumption we have $|||y||| = \sum \|y'_i\|$ for some $y'_i \sim y_i$, $\sum y_i = y$.

Let $x_i = \varepsilon \cdot Ay_i$ so that $x = \sum x_i$. It is easy to see, as explained below, that

$$\begin{gathered} \forall\, i \quad \exists\, \varepsilon_i \colon \mathbb{N} \to \{\pm 1\} \quad \exists\, A_i \subset \mathbb{N} \quad \text{such that} \\ x'_i = \varepsilon_i \cdot A_i y'_i \sim x_i \text{ and } \big(A_i \supset \operatorname{supp}(y'_i) \iff A \supset \operatorname{supp}(y_i)\big) \ . \end{gathered} \tag{9}$$

Since $K_U^{\|\cdot\|} = 1$, it follows that $\|x'_i\| \leq \|y'_i\|$ for all i. Moreover, since $A \not\supset \operatorname{supp}(y)$, there exists j such that $A \not\supset \operatorname{supp}(y_j)$, and hence $A_j \not\supset \operatorname{supp}(y'_j)$ and $\|x'_j\| < \|y'_j\|$. Thus

$$|||x||| \leq \sum \|x'_i\| < \sum \|y'_i\| = |||y||| \ .$$

This completes the proof.

To see (9) fix i, and write $y_i = z + \lambda\eta \cdot \mathbf{1}_B$ and $y'_i = z + \lambda\eta' \cdot \mathbf{1}_{B'}$, where $\eta \colon B \to \{\pm 1\}$ and $\eta' \colon B' \to \{\pm 1\}$ are functions on finite sets, $|B| = |B'|$, $B \cap \operatorname{supp}(z) = B' \cap \operatorname{supp}(z) = \emptyset$ and $\lambda \geq \|z\|_{\ell_\infty}$. Note that

$$x_i = \varepsilon \cdot Az + \lambda\varepsilon \cdot \eta \cdot \mathbf{1}_{A \cap B} \ .$$

Fix $B'' \subset B'$ such that $|B''| = |A \cap B|$. Set $A' = \big(A \cap \operatorname{supp}(z)\big) \cup B''$ and $x'_i = \varepsilon \cdot A' y'_i$. Then

$$x'_i = \varepsilon \cdot Az + \lambda\varepsilon \cdot \eta' \cdot \mathbf{1}_{B''} \sim x_i \ .$$

We also have $|\operatorname{supp}(y_i)| = |\operatorname{supp}(y'_i)|$ and $|A'| = |A' \cap \operatorname{supp}(y'_i)| = |A \cap \operatorname{supp}(y_i)|$. Hence $A \supset \operatorname{supp}(y_i)$ if and only if $A' \supset \operatorname{supp}(y'_i)$. Thus $\varepsilon_i = \varepsilon$ and $A_i = A'$ will do. □

As a corollary we obtain

Proposition 6.4. *Assume that in the definition of f and $|||\cdot|||$ the infimum is always attained and that (e_i) is strictly unconditional with respect to $\|\cdot\|$. Let $x \in c_{00}$ and assume that $|||x||| = \sum f(x_i)$, where $x = \sum x_i$. If in addition $x \geq 0$, then $x_i \geq 0$ for all i.*

Proof. As before, $K_U^{\|\cdot\|} = 1$ by strict unconditionality, and so $f(z) = f(|z|)$ for all $z \in c_{00}$. If x_j has negative coefficients for some j, then $\left|\sum x_i\right| \neq \sum |x_i|$ and, by the previous proposition, we have

$$|||x||| = \sum f(x_i) = \sum f(|x_i|) \geq \left|\left|\left| \sum |x_i| \right|\right|\right| > \left|\left|\left| \sum x_i \right|\right|\right| ,$$

which is a contradiction. □

6.3. Norms on $c_{00} \oplus c_{00}$.

The definitions and results of Subsections 6.1 and 6.2 extend to $c_{00}(S)$ for any countable set S. In particular, we can take $S = \mathbb{N} \sqcup \mathbb{N}$ (where $\sqcup$ denotes disjoint union) in which case $c_{00}(S)$ is identified with $c_{00} \oplus c_{00}$ in the obvious way. The unit vector basis of $c_{00}(S)$ is $(e_s)_{s \in S}$, where $e_s \colon S \to \mathbb{R}$ is the indicator function of $\{s\}$. This basis does not necessarily come with a natural ordering but as we only consider unconditional norms, the order is irrelevant.

Throughout this subsection we work with a fixed norm $\|\cdot\|$ on $c_{00} \oplus c_{00}$ which satisfies the following symmetry property: given $x_1, x_2, y_1, y_2 \in c_{00}$, if $|x_1| \overset{\text{dist}}{\equiv} |x_2|$ and $|y_1| \overset{\text{dist}}{\equiv} |y_2|$, then $\|(x_1, y_1)\| = \|(x_2, y_2)\|$. The resulting function f and seminorm $|||\cdot|||$ are defined as in Subsection 6.2.

The symmetry assumption on the norm implies that the infimum in the definition of f is always attained. More precisely, given $x, y \in c_{00}$

either $\exists\, E \subset M(x)\ \exists\, E' \subset \mathbb{N}$ such that $|E| = |E'|$, $E' \cap \operatorname{supp}(y) = \emptyset$ and

$$f(x, y) = \|(x - Ex, y + \lambda \mathbf{1}_{E'})\|$$

where $\lambda = \|x\|_{\ell_\infty} = \|x\|_{\ell_\infty} \vee \|y\|_{\ell_\infty}$,

or $\exists\, F \subset M(y)\ \exists\, F' \subset \mathbb{N}$ such that $|F| = |F'|$, $F' \cap \operatorname{supp}(x) = \emptyset$ and

$$f(x, y) = \|(x + \lambda \mathbf{1}_{F'}, y - Fy)\|$$

where $\lambda = \|y\|_{\ell_\infty} = \|x\|_{\ell_\infty} \vee \|y\|_{\ell_\infty}$.

We say that E (respectively, F) is *a set of moving coordinates of* (x, y) *on the left-hand side* (respectively, *right-hand side*).

The following is an easy consequence of the above observation.

Proposition 6.5. *Let $x_1, x_2, y_1, y_2 \in c_{00}$. Assume that $|x_1| \overset{\text{dist}}{\equiv} |x_2|$ and $|y_1| \overset{\text{dist}}{\equiv} |y_2|$. Then $f(x_1, y_1) = f(x_2, y_2)$ and $|||(x_1, y_1)||| = |||(x_2, y_2)|||$.*

The main result of this subsection is

Theorem 6.6. *The infimum in the definition of* $|||\cdot|||$ *is always attained.*

The precise statement of the theorem is as follows: given $x, y \in c_{00}$, there is a decomposition $(x,y) = \sum (x_i, y_i)$ such that $|||(x,y)||| = \sum f(x_i, y_i)$; we will call such a decomposition *a norm-attaining decomposition of* (x,y). The proof takes a number of steps.

Lemma 6.7. *Assume that* (x_1, y_1) *and* (x_2, y_2) *have an identical set* E *of moving coordinates on the left-hand side, and that there exists* $\varepsilon \colon E \to \{\pm 1\}$ *with* $\varepsilon \cdot Ex_i = |Ex_i|$ *for* $i = 1, 2$. *Then*

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2) \ .$$

The same holds if E *is a moving set of coordinates on the right-hand side and* $\varepsilon \cdot Ey_i = |Ey_i|$ *for* $i = 1, 2$.

Proof. Fix $E' \subset \mathbb{N}$ with $|E| = |E'|$ and $E' \cap \big(\operatorname{supp}(y_1) \cup \operatorname{supp}(y_2)\big) = \emptyset$. Then

$$f(x_i, y_i) = \|(x_i - Ex_i, y_i + \lambda_i \mathbf{1}_{E'})\|$$

where $\lambda_i = \|x_i\|_{\ell_\infty} = \|x_i\|_{\ell_\infty} \vee \|y_i\|_{\ell_\infty}$ $(i = 1, 2)$. Note that $\varepsilon \cdot Ex_i = |Ex_i| = \lambda_i \mathbf{1}_E$. It follows that $\varepsilon \cdot E(x_1 + x_2) = (\lambda_1 + \lambda_2)\mathbf{1}_E$ and $\lambda_1 + \lambda_2 = \|(x_1 + x_2, y_1 + y_2)\|_{\ell_\infty}$. Hence

$$\big(x_1 + x_2 - E(x_1 + x_2), y_1 + y_2 + (\lambda_1 + \lambda_2)\mathbf{1}_{E'}\big) \sim (x_1 + x_2, y_1 + y_2)$$

and

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &\leq \big\|\big(x_1 + x_2 - E(x_1 + x_2), y_1 + y_2 + (\lambda_1 + \lambda_2)\mathbf{1}_{E'}\big)\big\| \\ &\leq \|(x_1 - Ex_1, y_1 + \lambda_1 \mathbf{1}_{E'})\| + \|(x_2 - Ex_2, y_2 + \lambda_2 \mathbf{1}_{E'})\| \\ &= f(x_1, y_1) + f(x_2, y_2) \ . \end{aligned}$$

□

Corollary 6.8. *Given* $x, y \in c_{00}$, *there exists* $N \in \mathbb{N}$ *(depending only on* $|\operatorname{supp}(x)|$ *and* $|\operatorname{supp}(y)|$*) such that*

$$\begin{aligned} |||(x,y)||| = \inf \Big\{ \sum_{i=1}^{N} f(x_i, y_i) :\ (x,y) = \sum_{i=1}^{N} (x_i, y_i), \\ \operatorname{supp}(x_i) \subset \operatorname{supp}(x) \ \textit{and} \ \operatorname{supp}(y_i) \subset \operatorname{supp}(y) \ \forall\, i \Big\} \ . \end{aligned}$$

Proof. Assume that $(x,y) = \sum_{i \in I} (x_i, y_i)$ for some index set I, where $\operatorname{supp}(x_i) \subset \operatorname{supp}(x)$ and $\operatorname{supp}(y_i) \subset \operatorname{supp}(y)$ for all i. Write I as a disjoint union of sets $L_{E,\varepsilon}$, $E \subset \operatorname{supp}(x)$, $\varepsilon \colon E \to \{\pm 1\}$, and $R_{F,\varepsilon}$, $F \subset \operatorname{supp}(y)$, $\varepsilon \colon F \to \{\pm 1\}$, so that if $i \in L_{E,\varepsilon}$ then E is a set of moving coordinates of (x_i, y_i) on the left-hand side and $\varepsilon \cdot Ex_i = |Ex_i|$, and if $i \in R_{F,\varepsilon}$ then F is a set of moving coordinates of

(x_i, y_i) on the right-hand side and $\varepsilon \cdot Fy_i = |Fy_i|$. Then by the previous lemma we have

$$\sum_{\substack{E\subset \mathrm{supp}(x)\\ \varepsilon\colon E\to\{\pm 1\}}} f\Big(\sum_{i\in L_{E,\varepsilon}} (x_i, y_i)\Big) + \sum_{\substack{F\subset \mathrm{supp}(y)\\ \varepsilon\colon F\to\{\pm 1\}}} f\Big(\sum_{i\in R_{F,\varepsilon}} (x_i, y_i)\Big) \leq \sum_{i\in I} f(x_i, y_i)$$

from which the result follows by Proposition 6.1(vi). □

Proof of Theorem 6.6. Let $x, y \in \mathrm{c}_{00}$ and let $N \in \mathbb{N}$ be given by Corollary 6.8. Choose $x_i^{(n)}, y_i^{(n)}$ for $1 \leq i \leq N$ and $n \in \mathbb{N}$ such that $(x, y) = \sum_{i=1}^N \big(x_i^{(n)}, y_i^{(n)}\big)$ for all $n \in \mathbb{N}$, $\mathrm{supp}\,\big(x_i^{(n)}\big) \subset \mathrm{supp}(x)$, $\mathrm{supp}\,\big(y_i^{(n)}\big) \subset \mathrm{supp}(y)$ for all i, n, and

$$\sum_{i=1}^N f\big(x_i^{(n)}, y_i^{(n)}\big) \to |||(x,y)||| \ .$$

After passing to subsequences, we may assume that $x_i^{(n)} \to x_i$ and $y_i^{(n)} \to y_i$ as $n \to \infty$ for each i. This uses the fact that $\|\cdot\|$ dominates some multiple of the ℓ_∞-norm on $\mathrm{c}_{00} \oplus \mathrm{c}_{00}$.

Fix i. After passing to further subsequences, we may assume that $\big(x_i^{(n)}, y_i^{(n)}\big)$ has a set E of moving coordinates on, say, the left-hand side for all $n \in \mathbb{N}$. Fix $E' \subset \mathbb{N}$ with $|E'| = |E|$ and $E' \cap \mathrm{supp}(y) = \emptyset$. Then

$$f\big(x_i^{(n)}, y_i^{(n)}\big) = \big\|\big(x_i^{(n)} - Ex_i^{(n)}, y_i^{(n)} + \lambda_i^{(n)} \mathbf{1}_{E'}\big)\big\|$$

where $\lambda_i^{(n)} = \big\|\big(x_i^{(n)}, y_i^{(n)}\big)\big\|_{\ell_\infty}$.

Now $\|(x_i, y_i)\|_{\ell_\infty} = \lim_n \lambda_i^{(n)} = \lambda_i$, say, and $|Ex_i^{(n)}| = \lambda_i^{(n)} \mathbf{1}_E \to \lambda_i \mathbf{1}_E$. It follows that

$$f(x_i, y_i) \leq \|(x_i - Ex_i, y_i + \lambda_i\, \mathbf{1}_{E'})\| = \lim_n f\big(x_i^{(n)}, y_i^{(n)}\big) \ .$$

Since i was arbitrary, we obtain

$$|||(x,y)||| = \lim_n \sum_{i=1}^N f\big(x_i^{(n)}, y_i^{(n)}\big) \geq \sum_{i=1}^N f(x_i, y_i) \geq |||(x,y)||| \ ,$$

and this completes the proof. □

6.4. A 1-greedy non-subsymmetric basis

We let $\|\cdot\|$ denote the norm on $\mathrm{c}_{00} \oplus \mathrm{c}_{00}$ defined by $\|(x,y)\| = \|x\|_{\ell_2} + \|y\|_{\ell_{2,1}}$ for $x, y \in \mathrm{c}_{00}$. The definition of the Lorentz space $\ell_{2,1}$ was recalled in Section 4. We next define f and $|||\cdot|||$ as in Subsection 6.2. As explained at the beginning of Subsection 6.3, these definitions make sense for norms on $\mathrm{c}_{00}(S)$ for any countable set S including $S = \mathbb{N} \sqcup \mathbb{N}$, in which case $\mathrm{c}_{00}(S)$ is identified with $\mathrm{c}_{00} \oplus \mathrm{c}_{00}$ in the obvious way.

The following establishes Theorem B, which is the main result of this section.

Theorem 6.9. *$|||\cdot|||$ is a norm on $c_{00}\oplus c_{00}$ equivalent to $\|\cdot\|_{\ell_2}+\|\cdot\|_{\ell_{2,1}}$. Moreover, the unit vector basis of $c_{00}\oplus c_{00}$ is normalized, 1-greedy but not subsymmetric with respect to $|||\cdot|||$.*

Remark. What this theorem tells us is that the greedification process applied to $\|\cdot\|$ terminates after just one iteration, *i.e.,* $\|\cdot\|_{(\infty)} = |||\cdot|||$.

The main step in the proof will be showing the existence of norm-attaining decompositions, as defined after Theorem 6.6, of a particularly pleasant form. To begin with, we collect in a proposition what we already know about f and $|||\cdot|||$. First we note the following properties of $\|\cdot\|$:

- symmetry property: if $|x_1| \overset{\text{dist}}{\equiv} |x_2|$ and $|y_1| \overset{\text{dist}}{\equiv} |y_2|$, then $\|(x_1,y_1)\| = \|(x_2,y_2)\|$;
- the unit vector basis of $c_{00}\oplus c_{00}$ is normalized and 1-unconditional, and so $K_S^{\|\cdot\|} = K_U^{\|\cdot\|} = 1$;
- "strict unconditionality": if $|x|\le|y|$ and $|x|\neq|y|$, then $\|x\|<\|y\|$;
- ℓ_2-domination: $\|(x,y)\| \ge \|(x,y)\|_{\ell_2} = \left(\|x\|_{\ell_2}^2+\|y\|_{\ell_2}^2\right)^{1/2}$; in particular, if $\operatorname{supp}(x)\cap\operatorname{supp}(y)=\emptyset$, then $\|(x,y)\|\ge\|x+y\|_{\ell_2}$;
- For $A\subset\mathbb{N}$ we have $\|\mathbf{1}_A\|_{\ell_{2,1}} = \sqrt{|A|}$; it follows that the unit vector basis is democratic with $\Delta^{\|\cdot\|}=\sqrt{2}$.

Using Propositions 6.1, 6.4, 6.5 and Theorem 6.6 we obtain

Proposition 6.10. *(Properties of $|||\cdot|||$.)*

(i) *Symmetry property: if $|x_1| \overset{\text{dist}}{\equiv} |x_2|$ and $|y_1| \overset{\text{dist}}{\equiv} |y_2|$, then $f(x_1,y_1) = f(x_2,y_2)$ and $|||(x_1,y_1)||| = |||(x_2,y_2)|||$.*

(ii) *For all $x,y\in c_{00}$ there exists a norm-attaining decomposition $(x,y)=\sum(x_i,y_i)$ with $\operatorname{supp}(x_i)\subset\operatorname{supp}(x)$ and $\operatorname{supp}(y_i)\subset\operatorname{supp}(y)$ for all i. Moreover, if $x\ge0,\ y\ge0$, then $x_i\ge0,\ y_i\ge0$ for all i.*

(iii) *ℓ_2-domination: $|||(x,y)|||\ge\|(x,y)\|_{\ell_2}$. In particular, if $\|x\|_{\ell_\infty}\le\lambda$ and $A\subset\mathbb{N}$, then $|||(x,\lambda\mathbf{1}_A)||| = f(x,\lambda\mathbf{1}_A) = \|(x,\lambda\mathbf{1}_A)\|_{\ell_2} = \left(\|x\|_{\ell_2}^2+\lambda^2|A|\right)^{1/2}$.*

(iv) *$\|\cdot\|$ and $|||\cdot|||$ are equivalent. More precisely, $\|z\|\ge|||z|||\ge\frac{1}{1+4\sqrt{2}}\|z\|$ for all $z\in c_{00}\oplus c_{00}$.*

Corollary 6.11. *The unit vector basis of $c_{00}\oplus c_{00}$ is normalized, 1-unconditional and not subsymmetric with respect to $|||\cdot|||$.*

Remark. Thus, to prove Theorem 6.9, it is sufficient to show, by Theorem 0.2, that the unit vector basis of $c_{00}\oplus c_{00}$ satisfies Property (A) with respect to $|||\cdot|||$.

Remark. Let $y \in \mathrm{c}_{00}$ with $y \geq 0$. We can write $y = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{A_i}$ where $m \geq 0$, $\lambda_i > 0$ for all i and $\emptyset \neq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_m \subset \mathbb{N}$. This decomposition is unique and we shall call it the *wedding cake decomposition of* y. Note that $M(y) = A_1$, $\|y\|_{\ell_\infty} = \sum_{i=1}^{m} \lambda_i$ and $\|y\|_{\ell_{2,1}} = \sum_{i=1}^{m} \|\lambda_i \mathbf{1}_{A_i}\|_{\ell_2}$. This motivates the following attempt at finding norm-attaining decompositions.

Let $x \in \mathrm{c}_{00}$ be another vector with $x \geq 0$. Set $\lambda = \|x\|_{\ell_\infty} \vee \|y\|_{\ell_\infty}$ and $\lambda_0 = \lambda - \sum_{i=1}^{m} \lambda_i$. This yields a decomposition

$$(x, y) = (x_0, 0) + \sum_{i=1}^{m} (x_i, \lambda_i \mathbf{1}_{A_i}) \ , \qquad \text{where } x_i = \tfrac{\lambda_i}{\lambda} x \ , \quad i = 0, 1, \ldots, m \ .$$

Note that $\|x_i\|_{\ell_\infty} \leq \lambda_i$ for $i = 1, \ldots, m$, and hence (assuming, as we may, that $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$)

$$|||(x, y)||| \leq \|x_0\|_{\ell_2} + \sum_{i=1}^{m} \|x_i + \lambda_i \mathbf{1}_{A_i}\|_{\ell_2} \ .$$

It turns out that this is not a norm-attaining decomposition in general but, as the next two results show, a norm-attaining decomposition exists where on the right-hand side we use the wedding cake decomposition.

Theorem 6.12. *Let $x, y \in \mathrm{c}_{00}$ with $x \geq 0$, $y \geq 0$. Then there exists a norm-attaining decomposition $(x, y) = \sum (x_i, y_i)$ such that*

(i) For all i, either $y_i = 0$ or there exist $\lambda_i > 0$ and $A_i \subset \mathbb{N}$ ($A_i \neq \emptyset$) such that $y_i = \lambda_i \mathbf{1}_{A_i}$ and $x_i \leq \lambda_i$.

(ii) For all i, j with $y_i \neq 0$ and $y_j \neq 0$, we have $A_i \subset A_j$ or $A_j \subset A_i$.

(iii) Given $k, l \in \mathbb{N}$, if $\langle x, e_k^* \rangle \leq \langle x, e_l^* \rangle$, then $\langle x_i, e_k^* \rangle \leq \langle x_i, e_l^* \rangle$ for all i. In particular, $M(x) \subset M(x_i)$ for all i.

Proof. By the symmetry properties of $\|\cdot\|$, f and $|||\cdot|||$, we may assume without loss of generality that $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$.

(i) Let $u, v \in \mathrm{c}_{00}$ with $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$, $u \geq 0$, $v \geq 0$ and $v \neq 0$. We show that there is a decomposition $(u, v) = \sum (u_i, v_i)$ such that $\forall\, i$ either $v_i = 0$ or $\exists\, \lambda_i > 0$, $A_i \subset \mathbb{N}$ ($A_i \neq \emptyset$) such that $v_i = \lambda_i \mathbf{1}_{A_i}$ and $u_i \leq \lambda_i$, and moreover

$$f(u, v) \geq \sum f(u_i, v_i) \ .$$

Then (i) will follow: start with any norm-attaining decomposition of (x, y), as given by Proposition 6.10(ii), and replace each term (u, v) in that decomposition by a further decomposition as above.

Let $\lambda = \|u\|_{\ell_\infty} \vee \|v\|_{\ell_\infty}$ and let $v = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{A_i}$ be the wedding cake decomposition of v. Set $\lambda_0 = \lambda - \sum_{i=1}^{m} \lambda_i$.

We consider two cases. First we assume that $\|u\|_{\ell_\infty} \leq \|v\|_{\ell_\infty}$ (which implies that $\lambda_0 = 0$), and that (u, v) has a set F of moving coordinates on the right-hand side. Fix any $F' \subset \mathbb{N}$ with $|F'| = |F|$ and $F' \cap \operatorname{supp}(u) = \emptyset$ so that

$$f(u, v) = \|(u + \lambda \mathbf{1}_{F'}, v - Fv)\| \ .$$

Note that $F \subset A_1$ and $\sum_{i=1}^m \lambda_i \mathbf{1}_{A_i \setminus F}$ is the wedding cake decomposition of $v - Fv$ (we omit the first term if $F = A_1$). Set $u_i = \frac{\lambda_i}{\lambda} u$ and $v_i = \lambda_i \mathbf{1}_{A_i}$ for $i = 1, \ldots, m$. Then $(u, v) = \sum_{i=1}^m (u_i, v_i)$, $u_i \le \lambda_i$ for all i, and

$$\begin{aligned} f(u,v) &= \|(u + \lambda \mathbf{1}_{F'}, v - Fv)\| = \|u + \lambda \mathbf{1}_{F'}\|_{\ell_2} + \|v - Fv\|_{\ell_{2,1}} \\ &= \sum_{i=1}^m \left\| \tfrac{\lambda_i}{\lambda}\left(u + \lambda \mathbf{1}_{F'}\right) \right\|_{\ell_2} + \sum_{i=1}^m \|\lambda_i \mathbf{1}_{A_i \setminus F}\|_{\ell_2} \\ &= \sum_{i=1}^m \|(u_i + \lambda_i \mathbf{1}_{F'}, \lambda_i \mathbf{1}_{A_i \setminus F})\| \ge \sum_{i=1}^m f(u_i, v_i)\ , \end{aligned}$$

which proves the claim. In the second case we assume that $\|u\|_{\ell_\infty} \ge \|v\|_{\ell_\infty}$ and (u, v) has a set E of moving coordinates on the left-hand side. Fix any $E' \subset \mathbb{N}$ with $|E'| = |E|$ and $E' \cap \mathrm{supp}(v) = \emptyset$ so that

$$f(u,v) = \|(u - Eu, v + \lambda \mathbf{1}_{E'})\|\ .$$

Note that $\lambda_0 \mathbf{1}_{E'} + \sum_{i=1}^m \lambda_i \mathbf{1}_{A_i \cup E'}$ is the wedding cake decomposition of $v + \lambda \mathbf{1}_{E'}$ (with the first term omitted if $\lambda_0 = 0$ or $E' = \emptyset$). Set $u_i = \frac{\lambda_i}{\lambda} u$ and $v_i = \lambda_i \mathbf{1}_{A_i}$ for $i = 0, 1, \ldots, m$ (where $A_0 = \emptyset$, and so $v_0 = 0$). Then $(u, v) = \sum_{i=0}^m (u_i, v_i)$ and $u_i \le \lambda_i$ for all $i = 0, 1, \ldots, m$, and moreover

$$\begin{aligned} f(u,v) &= \|(u - Eu, v + \lambda \mathbf{1}_{E'})\| = \|u - Eu\|_{\ell_2} + \|v + \lambda \mathbf{1}_{E'}\|_{\ell_{2,1}} \\ &= \sum_{i=0}^m \|u_i - Eu_i\|_{\ell_2} + \sum_{i=0}^m \|\lambda_i \mathbf{1}_{A_i \cup E'}\|_{\ell_2} \\ &= \sum_{i=0}^m \|(u_i - Eu_i, \lambda_i \mathbf{1}_{A_i \cup E'})\| \ge \sum_{i=0}^m f(u_i, v_i)\ , \end{aligned}$$

which completes the proof of (i).

For (ii) we argue by contradiction. Assume that for some i, j the sets A_i, A_j are not comparable. Without loss of generality assume that $\lambda_i \le \lambda_j$ and note that

$$\begin{aligned} f(x_i, \lambda_i \mathbf{1}_{A_i}) + f(x_j, \lambda_j \mathbf{1}_{A_j}) = f(x_i, \lambda_i \mathbf{1}_{A_i}) &+ f\left(\tfrac{\lambda_i}{\lambda_j} x_j, \lambda_i \mathbf{1}_{A_j}\right) \\ &+ f\left(\tfrac{\lambda_j - \lambda_i}{\lambda_j} x_j, (\lambda_j - \lambda_i) \mathbf{1}_{A_j}\right)\ . \end{aligned}$$

Next write $(u, \mathbf{1}_A)$ and $(v, \mathbf{1}_B)$ for $\left(\frac{1}{\lambda_i} x_i, \mathbf{1}_{A_i}\right)$ and $\left(\frac{1}{\lambda_j} x_j, \mathbf{1}_{A_j}\right)$, respectively, and assume, without loss of generality, that $\|u\|_{\ell_2}^2 + |A| \le \|v\|_{\ell_2}^2 + |B|$. Set

$r = |A\backslash B|$. Then $(u, \mathbf{1}_A) + (v, \mathbf{1}_B) = (u, \mathbf{1}_{A\cap B}) + (v, \mathbf{1}_{A\cup B})$, and

$$\begin{aligned} f(u, \mathbf{1}_A) + f(v, \mathbf{1}_A) &= \|u + \mathbf{1}_A\|_{\ell_2} + \|v + \mathbf{1}_B\|_{\ell_2} \\ &= \Big(\|u\|_{\ell_2}^2 + |A|\Big)^{1/2} + \Big(\|v\|_{\ell_2}^2 + |B|\Big)^{1/2} \\ &> \Big(\|u\|_{\ell_2}^2 + |A| - r\Big)^{1/2} + \Big(\|v\|_{\ell_2}^2 + |B| + r\Big)^{1/2} \\ &= f(u, \mathbf{1}_{A\cap B}) + f(v, \mathbf{1}_{A\cup B}) \ . \end{aligned}$$

It follows that

$$\begin{aligned} f(x_i, \lambda_i \mathbf{1}_{A_i}) + f(x_j, \lambda_j \mathbf{1}_{A_j}) > f(x_i, \lambda_i \mathbf{1}_{A_i\cap A_j}) + f\Big(\tfrac{\lambda_i}{\lambda_j} x_j, \lambda_i \mathbf{1}_{A_i\cup A_j}\Big) \\ + f\Big(\tfrac{\lambda_j - \lambda_i}{\lambda_j} x_j, (\lambda_j - \lambda_i)\mathbf{1}_{A_j}\Big) \ . \end{aligned}$$

which contradicts the assumption that $(x, y) = \sum (x_i, y_i)$ was a norm-attaining decomposition.

It remains to show (iii). Let $k, l \in \mathbb{N}$ and assume that $\langle x, e_k^*\rangle \le \langle x, e_l^*\rangle$ but $\langle x_i, e_k^*\rangle > \langle x_i, e_l^*\rangle$ for some i. Then there exists j such that $\langle x_j, e_k^*\rangle < \langle x_j, e_l^*\rangle$. Fix $\eta > 0$. Define x_i' by $\langle x_i', e_k^*\rangle = \langle x_i, e_k^*\rangle - \eta$, $\langle x_i', e_l^*\rangle = \langle x_i, e_l^*\rangle + \eta$, and $\langle x_i', e_m^*\rangle = \langle x_i, e_m^*\rangle$ for $m \ne k, l$. Similarly, define x_j' by $\langle x_j', e_k^*\rangle = \langle x_j, e_k^*\rangle + \eta$, $\langle x_j', e_l^*\rangle = \langle x_j, e_l^*\rangle - \eta$, and $\langle x_j', e_m^*\rangle = \langle x_j, e_m^*\rangle$ for $m \ne k, l$. Finally, set $x_h' = x_h$ for $h \ne i, j$. Then $(x, y) = \sum (x_h', y_h')$ and, provided $\eta > 0$ is sufficiently small, we have

$$\sum f(x_h', y_h') = \sum \|x_h' + y_h'\|_{\ell_2} < \sum \|x_h + y_h\|_{\ell_2} = \sum f(x_h, y_h)$$

which contradicts the assumption that $(x, y) = \sum (x_i, y_i)$ was a norm-attaining decomposition. □

Theorem 6.13. *Let $x, y \in c_{00}$ with $x \ge 0$, $y \ge 0$. Let $y = \sum_{i=1}^m \lambda_i \mathbf{1}_{A_i}$ be the wedding cake decomposition of y. Then there exists a norm-attaining decomposition $(x, y) = \sum_{i\in I} (x_i, y_i)$ such that either*

(i) $I = \{0, 1, 2, \ldots, m\}$, $x_0 \ne 0$, $y_0 = 0$ and $y_i = \lambda_i \mathbf{1}_{A_i}$, $x_i \le \lambda_i$ for $i = 1, \ldots, m$.

(ii) $I = \{1, 2, \ldots, m\}$, $y_i = \lambda_i \mathbf{1}_{A_i}$, $x_i \le \lambda_i$ for $i = 1, \ldots, m$.

Moreover, if $\|x\|_{\ell_\infty} \ge \|y\|_{\ell_\infty}$, then $\|x_i\|_{\ell_\infty} = \lambda_i$ for all $i = 1, \ldots, m$.

Remark. The first part of Theorem 6.13 will be an easy consequence of Theorem 6.12. What it shows is that finding $|||(x, y)|||$ is finding the minimum of a convex function on a compact convex set in Euclidean space:

$$|||(x, y)||| = \min\Big\{ \sum_{i\in I} \|x_i + y_i\|_{\ell_2} : \sum_{i\in I} x_i = x,\ 0 \le x_i \le \lambda_i,\ M(x) \subset M(x_i)\ \forall\, i \Big\}$$

(where $\lambda_0 = \infty$ if (i) holds). The second part of the Theorem is what is important here: it shows that if $\|x\|_{\ell_\infty} \geq \|y\|_{\ell_\infty}$, then the minimum is attained on a certain face of this compact, convex set. There seems to be no nice geometric description of where the minimum occurs. The knowledge of this special face turns out to be sufficient for verifying Property (A).

Proof of Theorem 6.13. As usual, we will assume that $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$. Start with a norm-attaining decomposition $(x, y) = \sum_{i\in I}(x_i, y_i)$ satisfying (i), (ii) and (iii) of Theorem 6.12. Note that if for some $i \neq j$ we have $\mathrm{supp}(y_i) = \mathrm{supp}(y_j)$, then we may replace the terms (x_i, y_i) and (x_j, y_j) by the single term $(x_i + x_j, y_i + y_j)$ to obtain a new decomposition $(x, y) = \sum_{h\in I'}(x'_h, y'_h)$ such that $\sum_{h\in I'} f(x'_h, y'_h) \leq \sum_{h\in I} f(x_h, y_h)$. Indeed, we have

$$\begin{aligned} f(x_i, y_i) + f(x_j, y_j) &= \|x_i + y_i\|_{\ell_2} + \|x_j + y_j\|_{\ell_2} \\ &\geq \|x_i + x_j + y_i + y_j\|_{\ell_2} = f(x_i + x_j, y_i + y_j) \ . \end{aligned}$$

Moreover, the new decomposition also satisfies (i), (ii) and (iii) of Theorem 6.12. The first part of 6.13 now follows easily.

Now assume that $\|x\|_{\ell_\infty} \geq \|y\|_{\ell_\infty}$. If $0 \notin I$ (*i.e.,* if we are in alternative (ii)), then set $x_0 = y_0 = 0$. Now define $\mu_i = \|x_i\|_{\ell_\infty}$ for $i = 0, 1, \ldots, m$. Since $M(x) \subset M(x_i)$ for all i, it follows that

$$\|x\|_{\ell_\infty} = \sum_{i=0}^{m} \mu_i \leq \mu_0 + \sum_{i=1}^{m} \lambda_i = \mu_0 + \|y\|_{\ell_\infty} \leq \mu_0 + \|x\|_{\ell_\infty} \ .$$

Hence if $\mu_0 = 0$, then $\mu_i = \lambda_i$ for all $i = 1, \ldots, m$, and the theorem follows. Otherwise, set

$$\varepsilon_i = \frac{\lambda_i - \mu_i}{\mu_0} \qquad \text{for } i = 1, \ldots, m \ .$$

Note that

$$\sum_{i=1}^{m} \varepsilon_i = \frac{\|y\|_{\ell_\infty} - \|x\|_{\ell_\infty} + \mu_0}{\mu_0} \leq 1 \ .$$

Define

$$\bar{x}_i = x_i + \varepsilon_i x_0 \ , \qquad i = 1, \ldots, m \ ,$$

$$\bar{x}_0 = \Big(1 - \sum_{i=1}^{m} \varepsilon_i\Big) x_0 \ .$$

Then $\sum_{i=0}^{m} \bar{x}_i = x$, and for each $i = 1, \ldots, m$ we have

$$\begin{aligned} \langle \bar{x}_i, e_k^* \rangle &= \mu_i + \varepsilon_i \mu_0 = \lambda_i && \text{if } k \in M(x) \\ \langle \bar{x}_i, e_k^* \rangle &= \langle x_i, e_k^* \rangle + \varepsilon_i \langle x_0, e_k^* \rangle \\ &\leq \mu_i + \varepsilon_i \mu_0 = \lambda_i && \text{if } k \notin M(x) \ , \end{aligned}$$

and thus $\|\bar{x}_i\|_{\ell_\infty} = \lambda_i$. Finally, we have

$$\begin{aligned} f(\bar{x}_0, 0) + \sum_{i=1}^{m} f(\bar{x}_i, \lambda_i \mathbf{1}_{A_i}) &= \|\bar{x}_0\|_{\ell_2} + \sum_{i=1}^{m} \|\bar{x}_i + \lambda_i \mathbf{1}_{A_i}\|_{\ell_2} \\ &< \|x_0\|_{\ell_2} + \sum_{i=1}^{m} \|x_i + \lambda_i \mathbf{1}_{A_i}\|_{\ell_2} \end{aligned}$$

unless $\varepsilon_i = 0$ for all i. □

Remark. It is not difficult to see from the above proof that alternative (i) holds if and only if $\|x\|_{\ell_\infty} > \|y\|_{\ell_\infty}$.

Proof of Theorem 6.9. By Proposition 6.10(iv) and Corollary 6.11, it suffices to verify that the unit vector basis of $c_{00} \oplus c_{00}$ satisfies Property (A) with respect to $|||\cdot|||$. In fact, using Proposition 6.10(i), it is enough to show the following claim. Let $x, y \in c_{00}$ and $A \subset \mathbb{N}$; assume that $\text{supp}(x)$, $\text{supp}(y)$ and A are pairwise disjoint, and that $0 \le x \le 1$ and $0 \le y \le 1$; then

$$|||(x + \mathbf{1}_A, y)||| = |||(x, \mathbf{1}_A + y)||| \ .$$

We begin with the proof of the inequality $|||(x + \mathbf{1}_A, y)||| \le |||(x, \mathbf{1}_A + y)|||$ which only really uses Theorem 6.12(i). Let

$$(x, \mathbf{1}_A + y) = (x_0, 0) + \sum_{i=1}^{m} (x_i, \lambda_i \mathbf{1}_{A_i})$$

be a norm-attaining decomposition, where $0 \le x_i \le \lambda_i$ for all $i = 0, 1, \ldots, m$ (and $\lambda_0 = \infty$). Then

$$\begin{aligned} |||(x, \mathbf{1}_A + y)||| &= \|x_0\|_{\ell_2} + \sum_{i=1}^{m} \|x_i + \lambda_i \mathbf{1}_{A_i}\|_{\ell_2} \\ &= f(x_0, 0) + \sum_{i=1}^{m} f(x_i + \lambda_i \mathbf{1}_A, \lambda_i \mathbf{1}_{A_i \setminus A}) \\ &\ge |||(x + \mathbf{1}_A, y)||| \ , \end{aligned}$$

since $(x + \mathbf{1}_A, y) = (x_0, 0) + \sum_{i=1}^{m} (x_i + \lambda_i \mathbf{1}_A, \lambda_i \mathbf{1}_{A_i \setminus A})$.

We now turn to the reverse inequality. Let $y = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{A_i}$ be the wedding cake decomposition of y. By Theorem 6.13 there is a norm-attaining decomposition

$$(x + \mathbf{1}_A, y) = (x_0 + u_0, 0) + \sum_{i=1}^{m} (x_i + u_i, \lambda_i \mathbf{1}_{A_i})$$

where $0 \le x_i \le x$, $0 \le u_i \le \mathbf{1}_A$ for all $i = 0, 1, \ldots, m$, and moreover $x_i \le \lambda_i$, $u_i \le \lambda_i$ for $i = 1, \ldots, m$. By Theorem 6.12(iii) and by the last part of

Theorem 6.13 we have $u_i = \lambda_i \mathbf{1}_A$ for $i = 1, \dots, m$, and $u_0 = \lambda_0 \mathbf{1}_A$, where $\lambda_0 = \left(1 - \sum_{i=1}^m \lambda_i\right)$. It follows that

$$\begin{aligned} |||(x + \mathbf{1}_A, y)||| &= \|x_0 + u_0\|_{\ell_2} + \sum_{i=1}^m \|x_i + u_i + \lambda_i \mathbf{1}_{A_i}\|_{\ell_2} \\ &= f(x_0, u_0) + \sum_{i=1}^m f(x_i, u_i + \lambda_i \mathbf{1}_{A_i}) \\ &\geq |||(x, \mathbf{1}_A + y)||| \ . \end{aligned}$$

This completes the proof. □

We finish this subsection with a simple consequence.

Corollary 6.14. *There is a renorming of ℓ_2 with respect to which the unit vector basis is 1-greedy but not 1-symmetric.*

Proof. Let X be the completion of the subspace

$$\left\{(x, y) : \ x, y \in \mathrm{c}_{00}, \ \mathrm{supp}(y) \subset \{1\}\right\}$$

of $\mathrm{c}_{00} \oplus \mathrm{c}_{00}$ with respect to $|||\cdot|||$ (*i.e.,* we take all vectors on the left-hand side together with the span of e_1 on the right-hand side). It follows from Proposition 6.10(iii) that $|||(x, 0)||| = \|x\|_{\ell_2}$ for all $x \in \mathrm{c}_{00}$, and hence $|||\cdot|||$ restricted to X is equivalent to the ℓ_2-norm. The unit vector basis of X is 1-greedy by Theorem 6.9. We show that it is not 1-symmetric by computing $|||(2e_2, e_1)|||$. It follows from Theorem 6.13 (see also the subsequent remark) that

$$|||(2e_2, e_1)||| = f(e_2, 0) + f(e_2, e_1) = \|e_2\|_{\ell_2} + \|e_1 + e_2\|_{\ell_2} = 1 + \sqrt{2} \ ,$$

which differs from, say, $|||(2e_2 + e_1, 0)||| = \sqrt{5}$. □

Remark. Albiac and Wojtaszczyk gave a renorming of c_0 with respect to which the unit vector basis is 1-greedy but not 1-symmetric [1, Example 5.6].

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